## IX.2 Measurable calculus and spectral decomposition for normal operators

**Proposition 12** (Lax-Milgram). *Let $H$ be a Hilbert space and $B : H \times H \to \mathbb{C}$ a mapping satisfying the following properties.*

- *$x \mapsto B(x, y)$ is linear for each $y \in H$.*
- *$y \mapsto B(x, y)$ is conjugate linear for each $x \in H$.*
- *$\|B\| = \sup\{|B(x, y)| \,; x, y \in B_H\} < \infty$.*

*Then there is a unique $T \in L(H)$ such that $B(x, y) = \langle Tx, y\rangle$ for $h, k \in H$. Moreover, $\|T\| = \|B\|$.*

**Constructing the spectral measure of a normal operator - Step 1.** Let $H$ be a Hilbert space and let $T \in L(H)$ be a normal operator. Let $f \mapsto \tilde{f}(T)$, $f \in \mathcal{C}(\sigma(T))$, be the continuous functional calculus for $T$. For any $x, y \in H$ let $E_{x,y}$ denote the (unique) complex Radon measure on $\sigma(T)$ satisfying

$$\left\langle \tilde{f}(T)x, y \right\rangle = \int_{\sigma(T)} f \, \mathrm{d}E_{x,y}, \qquad f \in \mathcal{C}(\sigma(T)).$$

**Proposition 13** (properties of the measures $E_{x,y}$). *Using the above notation, the following holds:*

(a) *$x \mapsto E_{x,y}$ is linear for each $y \in H$.*
(b) *$y \mapsto E_{x,y}$ is conjugate linear for each $x \in H$.*
(c) *$E_{x,x}$ is a non-negative measure for each $x \in H$.*
(d) *$\|E_{x,y}\| \le \cancel{\|T\|} \cdot \|x\| \cdot \|y\|$ for $x, y \in H$.*
(e) *$E_{x,y} = \frac{1}{4}(E_{x+y,x+y} - E_{x-y,x-y} + iE_{x+iy,x+iy} - iE_{x-iy,x-iy})$ for $x, y \in H$.*

**Measurable calculus and the spectral mesure.** We use the above notation.

- Denote by $\mathcal{A}$ the $\sigma$-algebra of all the subsets of $\sigma(T)$ which are $E_{x,y}$-measurable for each $x, y \in H$. (Recall that $A$ is $E_{x,y}$-measurable if and only if there are Borel sets $B, C$ such that $B \subset A \subset C$ and $|E_{x,y}|\,(B \setminus C) = 0$.) Then $\mathcal{A}$ is the $\sigma$-algebra of all the subsets of $\sigma(T)$ which are $E_{x,x}$-measurable for each $x \in H$.
- Let $f : \sigma(T) \to \mathbb{C}$ be a bounded $\mathcal{A}$-measurable function By $\tilde{f}(T)$ we denote the operator in $L(H)$ satisfying

$$\left\langle \tilde{f}(T)x, y \right\rangle = \int_{\sigma(T)} f \, \mathrm{d}E_{x,y}, \qquad x, y \in H.$$

  Its existence and uniqueness is provided by Proposition 12. The assignement $f \mapsto \tilde{f}(T)$ is called the **measurable calculus** for $T$.
- For $A \in \mathcal{A}$ set $E_T(A) = \widetilde{\chi_A}(T)$. The assignement $E_T : A \mapsto E_T(A)$ is called the **spectral measure** of $T$.
- Denote by $\mathcal{N}$ the subfamily of $\mathcal{A}$ formed by the sets which are $|E_{x,y}|$-null for each $x, y \in H$. $\mathcal{N}$ is the family of all the sets which are $E_{x,x}$-null for each $x \in H$.
- Denote by $L^\infty(E_T)$ the space of all the bounded $\mathcal{A}$-measurable functions on $\sigma(T)$, where we identify the functions which are equal everywhere except on a set from $\mathcal{N}$. Equip $L^\infty(E_T)$ with the norm

$$\|f\| = \operatorname*{ess\,sup}_{\lambda \in \sigma(T)} |f(\lambda)| = \inf\{c > 0; \{\lambda \in \sigma(T); f(\lambda) > c\} \in \mathbb{N}\}.$$

  Then $L^\infty(E_T)$ is a commutative $C^*$-algebra (with the pointwise multiplication and the involution defined as the complex conjugation).
- $\tilde{f}(T)$ is defined exactly for $f \in L^\infty(E_T)$. Moreover, $\tilde{f}(T)$ is then well defined, i.e., $\tilde{f}(T) = \tilde{g}(T)$ whenever $f = g$ except on a set from $\mathcal{N}$.

**Lemma 14** (a consequence of Luzin's theorem).

(a) *Let $K$ be a compact metric space and let $\mu$ be a non-negative finite Borel measure on $K$. Let $f : K \to \mathbb{C}$ be a bounded $\mu$-measurable function. Then there is a uniformly bounded sequence $(f_n)$ in $\mathcal{C}(K)$ such that $f_n \to f$ $\mu$-almost everywhere. In particular, there is a bounded Borel function $g$ on $\sigma(T)$ such that $f = g$ $\mu$-almost everywhere.*
(g) *Let $H$ be a separable Hilbert space and let $T \in L(H)$ be a normal operator. Let $f \in L^\infty(E_T)$ Then there is a uniformly bounded sequence $(f_n)$ in $\mathcal{C}(\sigma(T))$ such that $f_n \to f$ except on a set from $\mathcal{N}$. In particular, there exists a bounded Borel function $g$ on $\sigma(T)$ such that $f = g$ except on a set form $\mathcal{N}$.*

**Theorem 15** (properties of the measurable calculus). *Let $H$ be a Hilbert space and $T \in L(H)$ be a normal operator.*

(a) *$f \mapsto \tilde{f}(T)$ is an isometric $*$-isomorphism of $L^\infty(E)$ into $L(H)$.*
(b) *If $(f_n)$ is a bounded sequence in $L^\infty(E)$ which pointwise converges to a function $f$ (except on a set from $\mathcal{N}$), then $f \in L^\infty(E)$ and, moreover,*

$$\left\langle \tilde{f}_n(T)x, y \right\rangle \to \left\langle \tilde{f}(T)x, y \right\rangle, \qquad x, y \in H.$$

(c) *$\sigma(\tilde{f}(T)) = \operatorname{ess\,rng}(f) = \{\lambda \in \mathbb{C}; \forall r > 0 : f^{-1}(U(\lambda, r)) \notin \mathcal{N}\}$ for $f \in L^\infty(E)$.*
(d) *$\tilde{f}(T)$ is a normal operator for each $f \in L^\infty(E)$. $\tilde{f}(T)$ is self-adjoint if and only if $f$ is essentially real-valued (i.e., $f(\lambda) \in \mathbb{R}$ except on a set from $\mathcal{N}$).*
(e) *$\tilde{g}(\tilde{f}(T)) = \widetilde{g \circ f}(T)$ whenever $f \in L^\infty(E)$ and $g$ is continuous on $\sigma(\tilde{f}(T))$ (see (c)).*
(f) *If $S \in L(H)$ commutes with $T$, then $S$ commutes with $\tilde{f}(T)$ for each $f \in L^\infty(E)$.*

**Definition.** An **abstract spectral measure** in a Hilbert space $H$ is a mapping $E$ with the following properties:

(i) The domain of $E$ is a $\sigma$-algebra $\mathcal{A}$ of subsets of $\mathbb{C}$ containing all the Borel sets.
(ii) $E(A)$ is an orthogonal projection on $H$ for each $A \in \mathcal{A}$.
(iii) $E(\emptyset) = 0$, $E(\mathbb{C}) = I$.
(iv) If $A \in \mathcal{A}$ satisfies $E(A) = 0$, then $B \in \mathcal{A}$ (and $E(B) = 0$) for each $B \subset A$.
(v) $E(A \cap B) = E(A)E(B)$ for $A, B \in \mathcal{A}$.
(vi) $E(A \cup B) = E(A) + E(B)$ whenever $A, B \in \mathcal{A}$, $A \cap B = \emptyset$.
(vii) For each pair $x, y \in H$ the mapping $E_{x,y} : A \mapsto \langle E(A)x, y \rangle$ is a complex Borel measure on $\mathbb{C}$.

The spectral measure $E$ is called **compactly supported** if there is a compact set $K \subset \mathbb{C}$ such that $E(\mathbb{C} \setminus K) = 0$.

Recall that $\mu$ is a **Borel measure** if it is a $\sigma$-additive measure defined on a $\sigma$-algebra $\mathcal{A}_\mu$ containing all Borel sets such that for any $A \in \mathcal{A}_\mu$ there are Borel sets $B, C$ such that $B \subset A \subset C$ and $|\mu|(B \setminus C) = 0$.

**Lemma 16.** *If $T \in L(H)$ is a normal operator, then $E_T$ is a compactly supported abstract spectral measure.*

**Proposition 17** (integral with respect to an abstract spectral measure). *Let $E$ be an abstract spectral measure defined on a $\sigma$-algebra $\mathcal{A}$. Let $f : \mathbb{C} \to \mathbb{C}$ a bounded $\mathcal{A}$-measurable function. Then there is a unique $T \in L(H)$ such that*

$$\langle Tx, y \rangle = \int f \,\mathrm{d}E_{x,y}, \qquad x, y \in H.$$

*Moreover, $\|T\| \le \|f\|_\infty$.*

**Theorem 18.** *Let $E$ be an abstract spectral measure defined on a $\sigma$-algebra $\mathcal{A}$. Define $\mathcal{N}$ and $L^\infty(E)$ in the same way as above (for $E_T$). Then the following holds:*

(a) *The mapping $\Psi : f \mapsto \int f \,\mathrm{d}E$ is an isometric $*$-isomorphism of the $C^*$-algebra $L^\infty(E)$ into $L(H)$.*
(b) *For each $f \in L^\infty(E)$ the operator $\Psi(f)$ is normal. Moreover, $\Psi(f)$ is self-adjoint if and only if $f$ is real-valued except on a set from $\mathcal{N}$ and $\Psi(f)$ is a positive operator if and only if $f \ge 0$ except on a set from $\mathcal{N}$.*
(c) *$\|\Psi(f)x\| = \left(\int |f|^2 \,\mathrm{d}E_{x,x}\right)^{\frac{1}{2}}$ for $f \in L^\infty(E)$ and $x \in H$.*
(d) *If $f \in L^\infty(E)$ and $g \in \mathcal{C}(\sigma(\Psi(f)))$, then $\Psi(g \circ f) = \tilde{g}(\Psi(f))$.*

**Lemma 19.** *Let $E$ be an abstract spectral measure, $f \in L^\infty(E)$ and $T = \int f \,\mathrm{d}E$. Then the spectral measure $E_T$ of $T$ is defined by $E_T(A) = E(f^{-1}(A))$.*

**Corollary 20** (spectral decomposition of a normal operator). *. Let $H$ be a Hilbert space and $T \in L(H)$ a normal operator. Then there is a unique abstract spectral measure such that $T = \int \mathrm{id} \,\mathrm{d}E$. Moreover, this is the measure $E_T$.*

**Theorem 21.** *Let $H$ be a Hilbert space and $T \in L(H)$ a normal operator. Then there is a nonnegative measure $\mu$ (defined on some measurable space), a unitary operator $U : H \to L^2(\mu)$ and a function $g \in L^\infty(\mu)$ such that*

$$Tx = U^*(g \cdot Ux), \qquad x \in H.$$

*If $H$ is separable, $\mu$ can be chosen to be $\sigma$-finite.*