V28 (a) $\mathcal{S} \subset C_0(\mathbb{R}^d) \cap \bigcap_{p \in [1,\infty]} L^p(\mathbb{R}^d)$

$\ulcorner$ $f \in \mathcal{S} \Rightarrow f$ je spojita in $t \mapsto (1+\|t\|^2) f(t)$ je omejena, torej f ima v ∞ limito 0

Pripomnimo: $C_0(\mathbb{R}^d) = \{ f : \mathbb{R}^d \to \mathbb{F}$ spojita, $\lim_{\|t\| \to \infty} f(t) = 0 \}$

$\forall \varepsilon > 0 \; \exists T > 0 \; \forall t : \|t\| > T \Rightarrow |f(t)| < \varepsilon$

Poleg $|f(t)| \leq \dfrac{M}{1+\|t\|^2}$, poleg tega f zvezna spojita

$\Rightarrow f \in L^\infty(\mathbb{R}^d)$ (f omejena)

$p \in [1,\infty) \Rightarrow$ pokaži $m > \dfrac{d}{2p}$, pač $t \mapsto \dfrac{1}{(1+\|t\|^2)^m} \in L^p$

$f(t) \cdot (1+\|t\|^2)^m$ omejena

$$\Rightarrow |f(t)| \leq \frac{M_m}{(1+\|t\|^2)^m} \Rightarrow f \in L^p \quad \lrcorner$$

(b) P polinom na $\mathbb{R}^d$
$\Rightarrow f \mapsto P \cdot f$ je spojita $\mathcal{S} \to \mathcal{S}$

$$|P(t)| = \Big|\sum_{|\alpha| \leq N} c_\alpha t^\alpha\Big| \leq \sum_{|\alpha| \leq N} |c_\alpha| \cdot \|t\|^{|\alpha|} \leq$$

$$\leq \Big(\sum_{|\alpha| \leq N} |c_\alpha|\Big) \cdot (1+\|t\|^2)^N$$

$\|t\| \geq 1 \Rightarrow \|t\| \leq \|t\|^2 \leq 1+\|t\|^2$

$\|t\| \leq 1 \Rightarrow \|t\| \leq 1 \leq 1+\|t\|^2$

Tedy $\forall P$ stupně $\leq N$ existuje C_P, že

$$|P(t)| \leq C_P (1+\|t\|^2)^N$$

Tedy: Nechť P je polynom stupně N a $M \geq 0$, $|\alpha| \leq M$

$$\left|(1+\|x\|^2)^M \cdot D^\alpha (P\cdot f)(t)\right| = (1+\|t\|^2)^M \cdot \Big|\sum_{\substack{\beta\leq\alpha \\ |\beta|\leq N}} d_{\alpha,\beta} \cdot D^\beta P(t) D^{\alpha-\beta} f(t)\Big|$$

$$\leq (1+\|t\|^2)^M \sum_{\substack{\beta\leq\alpha \\ |\beta|\leq N}} d_{\alpha,\beta} |D^\beta P(t)| \, |D^{\alpha-\beta} f(t)| \leq$$

$$\leq (1+\|t\|^2)^{M+N} \sum_{\substack{\beta\leq\alpha \\ |\beta|\leq N}} d_{\alpha,\beta} \cdot C_{D^\beta P} \cdot |D^{\alpha-\beta} f(t)| \leq$$

$$\leq \Big(\sum_{\substack{\beta\leq\alpha \\ |\beta|\leq N}} d_{\alpha\beta} \cdot C_{D^\beta P}\Big) \cdot \|f\|_{M+N}$$

$$\Rightarrow \|P\cdot f\|_M \leq \Big(\max_{|\alpha|\leq M} \sum_{\substack{\beta\leq\alpha \\ |\beta|\leq N}} d_{\alpha,\beta} \cdot C_{D^\beta P}\Big) \cdot \|f\|_{M+N}$$

Odtud snadno plyne: $f_n \to f$ v $\mathcal{S} \Rightarrow P\cdot f_n \to Pf$ v $\mathcal{S}$

<u>Dále:</u> $g \in \mathcal{S} \Rightarrow f \mapsto g\cdot f$ je spojité zobrazení $\mathcal{S} \to \mathcal{S}$

<u>Podobný výpočet:</u> $M \geq 0$, $|\alpha| \leq M$

$$|(1+\|t\|^2)^M D^\alpha (g\cdot f)(t)| \leq (1+\|t\|^2)^M \sum_{\beta\leq\alpha} d_{\alpha,\beta} |D^\beta g(t)| \, |D^{\alpha-\beta} f(t)|$$

$$\leq \sum_{\beta\leq\alpha} d_{\alpha,\beta} \cdot \|g\|_M \cdot \|f\|_M$$

$$\Rightarrow \|g\cdot f\|_M \leq \max_{|\alpha|\leq M}\Big(\sum_{\beta\leq\alpha} d_{\alpha\beta}\Big) \cdot \|g\|_M \cdot \|f\|_M$$

odtud je spojitost snadno vidět

Dokaz: α multiindeks $\Rightarrow f \mapsto D^\alpha f$ preslikava $\mathcal{S} \to \mathcal{S}$

$$\left|(1+\|t\|^2)^N D^\beta (D^\alpha f(t))\right| \leq \|f\|_{N+|\alpha|}$$

$|\beta| \leq N$

$$\text{Torej } \|D^\alpha f\|_N \leq \|f\|_{N+|\alpha|}$$

odkod je sledi zveznost

(c) $f \in \mathcal{S}$, P polinom $\Rightarrow \widehat{P(D)f} = \check{P} \cdot \hat{f}$

Stači pokazati indukcijo z dokazom $\widehat{\frac{\partial f}{\partial t_j}}(\xi) = i\xi_j \cdot \hat{f}(\xi)$

Dokažimo za $j=1$ (krajši zapis)

$f \in \mathcal{S} \Rightarrow \frac{\partial f}{\partial t_1} \in \mathcal{S}$, torej obe sta v L^1, potem ima Four. transf. smisel

$$\widehat{\frac{\partial f}{\partial t_1}}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{\partial f}{\partial t_1}(t) \cdot e^{-i\langle \xi, t\rangle} dt = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial t_1}(t)\, e^{-i\langle \xi, t\rangle} dt_1 \right) dt_2 \dots dt_d$$

$$= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^{d-1}} \underbrace{\left[f(t) \cdot e^{-i\langle \xi, t\rangle} \right]_{t_1=-\infty}^{+\infty}}_{\substack{\| \\ 0}\ f \in \mathcal{S} \subset C_0(\mathbb{R}^d)} - \int_{\mathbb{R}} f(t) \cdot (-i\xi_1)\, e^{-i\langle \xi, t\rangle} dt_1 \Big) dt_2 \dots dt_d$$

$$= \frac{i\xi_1}{(2\pi)^{d/2}} \int_{\mathbb{R}^{d-1}} f(t)\, e^{-i\langle \xi, t\rangle} dt = i\xi_1 \hat{f}(\xi).$$

Dle výpočtu v (b) existují konstanty

$$\left|(1+\|t\|^2)^d P(D)\check{Q} f(x)\right| \leq \|P(D)\check{Q} f\|_d \leq C_1 \|\check{Q} f\|_{d+2N}$$

$$\leq C_1 \cdot C_2 \cdot \|f\|_{d+2N+N}$$

Tedy

$$\|P(D)\check{Q} f\|_{L^1} \leq \int_{\mathbb{R}^d} \frac{C_1 C_2 \|f\|_{d+3N}}{(1+\|t\|^2)^d} \, d m_d(t)$$

$$= C_1 \cdot C_2 \cdot \underbrace{\left(\int_{\mathbb{R}^d} \frac{1}{(1+\|t\|^2)^d} \, d m_d(t) \right) \cdot \|f\|_{d+3N}}$$

Tedy $\|\hat{f}\|_N \leq$ — " —

Odtud plyne snadno spojitost.

<u>Důsledek 29</u> Fourierova transformace je spojité lineární zobrazení $L^1(\mathbb{R}^d)$ do $C_0(\mathbb{R}^d)$ s normou ≤ 1

<u>Dk:</u> Označme $\mathcal{F}(f) = \hat{f}$ pro $f \in L^1(\mathbb{R}^d)$

Víme, že $\mathcal{F}: L^1(\mathbb{R}^d) \to \ell^\infty(\mathbb{R}^d)$, $\|\mathcal{F}\| \leq 1$

(už z poznámky za definicí Fourierovy transformace.)

Zbývá ukázat, že $\mathcal{F}(L^1(\mathbb{R}^d)) \subset C_0(\mathbb{R}^d)$

Dle V 28 (d) a (a), je $\mathcal{F}(\mathcal{S}) \subset \mathcal{S} \subset C_0(\mathbb{R}^d)$

Navíc, $\mathcal{S}$ je hustý v $L^1(\mathbb{R}^d)$ [$\mathcal{S} \supset \mathcal{D}(\mathbb{R}^d)$ + Důsledek 3]

$C_0(\mathbb{R}^d)$ uzavřený v $\ell^\infty(\mathbb{R}^d)$ $\Rightarrow$ $\mathcal{F}(L^1(\mathbb{R}^d)) \subset C_0(\mathbb{R}^d)$