V.2 Continuous and bounded linear mappings

Proposition 6. *Let $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ be TVS over $\mathbb{F}$ and let $L : X \to Y$ be a linear mapping. The following assertions are equivalent:*

(i) *L is continuous.*
(ii) *L is continuous at $\boldsymbol{o}$.*
(iii) *L is* **uniformly continuous***, i.e.,*

$$\forall U \in \mathcal{U}(\boldsymbol{o})\, \exists V \in \mathcal{T}(\boldsymbol{o})\, \forall x, y \in X : x - y \in V \Rightarrow L(x) - L(y) \in U.$$

Proposition 7. *Let $(X,\mathcal{T})$ be a TVS over $\mathbb{F}$ and let $L : X \to \mathbb{F}$ be a linear mapping. The following assertions are equivalent:*

(i) *L is continuous.*
(ii) $\ker L$ *is a closed subspace of X.*
(iii) *There exists $U \in \mathcal{T}(\boldsymbol{o})$ such that $L(U)$ is a bounded subset of $\mathbb{F}$.*

If L is discontinuous, then $\ker L$ *is a dense subspace of X.*

Definition. Let $(X,\mathcal{T})$ be a TVS and let $A \subset X$. The set A is said to be **bounded** in $(X,\mathcal{T})$, if for any $U \in \mathcal{T}(\boldsymbol{o})$ there exists $\lambda > 0$ such that $A \subset \lambda U$.

Proposition 8. *Let $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ be TVS over $\mathbb{F}$ and let $L : X \to Y$ be a linear mapping. Consider the following two assertions:*

(i) *L is continuous.*
(ii) *For any bounded subset $A \subset X$ its image $L(A)$ is bounded in Y (i.e., L is a* **bounded mapping***).*

Then (i)$\Rightarrow$(ii). In case $\mathcal{T}$ is generated by a translation invariant metric on X, then (i)$\Leftrightarrow$(ii).

Remark. It follows from Theorem 12 in Section V.4 that, whenever a TVS $(X,\mathcal{T})$ is metrizable, i.e., the topology $\mathcal{T}$ is generated by a metric, then this metric can be chosen to be translation invariant.

Definition. Let $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ be TVS over $\mathbb{F}$ and let $L : X \to Y$ be a linear mapping. The mapping L is said to be

- an **isomorphism of X into Y** if L is continuous, one-to-one and L^{-1} is continuous on $L(X)$;
- an **isomorphism of X onto Y**, if L is continuous, one-to-one, onto and L^{-1} is continuous on Y.

The spaces X and Y are said to be **isomorphic** if there is an isomorphism of X onto Y.

V.3 Spaces of finite and infinite dimension

Proposition 9. *Let X be a HTVS of finite dimension.*

(a) *If Y is any TVS and $L : X \to Y$ is any linear mapping, then L is continuous.*
(b) *The space X is isomorphic to $\mathbb{F}^n$, where $n = \dim X$.*

Corollary 10. *Let X be a HTVS. Then any its finite-dimensional subspace is closed.*

Definition. Let $(X,\mathcal{T})$ be a TVS and let $A \subset X$. The set A is said to be **totally bounded** (or **precompact**), if for any $U \in \mathcal{T}(\boldsymbol{o})$ there exists a finite set $F \subset X$ such that $A \subset F + U$.

Remark: Any compact set in any TVS is totally bounded. Any totally bounded set is bounded.

Theorem 11. *Let X be a HTVS. The following assertions are equivalent:*

(i) $\dim X < \infty$.
(ii) *There exists a compact neighborhood of zero in X.*
(iii) *There exists a totally bounded neighborhood of zero in X.*