On optimality of constants
in the Little Grothendieck Theorem

by

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Abstract. We explore the optimality of the constants making valid the recently established little Grothendieck inequality for JB*-triples and JB*-algebras. In our main result we prove that for each bounded linear operator T from a JB*-algebra B into a complex Hilbert space H and $\varepsilon > 0$, there is a norm-one functional $\varphi \in B^*$ such that

$$\|Tx\| \leq (\sqrt{2} + \varepsilon)\|T\| \, \|x\|_\varphi \quad \text{for } x \in B.$$

The constant appearing in this theorem improves the best value known up to date (even for C*-algebras). We also present an easy example witnessing that the constant cannot be strictly smaller than $\sqrt{2}$, hence our main theorem is 'asymptotically optimal'. For type I JBW*-algebras we establish a canonical decomposition of normal functionals which may be used to prove the main result in this special case and also seems to be of an independent interest. As a tool we prove a measurable version of the Schmidt representation of compact operators on a Hilbert space.

1. Introduction. We investigate the optimal values of the constant in the Little Grothendieck Theorem for JB*-algebras. The story begins in 1953 when A. Grothendieck [20] proved his famous theorem on factorization of bilinear forms on spaces of continuous functions through Hilbert spaces. A weaker form of this result, called the Little Grothendieck Theorem, can be formulated as a canonical factorization of bounded linear operators from spaces of continuous functions into a Hilbert space. It was also proved by A. Grothendieck [20] (see also [43, Theorem 5.2]) and reads as follows.

THEOREM A. *There is a universal constant k such that for any bounded linear operator $T : C(K) \to H$, where K is a compact space and H is*

[1]

a Hilbert space, there is a Radon probability measure μ on K such that

$$\|Tf\| \leq k\|T\| \left(\int |f|^2 \, d\mu \right)^{1/2} \quad \text{for } f \in C(K).$$

Moreover, the optimal value of k is $2/\sqrt{\pi}$ in the complex case and $\sqrt{\pi/2}$ in the real case.

The Grothendieck Theorem was later extended to the case of C*-algebras by Pisier [42] and Haagerup [21]. The 'little version' of the extension reads as follows. Henceforth, all Hilbert spaces considered in this note will be over the complex field.

THEOREM B. *Let A be a C^*-algebra, H a Hilbert space and $T : A \to H$ a bounded linear operator. Then there are two states $\varphi_1, \varphi_2 \in A^*$ such that*

$$\|Tx\| \leq \|T\|(\varphi_1(x^*x) + \varphi_2(xx^*))^{1/2} \quad \text{for } x \in A.$$

Moreover, the constant 1 on the right-hand side is optimal.

The positive part of the previous theorem is due to Haagerup [21], the optimality result was proved by Haagerup and Itoh in [22] (see also [43, Section 11]). Let us recall that a *state* on a C^*-algebra is a positive functional of norm one, hence in the case of a complex $C(K)$ space (which is a commutative C^*-algebra), a state is just a functional represented by a probability measure. Hence, as a consequence of Theorem B we get a weaker version of the complex version of Theorem A with $k \leq \sqrt{2}$.

Let us point out that Theorem B is specific for (non-commutative) C^*-algebras due to the asymmetric role played there by the products xx^* and x^*x. To formulate its symmetric version recall that the Jordan product on a C^*-algebra A is defined by

$$x \circ y = \tfrac{1}{2}(xy + yx) \quad \text{for } x, y \in A.$$

Using this notation we may formulate the following consequence of Theorem B.

THEOREM C. *Let A be a C^*-algebra, H a Hilbert space and $T : A \to H$ a bounded linear operator. Then there is a state $\varphi \in A^*$ such that*

$$\|Tx\| \leq 2\|T\|\varphi(x \circ x^*)^{1/2} \quad \text{for } x \in A.$$

To deduce Theorem C from Theorem B it suffices to take $\varphi = \tfrac{1}{2}(\varphi_1 + \varphi_2)$ and to use positivity of the elements xx^* and x^*x. However, in this case the question on optimality of the constant remains open.

QUESTION 1.1. *Is the constant 2 in Theorem C optimal?*

It is easy to show that the constant should be at least $\sqrt{2}$ (see Example 7.10 below) and, to the best of our knowledge, no counterexample is known showing that $\sqrt{2}$ is not enough.

A further generalization of the Grothendieck Theorem, to the setting of JB*-triples (see Section 2 for basic definitions and properties), was suggested by J. T. Barton and Y. Friedman [3]. However, their proof contained a gap found later by Peralta and Rodríguez Palacios [39, 40] who proved a weaker variant of the theorem. A correct proof was recently provided in [25]. The 'little versions' of these results are summarized in the following theorem.

THEOREM D. *Let E be a JB*-triple, H a Hilbert space and $T : E \to H$ a bounded linear operator.*

(1) *If T^{**} attains its norm, there is a norm-one functional $\varphi \in E^*$ such that*
$$\|Tx\| \leq \sqrt{2}\,\|T\|\,\|x\|_\varphi \quad \text{for } x \in E.$$

(2) *Given $\varepsilon > 0$, there are norm-one functionals $\varphi_1, \varphi_2 \in E^*$ such that*
$$\|Tx\| \leq (\sqrt{2} + \varepsilon)\|T\|(\|x\|_{\varphi_1}^2 + \varepsilon\|x\|_{\varphi_2}^2)^{1/2} \quad \text{for } x \in E.$$

(3) *Given $\varepsilon > 0$, there is a norm-one functional $\varphi \in E^*$ such that*
$$\|Tx\| \leq (2 + \varepsilon)\|T\|\,\|x\|_\varphi \quad \text{for } x \in E.$$

The prehilbertian seminorms $\|\cdot\|_\varphi$ used in the statement are defined in Subsection 2.1 below.

Let us comment on the history and the differences of the three versions. It was claimed in [3, Theorem 1.3] that assertion (1) holds without the additional assumption on attaining the norm, because those authors assumed this assumption is satisfied automatically. In [39] and [40, Example 1 and Theorem 3] it was pointed out that this is not the case and assertion (2) was proved using a variational principle from [44]. In [40, Lemma 3] also assertion (1) was formulated.

Note that in (2) not only is the constant $\sqrt{2}$ replaced by a slightly larger one, but also the prehilbertian seminorm on the right-hand side is perturbed. This perturbation was recently avoided in [25, Theorem 6.2], at the cost of squaring the constant. Further, although the proof from [3] was not correct, up to now there is no counterexample to the statement itself. In particular, the following question remains open.

QUESTION 1.2. *What is the optimal constant in assertion* (3) *of Theorem* D*? In particular, does assertion* (1) *of the theorem hold without assuming the norm attainment?*

The main result of this note is the following partial answer.

THEOREM 1.3. *Let B be a JB*-algebra, H a Hilbert space and $T : B \to H$ a bounded linear operator. Given $\varepsilon > 0$, there is a norm-one functional $\varphi \in B^*$ such that*
$$\|Tx\| \leq (\sqrt{2} + \varepsilon)\|T\|\,\|x\|_\varphi \quad \text{for } x \in B.$$
In particular, this holds if B is a C*-algebra.

Note that JB*-algebras form a subclass of JB*-triples and can be viewed as a generalization of C*-algebras (see the next section). We further remark that the previous theorem is 'asymptotically optimal' as the constant cannot be strictly smaller than $\sqrt{2}$ by Example 7.2 below.

The paper is organized as follows. Section 2 contains basic background on JB*-triples and JB*-algebras. In Section 3 we formulate the basic strategy of the proof using majorization results for prehilbertian seminorms.

In Section 4 we deal with a large subclass of JBW*-algebras (finite ones and those of type I). The main result of this section is Proposition 4.2 which provides a canonical decomposition of normal functionals on the just commented JBW*-algebras. This statement may be used to prove the main result in this special case and, moreover, it seems to be of an independent interest. As a tool we further establish a measurable version of Schmidt decomposition of compact operators (see Theorem 4.4).

In Section 5 we address Jordan subalgebras of von Neumann algebras. Section 6 contains a synthesis of the previous sections, the proof of the main result and some consequences. In particular, we show that Theorem B (with the precise constant) follows easily from Theorem 1.3.

Section 7 contains several examples witnessing optimality of some results and related open problems. In Section 8 we discuss the possibility of extending our results to general JB*-triples.

2. Basic facts on JB*-triples and JB*-algebras. It is known that in most cases, like in $B(H)$, the hermitian part of a C*-algebra A need not be a subalgebra of A because it is not necessarily closed for the associative product. This instability can be avoided, at the cost of losing associativity, by replacing the associative product ab in A with the *Jordan product* defined by

$$(1) \qquad a \circ b := \tfrac{1}{2}(ab + ba).$$

This may be seen as an inspiration for the following abstract definitions. A real or complex *Jordan algebra* is a non-necessarily associative algebra B over $\mathbb{R}$ or $\mathbb{C}$ whose multiplication (denoted by $\circ$) satisfies the identities

$$(2) \qquad \begin{aligned} x \circ y &= y \circ x && \text{(commutative law)}, \\ (x \circ y) \circ x^2 &= x \circ (y \circ x^2) && \text{(Jordan identity)} \end{aligned}$$

for all $x, y \in B$, where $x^2 = x \circ x$.

Jordan algebras were the mathematical structures designed by the theoretical physicist P. Jordan to formalize the notion of an algebra of observables in quantum mechanics in 1933. The term 'Jordan algebra' was introduced by A. A. Albert in the 1940s. Promoted by the pioneering works of I. Kaplansky, E. M. Alfsen, F. W. Shultz, H. Hanche-Olsen, E. Størmer, J. D. M. Wright

and M. A. Youngson, JB*- and JBW*-algebras are Jordan extensions of C*- and von Neumann algebras. A *JB*-algebra* is a complex Jordan algebra $(B, \circ)$ equipped with a complete norm $\| \cdot \|$ and an involution $*$ satisfying the following axioms:

(a) $\|x \circ y\| \leq \|x\| \, \|y\|$ for $x, y \in B$;
(b) $\|U_x(x^*)\| = \|x\|^3$ for $x \in B$ (a *Gelfand–Naimark type axiom*),

where $U_x(y) = 2(x \circ y) \circ x - x^2 \circ y$ $(x, y \in B)$. These axioms guarantee that the involution of every JB*-algebra is an isometry (see [51, Lemma 4] or [9, Proposition 3.3.13]).

JB*-algebras were also called *Jordan C*-algebras* by I. Kaplansky and other authors at the early stages of the theory.

Every C*-algebra is a JB*-algebra with its original norm and involution and the Jordan product defined in (1). Actually, every norm-closed self-adjoint Jordan subalgebra of a C*-algebra is a JB*-algebra. Those JB*-algebras obtained as JB*-subalgebras of C*-algebras are called *JC*-algebras*. There exist JB*-algebras which are *exceptional* in the sense that they cannot be identified with a JB*-subalgebra of a C*-algebra: this is the case of the JB*-algebra $H_3(\mathbb{O})$ of all 3×3-hermitian matrices with entries in the algebra $\mathbb{O}$ of complex octonions (see, for example, [26, §7.2], [10, §§6.1 and 7.1] or [23, §§6.2 and 6.3]).

A *JBW*-algebra* (respectively, a *JW*-algebra*) is a JB*-algebra (respectively, a JC*-algebra) which is also a dual Banach space.

JB*-algebras are intrinsically connected with another mathematical object deeply studied in the literature. A *JB-algebra* is a real Jordan algebra J equipped with a complete norm satisfying

(3) $$\|a^2\| = \|a\|^2, \quad \text{and} \quad \|a^2\| \leq \|a^2 + b^2\| \quad \text{for all } a, b \in J.$$

In a celebrated lecture in St. Andrews in 1976, I. Kaplansky suggested the definition of JB*-algebra and pointed out that the self-adjoint part $B_{sa} = \{x \in B; \, x^* = x\}$ of a JB*-algebra is always a JB-algebra. One year later, J. D. M. Wright contributed one of the most influential results in the theory of JB*-algebras by proving that the complexification of every JB-algebra is a JB*-algebra (see [49]). A *JC-algebra* (respectively, a *JW-algebra*) is a norm-closed (respectively, a weak*-closed) real Jordan subalgebra of the hermitian part of a C*-algebra (respectively, of a von Neumann algebra).

Suppose B is a unital JB*-algebra. The smallest norm-closed real Jordan subalgebra $C(a)$ of B_{sa} containing a self-adjoint element a in B and 1 is associative. According to the usual notation, the *spectrum of a* in B, denoted by $\text{Sp}(a)$, is the the set of all real λ such that $a - \lambda 1$ does not have an inverse in $C(a)$ (cf. [26, 3.2.3]). If B is not unital, we consider the unitization of B to

define the spectrum of a self-adjoint element. It is known that the JB*-subalgebra of B generated by a single self-adjoint element $a \in B$ and the unit is isometrically JB*-isomorphic to the commutative C*-algebra $C(\mathrm{Sp}(a))$ of all complex-valued continuous functions on $\mathrm{Sp}(a)$ (see [26, 3.2.4. The spectral theorem]). An element $a \in B$ is called *positive* if $a = a^*$ and $\mathrm{Sp}(a) \subseteq \mathbb{R}_0^+$ (cf. [26, 3.3.3]).

Although there exist exceptional JB*-algebras which cannot be embedded as JB*-subalgebras of C*-algebras, the JB*-subalgebra of a JB*-algebra B generated by two hermitian elements (and the unit element) is a JC*-algebra (compare Macdonald's and Shirshov–Cohn's theorems [26, Theorems 2.4.13 and 2.4.14], [49, Corollary 2.2] or [9, Proposition 3.4.6]). Consequently, for each $x \in B$, the element $x \circ x^*$ is positive in B.

We refer to [26, 9, 10] for the basic background, notions and results on JB*-algebras.

C*- and JB*-algebras have been extensively employed as a framework for studying bounded symmetric domains in complex Banach spaces of infinite dimension, as an alternative notion to simply connected domains. The open unit ball of every C*-algebra is a bounded symmetric domain (see [27]) and the open unit balls of (unital) JB*-algebras are, up to a biholomorphic mapping, those bounded symmetric domains which have a realization as a tube domain, i.e., an upper half-plane (cf. [6]). These examples do not exhaust all possible bounded symmetric domains in arbitrary complex Banach spaces; a strictly wider class of Banach spaces is actually required. The most conclusive result was obtained by W. Kaup who proved in 1983 that every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a JB*-triple [32].

A complex Banach space E belongs to the class of *JB*-triples* if it admits a triple product (i.e., a continuous mapping) $\{\cdot, \cdot, \cdot\} : E^3 \to E$ which is symmetric and bilinear in the outer variables and conjugate linear in the middle variable and satisfies the next algebraic and geometric axioms:

(JB*-1) $\{x, y, \{a, b, c\}\} = \{\{x, y, a\}, b, c\} - \{a, \{y, x, b\}, c\} + \{a, b, \{x, y, c\}\}$
for any $x, y, a, b, c \in E$ (*Jordan identity*);

(JB*-2) For any $a \in E$ the operator $L(a, a) : x \mapsto \{a, a, x\}$ is hermitian with non-negative spectrum;

(JB*-3) $\|\{x, x, x\}\| = \|x\|^3$ for all $x \in E$ (a *Gelfand–Naimark type axiom*).

C*-algebras and JB*-algebras belong to the wide list of examples of JB*-triples when they are equipped with the triple products given by

(4)
$$\{a, b, c\} = \tfrac{1}{2}(ab^*c + cb^*a),$$
$$\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*,$$

respectively (see [6, Theorem 3.3] or [9, Theorem 4.1.45]). The first triple

product in (4) induces a structure of JB*-triple on every closed subspace of the space $B(H, K)$ of all bounded linear operators between complex Hilbert spaces H and K, which is closed under this triple product. In particular, $B(H, K)$ and every complex Hilbert space are JB*-triples with their canonical norms and the first triple product given in (4).

In a JB*-triple E the triple product is contractive, that is,

$$\|\{x, y, z\}\| \leq \|x\| \, \|y\| \, \|z\| \quad \text{for all } x, y, z \text{ in } E$$

(cf. [17, Corollary 3] or [10, Corollary 7.1.7], [12, p. 215]).

A linear bijection between JB*-triples is a triple-isomorphism if and only if it is an isometry (cf. [32, Proposition 5.5] or [12, Theorems 3.1.7, 3.1.20]). Thus, a complex Banach space admits at most one triple product under which it is a JB*-triple.

A *JBW*-triple* is a JB*-triple which is also a dual space. Every JBW*-triple admits a unique (in the isometric sense) predual and its triple product is separately weak*-continuous (see [5], [10, Theorems 5.7.20, 5.7.38]).

Each idempotent e in a Banach algebra A produces a Peirce decomposition of A as a sum of eigenspaces of the left and right multiplication operators by e. A. A. Albert extended the classical Peirce decomposition to the setting of Jordan algebras in the middle of the last century. The notion of idempotent might mean nothing in a general JB*-triple. The appropriate alternative is the concept of tripotent. An element e in a JB*-triple E is a *tripotent* if $\{e, e, e\} = e$. It is worth mentioning that when a C*-algebra A is regarded as a JB*-triple with respect to the first triple product given in (4), an element $e \in A$ is a tripotent if and only if it is a partial isometry (i.e., ee^*, or equivalently e^*e, is a projection) in A.

In case we fix a tripotent e in a JB*-triple E, the classical Peirce decomposition for associative and Jordan algebras extends to a *Peirce decomposition* of E associated with the eigenspaces of the mapping $L(e, e)$, whose eigenvalues are all contained in the set $\left\{0, \frac{1}{2}, 1\right\}$. For $j \in \{0, 1, 2\}$, the (linear) projection $P_j(e)$ of E onto the eigenspace, $E_j(e)$, of $L(e, e)$ corresponding to the eigenvalue $j/2$ admits a concrete expression in terms of the triple product as follows:

$$P_2(e) = L(e, e)(2L(e, e) - \mathrm{id}_E) = Q(e)^2,$$
$$P_1(e) = 4L(e, e)(\mathrm{id}_E - L(e, e)) = 2(L(e, e) - Q(e)^2),$$
$$P_0(e) = (\mathrm{id}_E - L(e, e))(\mathrm{id}_E - 2L(e, e)),$$

where $Q(e)(x) = \{e, x, e\}$ for $x \in E$. The projection $P_j(e)$ is known as the *Peirce-j projection* associated with e. Peirce projections are all contractive (cf. [16, Corollary 1.2]), and the JB*-triple E decomposes as the direct sum

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

which is termed the *Peirce decomposition* of E relative to e (see [16], [12, Definition 1.2.37] or [9, Subsection 4.2.2] and [10, Section 5.7] for more details). In the particular case in which e is a tripotent (i.e., a partial isometry) in a C*-algebra A with initial projection $p_i = e^*e$ and final projection $p_f = ee^*$, the subspaces in the Peirce decomposition are precisely

$$A_2(e) = p_f A p_i,$$
$$A_1(e) = p_f A (1 - p_i) \oplus (1 - p_f) A p_i,$$
$$A_0(e) = (1 - p_f) A (1 - p_i).$$

A tripotent e in a JB*-triple E is called *complete* if $E_0(e) = \{0\}$. We shall say that e is *unitary* if $E = E_2(e)$, or equivalently, if $\{e, e, x\} = x$ for all $x \in E$. Obviously, every unitary is a complete tripotent, but the converse implication is not always true: consider for example a non-surjective isometry e in $B(H)$. A non-zero tripotent e satisfying $E_2(e) = \mathbb{C}e$ is called *minimal*.

Note that in a unital JB* algebra there is another definition of a unitary element (cf. [9, Definition 4.2.25]). However, it is equivalent to the above-defined notion as witnessed by the following fact (where condition (3) is the alternative definition). We will work solely with the notion of unitary tripotent defined above (i.e., with condition (1) from the fact below) but we include these equivalences for the sake of completeness.

FACT 2.1. *Let B be a unital JB*-algebra and let $u \in B$. The following assertions are equivalent:*

(1) *u is a unitary tripotent, i.e., u is a tripotent with $B_2(u) = B$.*
(2) *u is a tripotent and $u \circ u^* = 1$.*
(3) *$u \circ u^* = 1$ and $u^2 \circ u^* = u$, i.e., u^* is the Jordan inverse of u.*

Proof. The equivalence (1)⇔(3) is proved in [6, Proposition 4.3] (see also [9, Theorem 4.2.28]).

To prove the equivalence (1)⇔(2) observe that assertion (2) means that $1 = \{u, u, 1\}$, i.e., $1 \in B_2(u)$. It remains to use [24, Proposition 6.6]. ∎

Complete tripotents in a JB*-triple E can be geometrically characterized since a norm-one element e in E is a complete tripotent if and only if it is an extreme point of its closed unit ball (cf. [6, Lemma 4.1], [34, Proposition 3.5] or [9, Theorem 4.2.34]). Consequently, every JBW*-triple contains an abundant collection of complete tripotents.

Given a unitary element u in a JB*-triple E, the latter becomes a unital JB*-algebra with Jordan product and involution defined by

(5) $x \circ_u y = \{x, u, y\}$ and $x^{*_u} = \{u, x, u\}$ for $x, y \in E$;

see [9, Theorem 4.1.55]. We even know that u is the unit of this JB*-algebra (i.e., $u \circ_u x = x$ for $x \in E$). Each tripotent e in a JB*-triple E is a unitary in the JB*-subtriple $E_2(e)$, and thus $(E_2(e), \circ_e, *_e)$ is a unital JB*-algebra.

Therefore, since the triple product is uniquely determined by the structure of a JB*-algebra, unital JB*-algebras are in one-to-one correspondence with those JB*-triples admitting a unitary element.

A linear subspace I of a JB*-triple E is called a *triple ideal* or simply an *ideal* of E if $\{I, E, E\} \subset I$ and $\{E, I, E\} \subset I$ (see [28]). Let I, J be two ideals of E. We will say that I and J are *orthogonal* if $I \cap J = \{0\}$ (and consequently $\{I, J, E\} = \{J, I, E\} = \{0\}$). It is known that every weak*-closed ideal I of a JBW*-triple M is *orthogonally complemented*, that is, there exists another weak*-closed ideal J of M which is orthogonal to I and $M = I \oplus^{\infty} J$ (see [28, Theorem 4.2(4) and Lemmata 4.3 and 4.4]). For each weak*-closed ideal I of M, we will denote by P_I the natural projection of M onto I. Let us observe that, in this case, P_I is always a weak*-continuous triple homomorphism.

2.1. Positive functionals and prehilbertian seminorms. As in the case of C*-algebras, a functional ϕ in the dual space, B^*, of a JB*-algebra B is called *positive* if it maps positive elements to non-negative real numbers. We will frequently apply that a functional ϕ in the dual space of a unital JB*-algebra B is positive if and only if $\|\phi\| = \phi(1)$ (cf. [26, Lemma 1.2.2]). The same conclusion holds for functionals in the predual of a JBW*-algebra.

A positive normal functional φ in the predual of a JBW*-algebra B is called *faithful* if from $\varphi(a) = 0$ for $a \geq 0$ in B it follows that $a = 0$.

If ϕ is a positive functional in the dual of a C*-algebra A, and 1 denotes the unit element in A^{**}, then the mapping

$$(a, b) \mapsto \phi\left(\frac{ab^* + b^*a}{2}\right) = \phi\{a, b, 1\} \quad (a, b \in A)$$

is a positive semidefinite sesquilinear form on $A \times A$; its associated prehilbertian seminorm is denoted by $\|x\|_{\phi} = (\phi\{x, x, 1\})^{1/2}$. If we consider a positive functional ϕ in the dual of a JB*-algebra B, the associated prehilbertian seminorm is defined by $\|x\|_{\phi}^2 = \phi\{x, x, 1\} = \phi(x \circ x^*)$, where 1 stands for the unit in B^{**}.

The lacking of a local order or positive cone in a general JB*-triple, and hence the lacking of positive functionals, makes a bit more complicated the definition of appropriate prehilbertian seminorms. Namely, let φ be a functional in the predual of a JBW*-triple M and let z be a norm-one element in M satisfying $\varphi(z) = \|\varphi\|$. Proposition 1.2 in [3] proves that the mapping $M \times M \to \mathbb{C}$, $(x, y) \mapsto \varphi\{x, y, z\}$, is a positive semidefinite sesquilinear form on M which does not depend on the choice of the element z (that is, $\varphi\{x, y, z\} = \varphi\{x, y, \tilde{z}\}$ for all $x, y \in M$ and all $\tilde{z} \in M$ with $\|\tilde{z}\| = 1$ and $\varphi(\tilde{z}) = \|\varphi\|$; see [10, Proposition 5.10.60]). The associated prehilbertian seminorm is denoted by $\|x\|_{\varphi} = (\varphi\{x, x, z\})^{1/2}$ $(x \in M)$. Since the triple

product of every JB*-triple is contractive, it follows that

(6) $$\|x\|_\varphi \leq \sqrt{\|\varphi\|}\,\|x\| \quad \text{for all } x \in M.$$

If φ is a non-zero functional in the dual E^* of a JB*-triple E, and we regard E^* as the predual of the JBW*-triple E^{**}, then the prehilbertian seminorm $\|\cdot\|_\varphi$ on E^{**} acts on E by mere restriction.

2.2. Comparison theory of projections and tripotents. Two projections p, q in a C*-algebra A (respectively, in a JB*-algebra B) are said to be *orthogonal* ($p \perp q$ for short) if $pq = 0$ (respectively, $p \circ q = 0$). The relation 'being orthogonal' can be used to define a natural partial ordering on the set of projections in A (respectively, in B) by declaring $p \leq q$ if $q - p$ is a projection and $q - p \perp p$. We write $p < q$ if $p \leq q$ and $p \neq q$.

Two tripotents e and u in a JB*-triple E are called *orthogonal* ($e \perp u$ for short) if $\{e, e, u\} = 0$ (equivalently, $u \in M_0(e)$). It is known that $e \perp u$ if and only if any of the following equivalent reformulations holds:

- $e \in E_0(u)$.
- $E_2(u) \subset E_0(e)$.
- $L(u, e) = 0$.
- $L(e, u) = 0$.
- Both $u + e$ and $u - e$ are tripotents.
- $\{u, u, e\} = 0$.

For proofs see [36, Lemma 3.9], [24, Proposition 6.7] or [23, Lemma 2.1]. The induced partial order defined by this orthogonality on the set of tripotents is given by $e \leq u$ if $u - e$ is a tripotent with $u - e \perp e$.

Let φ be a non-zero functional in the predual of a JBW*-triple M. By [16, Proposition 2] (or [10, Proposition 5.10.57]) there exists a unique tripotent $s(\varphi) \in M$, called the *support tripotent* of φ, such that $\varphi = \varphi \circ P_2(s(\varphi))$, and $\varphi|_{M_2(s(\varphi))}$ is a faithful positive functional on the JBW*-algebra $M_2(s(\varphi))$. In particular, $\|x\|_\varphi^2 = \varphi\{x, x, s(\varphi)\}$ for all $x \in M$.

The support tripotent of a non-zero functional φ in the predual of a JBW*-triple M is also the smallest tripotent in M at which φ attains its norm, that is,

(7) $$\varphi(u) = \|\varphi\| \text{ for some tripotent } u \in M \implies s(\varphi) \leq u.$$

Namely, the element $P_2(s(\varphi))(u)$ lies in the unit ball of $M_2(s(\varphi))$ because $P_2(s(\varphi))$ is contractive. Since $\varphi = \varphi \circ P_2(s(\varphi))$ and $\varphi|_{M_2(s(\varphi))}$ is a faithful functional on the JBW*-algebra $M_2(s(\varphi))$, we deduce that $P_2(s(\varphi))(u) = s(\varphi)$. It follows from [16, Lemma 1.6 or Corollary 1.7] that $s(\varphi) \leq u$. Actually the previous arguments prove

(8) $\varphi(a) = \|\varphi\|$ for some element $a \in M$ with $\|a\| = 1$
$$\implies a = s(\varphi) + P_0(s(\varphi))(a).$$

Two projections p and q in a von Neumann algebra W are called (*Murray–von Neumann*) *equivalent* (written $p \sim q$) if there is a partial isometry $e \in W$ whose initial projection is p and whose final projection is q. This Murray–von Neumann equivalence is employed to classify projections and von Neumann algebras in terms of their properties. For example a projection p in W is said to be *finite* if there is no projection $q < p$ that is equivalent to p. For example, all finite-dimensional projections in $B(H)$ are finite, but the identity operator on H is not finite when H is an infinite-dimensional complex Hilbert space. The von Neumann algebra W is called *finite* if its unit element is a finite projection. The set of all finite projections in the sense of Murray–von Neumann in W forms a (modular) sublattice of the set of all projections in W (see e.g. [47, Theorem V.1.37]). We recall that a projection p in W is *infinite* if it is not finite, and *properly infinite* if $p \neq 0$ and zp is infinite whenever z is a central projection such that $zp \neq 0$ (cf. [47, Definition V.1.15]).

In the setting of JBW*-algebras the notion of finiteness was replaced by the concept of modularity, and the Murray–von Neumann equivalence by the relation 'being equivalent by symmetries', that is, two projections p, q in a JBW*-algebra N are called *equivalent* (*by symmetries*) (denoted by $p \overset{s}{\sim} q$) if there is a finite set $s_1, \ldots, s_n$ of self-adjoint symmetries (i.e. $s_j = 1 - 2p_j$ for certain projections p_j) such that $Q(s_1) \cdots Q(s_n)(p) = q$, where $Q(s_j)(x) = \{s_j, x, s_j\} = 2(s_j \circ x^*) \circ s_j - s_j^2 \circ x^*$ for all $x \in N$ (cf. [48, §10], [26, 5.1.4], [2, §3] or [23, §7.1]). Unlike Murray–von Neumann equivalence, $p \overset{s}{\sim} q$ in N implies $1 - p \overset{s}{\sim} 1 - q$. If M is a von Neumann algebra regarded as a JBW*-algebra, and p, q are projections in M, then $p \overset{s}{\sim} q$ if and only if p and q are *unitarily equivalent*, i.e., there exists a unitary $u \in M$ such that $upu^* = q$ (see [1, Proposition 6.56]). In particular, $p \overset{s}{\sim} q$ implies $p \sim q$.

In a recent contribution we study the notion of finiteness in JBW*-algebras and JBW*-triples from a geometric point of view. In the setting of von Neumann algebras, the results by H. Choda, Y. Kijima, and Y. Nakagami assert that a von Neumann algebra W is finite if and only if all the extreme points of its closed unit ball are unitary (see [11, Theorem 2] or [38, proof of Theorem 4]). Therefore, a projection p in W is finite if and only if every extreme point of the closed unit ball of pWp is a unitary in the latter von Neumann algebra. This is the motivation for the notion of finiteness introduced in [23]. According to the just quoted reference, a tripotent e in a JBW*-triple M is called

- *finite* if any tripotent $u \in M_2(e)$ which is complete in $M_2(e)$ is already unitary in $M_2(e)$;
- *infinite* if it is not finite;
- *properly infinite* if $e \neq 0$ and for each weak*-closed ideal I of M the tripotent $P_I(e)$ is infinite whenever it is non-zero.

If any tripotent in M is finite, we say that M itself is *finite*. Finite-dimensional JBW*-triples are always finite [23, Proposition 3.4]. A JBW*-triple M is said to be *infinite* if it is not finite. Finally, M is *properly infinite* if each non-zero weak*-closed ideal of M is infinite.

Every JBW*-triple decomposes as an orthogonal sum of weak*-closed ideals M_1, M_2, M_3 and M_4, where M_1 is a finite JBW*-algebra, M_2 is either a trivial space or a properly infinite JBW*-algebra, M_3 is a finite JBW*-triple with no non-zero direct summand isomorphic to a JBW*-algebra, and M_4 is either a trivial space or $M_4 = qV_4$, where V_4 is a von Neumann algebra and $q \in V_4$ is a properly infinite projection such that qV_4 has no direct summand isomorphic to a JBW*-algebra; we further know that M_4 is properly infinite provided that it is not zero (see [23, Theorem 7.1] where a more detailed description is presented). This decomposition applies in the particular case in which M is a JBW*-algebra with the appropriate modifications and simplifications on the summands to avoid those which are not JBW*-algebras.

In a von Neumann algebra W the two notions of finiteness coincide for projections (see [23, Lemma 3.2(a)]). Every modular projection in a JBW*-algebra is a finite tripotent in the sense above, but the reciprocal is not always true (cf. [23, Lemma 7.12 and Remark 7.13]).

Finite JBW*-triples enjoy formidable properties. For example, for each finite tripotent u in a JBW*-algebra M there is a unitary element $e \in M$ with $u \le e$ (cf. [23, Proposition 7.5]). More details and properties can be found in [23].

A projection p in a von Neumann algebra W is called *abelian* if the subalgebra pWp is abelian (see [47, Definition V.1.15]). The von Neumann algebra W is said to be of *type I* or *discrete* if every non-zero (central) projection contains a non-zero abelian subprojection [47, Definition V.1.17]. In the previous definition the word central can be relaxed (see, for example, [46, Corollary 4.20]).

A tripotent e in a JB*-triple is said to be *abelian* if the JB*-algebra $E_2(u)$ is associative, or equivalently, $(E_2(u), \circ_u, *_u)$ is a unital abelian C*-algebra. Obviously, any minimal tripotent is abelian. We further know that every abelian tripotent is finite [23, Lemma 3.2(e)].

According to [29, 30, 28], a JBW*-triple M is said to be of *type I* (respectively, *continuous*) if it coincides with the weak*-closure of the span of all its abelian tripotents (respectively, it contains no non-zero abelian tripotents). Every JBW*-triple can be written as the orthogonal sum of two weak*-closed ideals M_1 and M_2 such that M_1 is of type I and M_2 is continuous (any of these summands might be trivial). G. Horn and E. Neher established in [29, 30] structure results describing type I and continuous JBW*-triples.

Concretely, every JBW*-triple of type I may be represented in the form

$$(9) \qquad \bigoplus_{j \in J}^{\ell_\infty} A_j \overline{\otimes} C_j,$$

where the A_j's are abelian von Neumann algebras and the C_j's are Cartan factors (the concrete definitions will be presented below in Section 4, the reader can also consult [36, 31, 33] for details). To reassure the reader we simply note that every Cartan factor C is a JBW*-triple. When C is a JW*-subtriple of some $B(H)$ and A is an abelian von Neumann algebra, the symbol $A \overline{\otimes} C$ denotes the weak*-closure of the algebraic tensor product $A \otimes C$ in the von Neumann tensor product $A \overline{\otimes} B(H)$ (see [47, Section IV.1] and [29, §1]). In the remaining cases C is finite-dimensional and $A \overline{\otimes} C$ will stand for the completed injective tensor product (see [45, Chapter 3]).

3. Majorizing certain seminorms. The main result will be proved using its dual version. The starting point is the following dual version of Theorem D(2).

THEOREM 3.1 ([40, Theorem 3]). *Let M be a JBW*-triple, H a Hilbert space and $T : M \to H$ a weak*-to-weak continuous linear operator. Given $\varepsilon > 0$, there are norm-one functionals $\varphi_1, \varphi_2 \in M_*$ such that*

$$\|Tx\| \leq (\sqrt{2} + \varepsilon)\|T\|(\|x\|_{\varphi_1}^2 + \varepsilon\|x\|_{\varphi_2}^2)^{1/2} \quad \text{for } x \in M.$$

We continue by recalling two results from [25]. The first one is essentially the main result and easily implies Theorem D(3). The second one was used to prove one of the particular cases and we will use it several times as well.

PROPOSITION 3.2 ([25, Theorem 2.4]). *Let M be a JBW*-triple. Then given any two functionals φ_1, φ_2 in M_*, there exists a norm-one functional $\psi \in M_*$ such that*

$$\|x\|_{\varphi_1}^2 + \|x\|_{\varphi_2}^2 \leq 2(\|\varphi_1\| + \|\varphi_2\|) \cdot \|x\|_\psi^2$$

for all $x \in M$.

LEMMA 3.3 ([25, Proposition 3.2]). *Let M be a JBW*-triple and let $\varphi \in M_*$. Assume that $p \in M$ is a tripotent such that $s(\varphi) \in M_2(p)$. Then there exists a functional $\tilde{\varphi} \in M_*$ such that $\|\tilde{\varphi}\| = \|\varphi\|$, $s(\tilde{\varphi}) \leq p$ and $\|x\|_\varphi \leq \sqrt{2}\,\|x\|_{\tilde{\varphi}}$ for all $x \in M$.*

The key step to prove our main result is the following proposition which says that for JBW*-algebras a stronger version of Proposition 3.2 is achievable.

PROPOSITION 3.4. *Let M be a JBW*-algebra. Then given any two functionals φ_1, φ_2 in M_* and $\varepsilon > 0$, there exists a norm-one functional $\psi \in M_*$*

$$\|x\|_{\varphi_1}^2 + \|x\|_{\varphi_2}^2 \leq (\|\varphi_1\| + 2\|\varphi_2\| + \varepsilon)\|x\|_\psi^2 \quad for\ x \in M.$$

Using this proposition we will easily deduce the main result in Section 6 below. Proposition 3.4 will be proved using the following result.

PROPOSITION 3.5. *Let M be a JBW*-algebra, $\varphi \in M_*$ and $\varepsilon > 0$. Then there are a functional $\tilde{\varphi} \in M_*$ and a unitary element $w \in M$ such that*

$$\|\tilde{\varphi}\| \leq \|\varphi\|, \quad s(\tilde{\varphi}) \leq w \quad and \quad \|\cdot\|_\varphi^2 \leq (1+\varepsilon)\|\cdot\|_{\tilde{\varphi}}^2.$$

This proposition will be proved at the beginning of Section 6 using the results from Sections 4 and 5. Let us now show that it implies Proposition 3.4.

Proof of Proposition 3.4 from Proposition 3.5. Let $\tilde{\varphi}_1 \in M_*$ and $w \in M$ correspond to φ_1 and $\varepsilon/\|\varphi_1\|$ by Proposition 3.5. Since w is unitary, we have $M_2(w) = M$, hence we may apply Lemma 3.3 to get $\psi_2 \in M_*$ such that

$$s(\psi_2) \leq w, \quad \|\psi_2\| \leq \|\varphi_2\|, \quad \|\cdot\|_{\varphi_2} \leq \sqrt{2}\,\|\cdot\|_{\psi_2}.$$

Then

$$\|\cdot\|_{\varphi_1}^2 + \|\cdot\|_{\varphi_2}^2 \leq \left(1 + \frac{\varepsilon}{\|\varphi_1\|}\right)\|\cdot\|_{\tilde{\varphi}_1}^2 + \|\cdot\|_{\varphi_2}^2$$

$$\leq \left(1 + \frac{\varepsilon}{\|\varphi_1\|}\right)\|\cdot\|_{\tilde{\varphi}_1}^2 + 2\|\cdot\|_{\psi_2}^2$$

$$= \|\cdot\|_{(1+\varepsilon/\|\varphi_1\|)\tilde{\varphi}_1 + 2\psi_2}^2$$

$$= \left(\left(1 + \frac{\varepsilon}{\|\varphi_1\|}\right)\|\tilde{\varphi}_1\| + 2\|\psi_2\|\right)\|\cdot\|_\psi^2,$$

where

$$\psi = \frac{(1+\varepsilon/\|\varphi_1\|)\tilde{\varphi}_1 + 2\psi_2}{(1+\varepsilon/\|\varphi_1\|)\|\tilde{\varphi}_1\| + 2\|\psi_2\|}.$$

(Note that the first equality follows from the fact that the support tripotents of both functionals are below w.) Since the functionals $\tilde{\varphi}_1$ and ψ_2 attain their norms at w, we deduce that $\|\psi\| = 1$. It remains to observe that

$$\left(1 + \frac{\varepsilon}{\|\varphi_1\|}\right)\|\tilde{\varphi}_1\| + 2\|\psi_2\| \leq \|\varphi_1\| + \varepsilon + 2\|\varphi_2\|. \quad \blacksquare$$

4. Finite or type I JBW*-algebras. The aim of this section is to prove a stronger version of Proposition 3.5 for a large subclass of JBW*-algebras (see Proposition 4.2). We follow the notation from [23] recalled in Section 2.

Since in a finite JBW*-algebra any tripotent is majorized by a unitary one (cf. [23, Lemma 3.2(d)]), we get the following observation.

OBSERVATION 4.1. *Let M be a finite JBW*-algebra. Then Proposition 3.5 holds for M in a very strong version: one can take $\tilde{\varphi} = \varphi$ and $\varepsilon = 0$.*

There is a larger class of JBW*-algebras for which we get a stronger and canonical version of Proposition 3.5. The concrete result appears in the content of the following proposition. The exact relationship with Proposition 3.5 will be explained in Remark 5.7(1) below.

We first recall that, in the setting of JBW*-triples, two normal functionals φ and ψ in the predual of a JBW*-triple M are called (*algebraically*) *orthogonal* (written $\varphi \perp \psi$) if their support tripotents are orthogonal in M —that is, $s(\varphi) \perp s(\psi)$ (cf. [18, 14]). It is shown in [18, Lemma 2.3] (see also [14, Theorem 5.4]) that $\varphi, \psi \in M_*$ are orthogonal if and only if they are 'geometrically' *L-orthogonal* in M_* i.e., $\|\varphi \pm \psi\| = \|\varphi\| + \|\psi\|$. In particular $\|\cdot\|_{\varphi+\psi}^2 = \|\cdot\|_\varphi^2 + \|\cdot\|_\psi^2$ if φ and ψ are orthogonal because in this case φ, ψ and $\varphi + \psi$ attain their respective norms at $s(\varphi) + s(\psi)$.

PROPOSITION 4.2. *Let M be a JBW*-algebra which is triple-isomorphic to a direct sum $M_1 \oplus^{\ell\infty} M_2$, where M_1 is a finite JBW*-algebra and M_2 is a type I JBW*-algebra. Let $\varphi \in M_*$ be arbitrary. Then for each $\varepsilon > 0$ there are two functionals $\varphi_1, \varphi_2 \in M_*$ such that*

(i) $\varphi = \varphi_1 + \varphi_2$;
(ii) $\varphi_1 \perp \varphi_2$;
(iii) $\|\varphi_2\| < \varepsilon$;
(iv) $s(\varphi_1)$ *is a finite tripotent in M.*

The rest of this section is devoted to proving Proposition 4.2. To this end we will use the following decomposition result which was essentially established in [23]. Let us note that the concrete definition of a type 2 Cartan factor can be found in the next subsection.

PROPOSITION 4.3. *Let M be a JBW*-algebra which is triple-isomorphic to a direct sum $M_1 \oplus^{\ell\infty} M_2$, where M_1 is a finite JBW*-algebra and M_2 is a type I JBW*-algebra. Then M is triple-isomorphic to a JBW*-algebra of the form*

$$N \oplus^{\ell\infty} \left(\bigoplus_{j \in J} L^\infty(\mu_j) \,\overline{\otimes}\, C_j \right) \oplus^{\ell\infty} \left(\bigoplus_{\lambda \in \Lambda} L^\infty(\nu_\lambda) \,\overline{\otimes}\, B(H_\lambda) \right),$$

where

- N *is a finite JBW*-algebra;*
- J *and Λ are (possibly empty) sets;*
- μ_j*'s and ν_λ's are probability measures;*
- C_j *is an infinite-dimensional type 2 Cartan factor for each $j \in J$;*
- H_λ *is an infinite-dimensional Hilbert space for each $\lambda \in \Lambda$.*

Proof. By [23, Theorem 7.1], M is triple-isomorphic to $N \oplus^{\ell_\infty} N_1$, where N is a finite JBW*-algebra and N_1 is (either trivial or) a properly infinite JBW*-algebra. By the same theorem, N_1 is triple-isomorphic to

$$\left(\bigoplus_{j \in J} L^\infty(\mu_j) \,\overline{\otimes}\, C_j \right) \oplus^{\ell_\infty} N_2,$$

where the first summand has the above-mentioned form and N_2 is (either trivial or) a properly infinite von Neumann algebra. Since by the assumptions N_2 is clearly of type I, we may conclude using [47, Theorem V.1.27]. ∎

We observe that the validity of Proposition 4.2 is preserved by ℓ_∞-sums, so it is enough to prove it for the individual summands from Propostion 4.3. For the finite JBW*-algebra N we may use Observation 4.1. We will prove the desired conclusion for the summands $L^\infty(\mu_j) \,\overline{\otimes}\, C_j$. For the remaining summands an easier version of the same proof works as we will explain below.

4.1. The case of type 2 Cartan factors. Let us start by recalling the definition of type 2 Cartan factors. Let H be a Hilbert space with a fixed orthonormal basis $(e_\gamma)_{\gamma \in \Gamma}$. Then H is canonically represented as $\ell^2(\Gamma)$. For $\xi \in H$ let $\overline{\xi}$ be the coordinatewise complex conjugate of ξ. Further, for $x \in B(H)$ we denote by x^t the operator defined by

$$x^t \xi = \overline{x^* \overline{\xi}}, \quad \xi \in H.$$

Then x^t is the transpose of x with respect to the fixed orthonormal basis, i.e.,

$$\langle x^t e_\gamma, e_\delta \rangle = \langle x e_\delta, e_\gamma \rangle \quad \text{for } \gamma, \delta \in \Gamma$$

(see, e.g., [23, Section 5.3] for the easy computation). Then

$$B(H)_s = \{ x \in B(H) \,;\, x^t = x \} \quad \text{and} \quad B(H)_a = \{ x \in B(H) \,;\, x^t = -x \}$$

are the so-called *Cartan factors* of *type* 3 and *type* 2, respectively. They are formed by operators with symmetric (antisymmetric, respectively) 'representing matrices' with respect to the fixed orthonormal basis. We will deal with the second case, i.e., with 'antisymmetric operators'.

So, assume that H has infinite dimension (or, equivalently, Γ is an infinite set). Let $M = B(H)_a$. Define $\pi : B(H) \to M$ by $\pi(x) = \frac{1}{2}(x - x^t)$. Then π is a norm-one projection which is moreover weak*-to-weak* continuous. Hence $\pi_* : M_* \to B(H)_*$ defined by $\pi_*(\varphi) = \varphi \circ \pi$ is an isometric injection. Moreover,

$$\begin{aligned} \pi_* M_* &= \{ \varphi \in B(H)_* \,;\, \varphi(x^t) = -\varphi(x) \text{ for } x \in B(H) \} \\ &= \{ \varphi \in B(H)_* \,;\, \varphi|_{B(H)_s} = 0 \}. \end{aligned}$$

Recall that $B(H)_*$ is isometric to the space $N(H)$ of nuclear operators via the trace duality (cf. [47, Theorem II.1.8]). Moreover, any $y \in N(H)$ is

represented in the form

$$y = \sum_{k \geq 1} \lambda_k \langle \cdot, \eta_k \rangle \xi_k$$

where (ξ_k) and (η_k) are orthonormal sequences in H and the λ_k are positive numbers with $\sum_{k \geq 1} \lambda_k = \|y\|_N$. Then clearly

$$y^* = \sum_{k \geq 1} \lambda_k \langle \cdot, \xi_k \rangle \eta_k,$$

hence for any $\xi \in H$ we have

$$y^t \xi = \overline{y^* \overline{\xi}} = \overline{\sum_{k \geq 1} \lambda_k \langle \overline{\xi}, \xi_k \rangle \eta_k} = \sum_{k \geq 1} \lambda_k \langle \xi, \overline{\xi_k} \rangle \overline{\eta_k},$$

thus

$$y^t = \sum_{k \geq 1} \lambda_k \langle \cdot, \overline{\xi_k} \rangle \overline{\eta_k}.$$

In particular

$$(10) \qquad \operatorname{tr}(y^t) = \sum_{k \geq 1} \lambda_k \langle \overline{\eta_k}, \overline{\xi_k} \rangle = \sum_{k \geq 1} \lambda_k \langle \xi_k, \eta_k \rangle = \operatorname{tr}(y).$$

Hence, given $\varphi \in B(H)_*$ represented by $y \in N(H)$, the functional $\varphi^t(x) = \varphi(x^t)$, $x \in B(H)$, is represented by y^t. Indeed,

$$\varphi^t(x) = \varphi(x^t) = \operatorname{tr}(x^t y) = \operatorname{tr}(y^t x) = \operatorname{tr}(x y^t) \quad \text{for } x \in B(H).$$

It follows that

$$\pi_* M_* = \{\varphi \in B(H)_* \,;\, \varphi \text{ is represented}$$
$$\text{by an antisymmetric nuclear operator}\}.$$

Proof of Proposition 4.2 for $M = B(H)_a$. Fix $\varphi \in M_*$ of norm one and $\varepsilon > 0$. Let $u = s(\varphi) \in M$. Set $\tilde{\varphi} = \pi_* \varphi$. Fix $y \in N(H)$ representing $\tilde{\varphi}$. Then

$$y = \sum_{k \geq 1} \lambda_k \langle \cdot, \eta_k \rangle \xi_k$$

where (ξ_k) and (η_k) are orthonormal sequences in H and the λ_k are strictly positive numbers with $\sum_{k \geq 1} \lambda_k = 1$. Observe that

$$s(\tilde{\varphi}) = \sum_{k \geq 1} \langle \cdot, \xi_k \rangle \eta_k.$$

Moreover, since y is antisymmetric, we deduce that so is $s(\tilde{\varphi})$. Indeed, by the above we have

$$y = -y^t = -\sum_{k \geq 1} \lambda_k \langle \cdot, \overline{\xi_k} \rangle \overline{\eta_k}.$$

Hence
$$s(\tilde{\varphi}) = -\sum_{k \geq 1} \langle \cdot, \overline{\eta_k} \rangle \overline{\xi_k} = -s(\tilde{\varphi})^t.$$

For $\delta > 0$ set
$$y_\delta = \sum_{\lambda_k \geq \delta} \lambda_k \langle \cdot, \eta_k \rangle \xi_k.$$

Then y_δ is a finite rank operator and
$$y_\delta^t = \sum_{\lambda_k \geq \delta} \lambda_k \langle \cdot, \overline{\xi_k} \rangle \overline{\eta_k}.$$

By uniqueness of the nuclear representation (the sequence (λ_k) is unique and for any fixed $\lambda > 0$ the linear spans of those η_k, resp. ξ_k, for which $\lambda_k = \lambda$ are uniquely determined) we deduce that y_δ is antisymmetric and hence its support tripotent
$$u_\delta = \sum_{\lambda_k \geq \delta} \langle \cdot, \xi_k \rangle \eta_k$$

is antisymmetric as well.

Fix $\delta > 0$ such that $\sum_{\lambda_k < \delta} \lambda_k < \varepsilon$. Then $\|y - y_\delta\|_N < \varepsilon$.

Let $\tilde{\varphi}_1$ be the functional represented by y_δ and $\tilde{\varphi}_2 = \tilde{\varphi} - \tilde{\varphi}_1$ (i.e., the functional represented by $y - y_\delta$). As y_δ is antisymmetric, both $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ belong to $\pi_* M_*$. Moreover, $s(\tilde{\varphi}_1) = u_\delta$ and $s(\tilde{\varphi}_2) = u - u_\delta$. As $u_\delta \perp u - u_\delta$, we deduce that $\tilde{\varphi}_1 \perp \tilde{\varphi}_2$. Further, u_δ is a finite tripotent, being a finite rank partial isometry.

Since we are in $\pi_* M_*$, we have functionals $\varphi_1, \varphi_2 \in M_*$ such that $\tilde{\varphi}_j = \pi_* \varphi_j$. It is now clear that they provide the sought decomposition of φ. ∎

We have settled the case of $B(H)_a$. Note that for $M = B(H)$ the same proof works—we just do not use the mapping π and are not obliged to check the antisymmetry. The proof was done using the Schmidt decomposition of nuclear operators. To prove the result for the tensor product we will use a measurable version of the Schmidt decomposition established in the following subsection.

4.2. Measurable version of Schmidt decomposition. In this subsection we are going to prove the following result (note that $K(H)$ denotes the C*-algebra of compact operators on H).

THEOREM 4.4. *Let H be a Hilbert space. Then there are sequences $(\lambda_n)_{n=0}^\infty$ and $(\boldsymbol{u}_n)_{n=0}^\infty$ of mappings such that the following properties are fulfilled for $n \in \mathbb{N}$ and $x \in K(H)$:*

(a) $\lambda_n : K(H) \to [0, \infty)$ *is a lower semicontinuous mapping;*
(b) $\lambda_{n+1}(x) < \lambda_n(x)$ *whenever $\lambda_n(x) > 0$;*
(c) $\boldsymbol{u}_n : K(H) \to K(H)$ *is a Borel measurable mapping;*

(d) $\boldsymbol{u}_n(x)$ *is a finite rank partial isometry on* H;
(e) $\boldsymbol{u}_n(x) = 0$ *whenever* $\lambda_n(x) = 0$;
(f) *the partial isometries* $\boldsymbol{u}_k(x)$, $k \in \mathbb{N} \cup \{0\}$, *are pairwise orthogonal;*
(g) $x = \sum_{n=0}^{\infty} \lambda_n(x)\boldsymbol{u}_n(x)$, *where the series converges in the operator norm.*

Let us point out that the Borel measurability in this theorem and in the lemmata used in the proof is considered with respect to the norm-topology. However, if X is a separable Banach space, it is well known and easy to see that any norm-open set is weakly F_σ, hence the norm Borel sets coincide with the weak Borel sets (cf. [35, pp. 74 and 75]). This applies in particular to H, $K(H)$ and $K(H) \times H$ where H is a separable Hilbert space.

The proof will be done in several steps contained in the following lemmata.

LEMMA 4.5. *Let* H *be a Hilbert space* (*not necessarily separable*). *For* $x \in K(H)$ *let* $(\alpha_n(x))$ *be the sequence of its singular numbers. Moreover, let* $(\lambda_n(x))$ *be the strictly decreasing version of* $(\alpha_n(x))$ (*recall that the sequence* $(\alpha_n(x))$ *itself is non-increasing*), *completed by zeros if necessary. That is,*

$$\lambda_n(x) = \begin{cases} \alpha_k(x) & \text{if } \operatorname{card}\{\alpha_0(x), \alpha_1(x), \ldots, \alpha_k(x)\} = n+1, \\ 0 & \text{if such a } k \text{ does not exist.} \end{cases}$$

Then the following assertions are valid for each $n \in \mathbb{N} \cup \{0\}$:

(i) α_n *is a 1-Lipschitz function on* $K(H)$;
(ii) λ_n *is a lower semicontinuous function on* $K(H)$, *in particular it is Borel measurable and of the first Baire class.*

Proof. (i) This is proved in [19, Corollary VI.1.6] and easily deduced from the following well-known formula for singular numbers:

$$\alpha_n(x) = \operatorname{dist}(x, \{y \in K(H)\,;\, \dim yH \le n\}), \qquad x \in K(H),\, n \in \mathbb{N} \cup \{0\}$$

(cf. [19, Theorem VI.1.5]).

(ii) Clearly $\lambda_n \ge 0$. Moreover, for each $c > 0$ we have $\lambda_n(x) > c$ if and only if

$$\exists c_0 > c_1 > \cdots > c_n > c_{n+1} = c,\ \exists k_0, k_1, \ldots k_n \in \mathbb{N} \text{ such that}$$
$$\alpha_{k_j}(x) \in (c_{j+1}, c_j) \quad \forall j \in \{0, 1, \ldots, n\}.$$

Since the functions α_k are continuous by (i), $\{x\,;\, \lambda_n(x) > c\}$ is open. Now the lower semicontinuity easily follows.

Finally, any lower semicontinuous function on a metric space is clearly F_σ-measurable, hence Borel measurable and also of the first Baire class (cf. [37, Corollary 3.8(a)]). ∎

LEMMA 4.6. *Let* H *be a Hilbert space. For any* $x \in K(H)_+$ *and* $n \in \mathbb{N} \cup \{0\}$ *let* $p_n(x)$ *be the projection onto the eigenspace with respect to the eigen-*

value $\lambda_n(x)$ provided $\lambda_n(x) > 0$, and $p_n(x) = 0$ otherwise. Then the mapping p_n is Borel measurable.

Proof. We start by proving that the mapping p_0 is Borel measurable. For $x \in K(H)_+ \setminus \{0\}$ we set

$$\psi(x) = \frac{x - \lambda_0(x) \cdot I}{2(\lambda_0(x) - \lambda_1(x))} + I.$$

Then the mapping $\psi : K(H)_+ \setminus \{0\} \to B(H)_{\mathrm{sa}}$ is Borel measurable (by Lemma 4.5(ii), note that for $x \in K(H)_+ \setminus \{0\}$ we have $\lambda_0(x) > \lambda_1(x)$).

Moreover, since

$$x = \sum_{n \geq 0} \lambda_n(x) p_n(x),$$

by the Hilbert–Schmidt Theorem, we deduce that

$$\psi(x) = p_0(x) + \sum_{n \geq 1} \frac{\lambda_0(x) - 2\lambda_1(x) + \lambda_n(x)}{2(\lambda_0(x) - \lambda_1(x))} p_n(x)$$

$$+ \frac{\lambda_0(x) - 2\lambda_1(x)}{2(\lambda_0(x) - \lambda_1(x))} \Big(I - \sum_{n \geq 0} p_n(x) \Big),$$

hence the spectrum of $\psi(x)$ is

$$\sigma(\psi(x)) = \big\{ 1, \tfrac{\lambda_0(x) - 2\lambda_1(x)}{2(\lambda_0(x) - \lambda_1(x))} \big\} \cup \big\{ \tfrac{\lambda_0(x) - 2\lambda_1(x) + \lambda_n(x)}{2(\lambda_0(x) - \lambda_1(x))} ; n \geq 1 \big\} \subset \{1\} \cup \big(-\infty, \tfrac{1}{2} \big].$$

It follows that $p_0(x) = f(\psi(x))$ whenever f is a continuous function on $\mathbb{R}$ with $f = 0$ on $\big(-\infty, \tfrac{1}{2} \big]$ and $f(1) = 1$.

Since the mapping $y \mapsto f(y)$ is continuous on $B(H)_{\mathrm{sa}}$ by [47, Proposition I.4.10], we deduce that p_0 is a Borel measurable mapping.

Further, for $n \in \mathbb{N}$ we have

$$p_n(x) = \begin{cases} 0 & \text{if } \lambda_n(x) = 0, \\ p_0(x - \sum_{k=0}^{n-1} \lambda_k(x) p_k(x)) & \text{if } \lambda_n(x) > 0, \end{cases}$$

hence by the obvious induction we see that p_n is Borel measurable as well. ∎

Proof of Theorem 4.4. Fix any $x \in K(H)$. Let $x = u(x)|x|$ be the polar decomposition. By the Hilbert–Schmidt Theorem we have

$$|x| = \sum_n \lambda_n(x) \, p_n(|x|)$$

(note that $\lambda_n(x) = \lambda_n(|x|)$). Hence

$$x = \sum_n \lambda_n(x) u(x) p_n(|x|) = \sum_n \lambda_n(x) u_n(x),$$

where $u_n(x) = u(x) p_n(|x|)$ are mutually orthogonal partial isometries (of finite rank). The mappings λ_n are lower semicontinuous by Lemma 4.5.

Further, the assignment $x \mapsto |x| = \sqrt{x^*x}$ is continuous by the properties of the functional calculus. Indeed, the mapping $x \mapsto x^*x$ is obviously continuous and the mapping $y \mapsto \sqrt{y}$ is continuous on the positive cone of $K(H)$ by [47, Proposition I.4.10].

Hence, we can deduce from Lemma 4.6 that the assignments $x \mapsto p_n(|x|)$ are Borel measurable. Since $u_n(x) = 0$ whenever $\lambda_n(x) = 0$ and $u_n(x) = \frac{1}{\lambda_n(x)} x p_n(|x|)$ if $\lambda_n(x) > 0$, it easily follows that the mapping u_n is Borel measurable. ∎

PROPOSITION 4.7. *Let H be a separable Hilbert space. Consider the mappings λ_n and u_n provided by Theorem 4.4 restricted to $N(H)$. Then λ_n and u_n are also Borel measurable with respect to the nuclear norm. Moreover, the series from assertion* (g) *converges absolutely in the nuclear norm, and*

$$\|x\| = \sum_{n=0}^{\infty} \lambda_n \|u_n(x)\|$$

where the norm is the nuclear one.

Proof. The Borel measurability of λ_n and u_n follows from the continuity of the canonical inclusion of $N(H)$ into $K(H)$ together with Theorem 4.4. The rest follows from the Schmidt representation of nuclear operators. ∎

4.3. Proof of the remaining cases of Proposition 4.2. Let us adopt the notation from Subsection 4.1. Moreover, let μ be a probability measure and $A = L^{\infty}(\mu)$. Set $W = A \overline{\otimes} B(H)$. Then W is a von Neumann algebra canonically represented in $B(L^2(\mu, H))$ (for a detailed description see e.g. [23, Section 5.3]). Moreover, on $L^2(\mu, H)$ we have a canonical conjugation (the pointwise one—recall that $H = \ell^2(\Gamma)$ is equipped with the coordinatewise conjugation). Therefore a natural transpose of any $x \in W$ is defined by

$$x^t(\boldsymbol{f}) = \overline{x^*(\overline{\boldsymbol{f}})}, \quad \boldsymbol{f} \in L^2(\mu, H).$$

Then we have a canonical identification

$$M = A \overline{\otimes} B(H)_a = W_a = \{x \in W \,;\, x^t = -x\}.$$

Similarly to Subsection 4.1 we denote by π the canonical projection of W onto M, i.e., $x \mapsto \frac{1}{2}(x - x^t)$.

Recall that, by [47, Theorem IV.7.17], $W_* = L^1(\mu, N(H))$ (the Lebesgue–Bochner space). Since π is a weak*-weak* continuous norm-one projection, we have an isometric embedding $\pi_* : M_* \to W_*$ defined by $\pi_* \omega = \omega \circ \pi$. Moreover, clearly

$$\pi_*(M_*) = \{\omega \in W_* \,;\, \omega^t = -\omega\}.$$

LEMMA 4.8. *Assume that $\boldsymbol{g} \in L^1(\mu, N(H)) = W_*$. Then*

(i) $\boldsymbol{g}^*(\omega) = (\boldsymbol{g}(\omega))^*$ *μ-a.e.;*
(ii) $\boldsymbol{g}^t(\omega) = (\boldsymbol{g}(\omega))^t$ *μ-a.e.*

Proof. Let us start by explaining the meaning. On the left-hand side we consider the involution and transpose applied to $\boldsymbol{g}$ as to a functional on W, while on the right-hand side these operations are applied to the nuclear operators $\boldsymbol{g}(\omega)$.

Observe that it is enough to prove the equality for $\boldsymbol{g} = \chi_E y$ (where E is a measurable set and $y \in N(H)$) as functions of this form are linearly dense in $L^1(\mu, N(H))$, i.e., we want to prove

$$(\chi_E y)^* = \chi_E y^* \quad \text{and} \quad (\chi_E y)^t = \chi_E y^t.$$

It is clear that the elements on the right-hand side belong to $L^1(\mu, N(H))$ $= W_*$, so the equality may be proved as equality of functionals. Since these functionals are linear and weak*-continuous on W, it is enough to prove the equality on the generators $f \otimes x$, $f \in L^\infty(\mu)$, $x \in B(H)$.

So, fix such f and x and recall that

$$(f \otimes x)^* = \bar{f} \otimes x^* \quad \text{and} \quad (f \otimes x)^t = f \otimes x^t.$$

Indeed, the first equality follows from the very definition of the von Neumann tensor product, the second one is proved in [23, the computation before Lemma 5.10]. Hence we have

$$\langle (\chi_E y)^*, f \otimes x \rangle = \overline{\langle \chi_E y, \bar{f} \otimes x^* \rangle} = \overline{\int_E \bar{f} \, d\mu \cdot \mathrm{tr}(yx^*)} = \int_E f \, d\mu \cdot \mathrm{tr}((yx^*)^*)$$

$$= \int_E f \, d\mu \cdot \mathrm{tr}(xy^*) = \int_E f \, d\mu \cdot \mathrm{tr}(y^*x) = \langle \chi_E y^*, f \otimes x \rangle,$$

and similarly, by (10),

$$\langle (\chi_E y)^t, f \otimes x \rangle = \langle \chi_E y, f \otimes x^t \rangle = \int_E f \, d\mu \cdot \mathrm{tr}(yx^t) = \int_E f \, d\mu \cdot \mathrm{tr}((yx^t)^t)$$

$$= \int_E f \, d\mu \cdot \mathrm{tr}(xy^t) = \int_E f \, d\mu \cdot \mathrm{tr}(y^t x) = \langle \chi_E y^t, f \otimes x \rangle. \quad \blacksquare$$

It easily follows that

$$\pi_*(M_*) = L^1(\mu, N(H)_a).$$

LEMMA 4.9. *Let $\boldsymbol{g} \in L^1(\mu, N(H)) = W_*$. Then*

(i) $\langle f \otimes x, \boldsymbol{g} \rangle = \int f(\omega) \, \mathrm{tr}(x\boldsymbol{g}(\omega)) \, d\mu(\omega)$ *for $f \in L^\infty(\mu)$ and $x \in B(H)$.*

(ii) *There exists a projection $p \in B(H)$ with separable range such that $p\boldsymbol{g}(\omega)p = \boldsymbol{g}(\omega)$ μ-a.e. In this case $(1 \otimes p)\boldsymbol{g}(1 \otimes p) = \boldsymbol{g}$, i.e.,*

$$\langle T, \boldsymbol{g} \rangle = \langle (1 \otimes p)T(1 \otimes p), \boldsymbol{g} \rangle \quad \text{for } T \in W.$$

Proof. (i) Fix $f \in L^\infty(\mu)$ and $x \in B(H)$. Consider both the left-hand side and the right-hand side as functionals depending on $\boldsymbol{g}$. Since both functionals are linear and continuous on $L^1(\mu, N(H))$, it is enough to prove the equality

for $\boldsymbol{g} = \chi_E y$ where E is a measurable set and $y \in N(H)$. In this case we have

$$\langle f \otimes x, \chi_E y \rangle = \int\limits_E f \, \mathrm{d}\mu \, \mathrm{tr}(xy),$$

so the equality holds.

(ii) Note that $\boldsymbol{g}$ is essentially separably-valued, so there is a separable subspace $Y \subset N(H)$ with $\boldsymbol{g}(\omega) \in Y$ μ-a.e. Since for any $y \in N(H)$ there is a projection q with separable range such that $qyq = y$ (due the the Schmidt representation), the existence of p easily follows.

To prove the last equality it is enough to verify it for the generators $T = f \otimes x$ and this easily follows from (i). ■

PROPOSITION 4.10. *Let $\boldsymbol{g} \in L^1(\mu, N(H))$. Then there are a separable subspace $H_0 \subset H$, a sequence (ζ_n) of non-negative measurable functions and a sequence $(\boldsymbol{u}_n)$ of measurable mappings with values in $K(H_0)$ such that the following hold for each ω:*

(a) *$\zeta_{n+1}(\omega) < \zeta_n(\omega)$ whenever $\zeta_n(\omega) > 0$;*
(b) *$\boldsymbol{u}_n(\omega)$ is a finite rank partial isometry on H_0;*
(c) *$\boldsymbol{u}_n(\omega) = 0$ whenever $\zeta_n(x) = 0$;*
(d) *the partial isometries $\boldsymbol{u}_k(\omega)$, $k \in \mathbb{N} \cup \{0\}$, are pairwise orthogonal;*
(e) *$\boldsymbol{g} = \sum_{n=0}^{\infty} \zeta_n \boldsymbol{u}_n$ where the series converges absolutely almost everywhere and also in the norm of $L^1(\mu, N(H))$.*

Proof. Let $p \in B(H)$ be a projection with separable range provided by Lemma 4.9(ii) and set $H_0 = pH$. Let (λ_n) and $(\boldsymbol{u}_n)$ be the mappings provided by Theorem 4.4. Let $\boldsymbol{u}_n(\omega) = \boldsymbol{u}_n(\boldsymbol{g}(\omega))$ and $\zeta_n(\omega) = \lambda_n(\boldsymbol{g}(\omega))$. Then these functions are measurable due to measurability of $\boldsymbol{g}$ and Proposition 4.7. Assertions (a)–(d) now follow from Theorem 4.4.

By Proposition 4.7 we get the first statement of (e) and, moreover,

$$\sum_n \|\zeta_n(\omega) \boldsymbol{u}_n(\omega)\| = \|\boldsymbol{g}(\omega)\| \quad \mu\text{-a.e.,}$$

hence the convergence holds also in the norm of $L^1(\mu, N(H))$, by the Lebesgue dominated convergence theorem for Bochner integral. ■

Set

$W_0 = \{\boldsymbol{f} : \Omega \to B(H) \,;\, \boldsymbol{f}$ is bounded, measurable and has separable range$\}$.

By a measurable function we mean a strongly measurable one, i.e., an almost everywhere limit of simple functions. However, note that weak measurability is equivalent to strong measurability in this case by the Pettis measurability theorem as we consider only functions with separable range.

Then W_0 is clearly a C*-algebra when equipped with the pointwise operations and the supremum norm.

We remark that the following lemma seems to be close to the results of [47, Section IV.7]. However, it is not clear how to apply those results in our situation, so we give the proofs.

LEMMA 4.11. *For $\boldsymbol{f} \in W_0$ and $\boldsymbol{h} \in L^2(\mu, H)$ define the function $T_{\boldsymbol{f}}\boldsymbol{h}$ by*

$$T_{\boldsymbol{f}}\boldsymbol{h}(\omega) = \boldsymbol{f}(\omega)(\boldsymbol{h}(\omega)), \quad \omega \in \Omega.$$

(i) *For each $\boldsymbol{f} \in W_0$ the mapping $T_{\boldsymbol{f}}$ is a bounded linear operator on $L^2(\mu, H)$ which belongs to W and satisfies $\|T_{\boldsymbol{f}}\| \leq \|\boldsymbol{f}\|_\infty$.*
(ii) *If $\boldsymbol{f} \in W_0$ and $\boldsymbol{g} \in W_* = L^1(\mu, N(H))$, then*

$$\langle T_{\boldsymbol{f}}, \boldsymbol{g} \rangle = \int \mathrm{tr}(\boldsymbol{f}(\omega)\boldsymbol{g}(\omega)) \, \mathrm{d}\mu(\omega).$$

(iii) *$T_{\boldsymbol{f}}$ is a partial isometry (a projection) in W whenever $\boldsymbol{f}(\omega)$ is a partial isometry (a projection) μ-a.e.*
(iv) *If $\boldsymbol{g} \in L^1(\mu, N(H))$ is represented as in Proposition 4.10(e), then $s(\boldsymbol{g}) \leq \sum_n T_{\boldsymbol{u}_n^*}$ where series converges in the SOT topology in W.*

Proof. (i) It is clear that $\boldsymbol{h} \mapsto T_{\boldsymbol{f}}\boldsymbol{h}$ is a linear mapping assigning to each H-valued function another H-valued function. Moreover,

$$\|T_{\boldsymbol{f}}\boldsymbol{h}(\omega)\| = \|\boldsymbol{f}(\omega)(\boldsymbol{h}(\omega))\| \leq \|\boldsymbol{f}(\omega)\| \, \|\boldsymbol{h}(\omega)\| \leq \|\boldsymbol{f}\|_\infty \|\boldsymbol{h}(\omega)\|.$$

In particular, if a sequence $(\boldsymbol{h}_n)$ converges almost everywhere to a function $\boldsymbol{h}$, then $(T_{\boldsymbol{f}}\boldsymbol{h}_n)$ converges almost everywhere to $T_{\boldsymbol{f}}\boldsymbol{h}$. Therefore $T_{\boldsymbol{f}}$ is well defined on $L^2(\mu, H)$ (in the sense that if $\boldsymbol{h}_1 = \boldsymbol{h}_2$ a.e., then $T_{\boldsymbol{f}}\boldsymbol{h}_1 = T_{\boldsymbol{f}}\boldsymbol{h}_2$ a.e.).

The next step is to observe that $T_{\boldsymbol{f}}\boldsymbol{h}$ is measurable whenever $\boldsymbol{h}$ is measurable. This is easy for simple functions. Any measurable function is an a.e. limit of a sequence of simple functions, hence the measurability follows by the above.

Further, it follows from the above inequality that $\|T_{\boldsymbol{f}}\boldsymbol{h}\|_2 \leq \|\boldsymbol{f}\|_\infty \|\boldsymbol{h}\|_2$, thus $\|T_{\boldsymbol{f}}\| \leq \|\boldsymbol{f}\|_\infty$. Finally, by [23, Lemma 5.12] we deduce that $T_{\boldsymbol{f}} \in W$.

(ii) Let us first show that $\boldsymbol{f}\boldsymbol{g} \in L^1(\mu, N(H))$ whenever $\boldsymbol{f} \in W_0$ and $\boldsymbol{g} \in L^1(\mu, N(H))$. By the obvious inequalities the only thing to prove is measurability of this mapping. This is easy if $\boldsymbol{g}$ is a simple function. The general case follows from the facts that any measurable function is an a.e. limit of simple functions and that measurability is preserved by a.e. limits of sequences.

It remains to prove the equality. Since the functions from W_0 are separably-valued, functions which are countably-valued are dense in W_0. So, it is enough to prove the equality for countably-valued functions. To this end let

$$\boldsymbol{f} = \sum_{k \in \mathbb{N}} \chi_{E_k} x_k,$$

where (E_k) is a disjoint sequence of measurable sets and (x_k) is a bounded

sequence in $B(H)$. For any $\boldsymbol{h} \in L^2(\mu, H)$ we have

$$T_{\boldsymbol{f}}\boldsymbol{h}(\omega) = \sum_{k \in \mathbb{N}} \chi_{E_k}(\omega) x_k(\boldsymbol{h}(\omega)), \quad \omega \in \Omega.$$

As $T_{\boldsymbol{f}}\boldsymbol{h} \in L^2(\mu, H)$ by (i) and the sets E_k are pairwise disjoint, we see that

$$T_{\boldsymbol{f}}\boldsymbol{h} = \sum_{k \in \mathbb{N}} T_{\chi_{E_k} x_k}\boldsymbol{h},$$

where the series converges in $L^2(\mu, H)$. Since this holds for any $\boldsymbol{h} \in L^2(\mu, H)$, we deduce that

$$T_{\boldsymbol{f}} = \sum_{k \in \mathbb{N}} T_{\chi_{E_k} x_k}$$

unconditionally in the SOT topology, hence also in the weak* topology of W. Thus, for any $\boldsymbol{g} \in W_* = L^1(\mu, N(H))$ we get

$$\langle T_{\boldsymbol{f}}, \boldsymbol{g} \rangle = \sum_{k \in \mathbb{N}} \langle T_{\chi_{E_k} x_k}, \boldsymbol{g} \rangle = \sum_{k \in \mathbb{N}} \int_{E_k} \operatorname{tr}(x_k \boldsymbol{g}(\omega)) \, d\mu(\omega) = \int \operatorname{tr}(\boldsymbol{f}(\omega)\boldsymbol{g}(\omega)) \, d\mu(\omega),$$

where in the second equality we used Lemma 4.9(i).

(iii) This is obvious as the mapping $\boldsymbol{f} \mapsto T_{\boldsymbol{f}}$ is clearly a $*$-homomorphism of W_0 into W.

(iv) First observe that the mappings $\boldsymbol{u}_n^*$ belong to W_0. Indeed, by Proposition 4.10 the mapping $\boldsymbol{u}_n$ is measurable and has separable range (as $K(H_0)$ is separable). Moreover, $\|\boldsymbol{u}_n\|_\infty \leq 1$ for each $n \in \mathbb{N}$. These properties are shared by $\boldsymbol{u}_n^*$, hence $\boldsymbol{u}_n^* \in W_0$.

By (iii) we deduce that $T_{\boldsymbol{u}_n^*}$ is a partial isometry for any $n \in \mathbb{N}$. Moreover, these partial isometries are pairwise orthogonal (cf. property (d) from Proposition 4.10), hence $U = \sum_n T_{\boldsymbol{u}_n^*}$ is a well-defined partial isometry in W. Moreover, by taking $\boldsymbol{g}$ as in Proposition 4.10(e), we have

$$\langle U, \boldsymbol{g} \rangle = \sum_{n=0}^{\infty} \langle T_{\boldsymbol{u}_n^*}, \boldsymbol{g} \rangle = \sum_{n=0}^{\infty} \int \operatorname{tr}(\boldsymbol{u}_n^*(\omega)\boldsymbol{g}(\omega)) \, d\mu(\omega)$$

$$= \sum_{n=0}^{\infty} \int \zeta_n(\omega) \operatorname{tr}(\boldsymbol{u}_n^*(\omega)\boldsymbol{u}_n(\omega)) \, d\mu(\omega)$$

$$= \int \sum_{n=0}^{\infty} \zeta_n(\omega) \operatorname{tr}(\boldsymbol{u}_n^*(\omega)\boldsymbol{u}_n(\omega)) \, d\mu(\omega) = \int \|\boldsymbol{g}(\omega)\| \, d\mu(\omega) = \|\boldsymbol{g}\|,$$

thus $s(\boldsymbol{g}) \leq U$. ∎

Proof of Proposition 4.2 for $A \overline{\otimes} B(H)_a$. Fix any $\boldsymbol{g} \in M_* = L^1(\mu, N(H)_a)$ and $\varepsilon > 0$. Fix its representation from Proposition 4.10. Fix $N \in \mathbb{N}$ with

$$\left\| \sum_{n > N} \zeta_n \boldsymbol{u}_n \right\| < \varepsilon.$$

This is possible by the convergence established in Proposition 4.10. Note that

$$-\boldsymbol{g} = \boldsymbol{g}^t = \sum_{n=1}^{\infty} \zeta_n \boldsymbol{u}_n^t,$$

hence $\boldsymbol{u}_n^t = -\boldsymbol{u}_n^t$. (Note also that the representation from Proposition 4.10 is unique due to the uniqueness of the Hilbert–Schmidt representation). Let

$$\boldsymbol{g}_1 = \sum_{n=1}^{N} \zeta_n \boldsymbol{u}_n.$$

Then $\boldsymbol{g}_1 \in M_*$ as $\boldsymbol{g}_1^t = -\boldsymbol{g}_1$. Further, let

$$\boldsymbol{v} = \sum_{n=1}^{N} \boldsymbol{u}_n.$$

We have $\boldsymbol{g} - \boldsymbol{g}_1 \perp \boldsymbol{g}_1$ as

$$s(\boldsymbol{g}_1) \leq T_{\boldsymbol{v}^*} \quad \text{and} \quad s(\boldsymbol{g} - \boldsymbol{g}_1) \leq \sum_{n>N} T_{\boldsymbol{u}_n^*}$$

and the two tripotents on the right-hand sides are orthogonal. Moreover, $T_{\boldsymbol{v}^*}$ is a finite tripotent in M by [23, Proposition 5.31(i) and Lemma 5.16(ii)]. ∎

Proof of Proposition 4.2 for $A \overline{\otimes} B(H)$. The proof is an easier version of the previous case. Fix $\boldsymbol{g} \in W_* = L^1(\mu, N(H))$ and $\varepsilon > 0$. In the same way we find N and define $\boldsymbol{g}_1$ and $\boldsymbol{v}$. We omit the considerations of the transpose and antisymmetry. Finally, $T_{\boldsymbol{v}^*}$ is a finite tripotent in W by [23, Proposition 4.7 and Lemma 5.16(ii)]. ∎

5. JW*-algebras. The aim of this section is to show the following proposition which will be used to prove Proposition 3.5.

PROPOSITION 5.1. *Let M be a JBW*-algebra, $\varphi \in M_*$ and $\varepsilon > 0$. Then there are functionals $\varphi_1, \varphi_2 \in M_*$ and a unitary element $w \in M$ satisfying:*

(i) $\|\varphi_1\| \leq \|\varphi\|$;
(ii) $\|\varphi_2\| < \varepsilon$;
(iii) $s(\varphi_1) \leq w$;
(iv) $\|\cdot\|_\varphi^2 \leq \|\cdot\|_{\varphi_1}^2 + \|\cdot\|_{\varphi_2}^2$.

The proof will be given at the end of the section with the help of several lemmata.

We focus mainly on JW*-algebras, i.e., on weak*-closed Jordan *-subalgebras of von Neumann algebras. To this end we recall some notation (cf. [47, Section III.2]).

Let A be a C*-algebra and let $\phi \in A^*$. Then we define functionals $a\phi$ and ϕa by

(11) $$a\phi(x) = \phi(xa) \quad \text{and} \quad \phi a(x) = \phi(ax) \quad \text{for } x \in A.$$

Note that $a\phi, \phi a \in A^*$ and $\|a\phi\| \le \|a\| \|\phi\|$, $\|\phi a\| \le \|a\| \|\phi\|$. We recall the natural isometric involution $\phi \mapsto \phi^*$ defined by $\phi^*(x) = \overline{\phi(x^*)}$. Then clearly $(a\phi)^* = \phi^* a^*$, $(\phi a)^* = a^* \phi^*$.

If W is a von Neumann algebra and if $\phi \in W_*$, $a \in W$ then $a\phi, \phi a \in W_*$. Further, given $\phi \in W_*$ we set $|\phi| = s(\phi)\phi$ where $s(\phi) \in W$ is the support tripotent of ϕ. Then $\phi = s(\varphi)^* |\phi|$ is the polar decomposition of ϕ (cf. [47, Section III.4]). More generally, if $a \in W$ is a norm-one element at which ϕ attains its norm then $|\phi| = a\phi$, $\phi = a^* |\phi|$, $|\phi^*| = \phi a$ (cf. (8)). Note that $|\phi| = |\phi|^*$ since $|\phi|$ is positive. All this is stable by small perturbations as witnessed by the following lemma.

LEMMA 5.2 ([41, Lemma 3.3]). *Let A be a C*-algebra, ϕ a functional on A and a, b in the unit ball of A. Then*

(12) $$\left\|\phi - a^* |\phi|\right\| \le (2\|\phi\|)^{1/2} \big| \|\phi\| - \phi(a) \big|^{1/2},$$

(13) $$\left\| |\phi| - a\phi \right\| \le (2\|\phi\|)^{1/2} \big| \|\phi\| - \phi(a) \big|^{1/2},$$

(14) $$\left\| |\phi^*| - \phi a \right\| \le (2\|\phi\|)^{1/2} \big| \|\phi\| - \phi(a) \big|^{1/2}.$$

(Formula (14), which is not stated explicitly in [41, Lemma 3.3], follows easily from (13) by $\left\| |\phi^*| - \phi a \right\| = \left\| |\phi^*| - a^* \phi^* \right\| \le (2\|\phi^*\|)^{1/2} \big| \|\phi^*\| - \phi^*(a^*) \big|^{1/2} = (2\|\phi\|)^{1/2} \big| \|\phi\| - \phi(a) \big|^{1/2}$.)

There is another way to obtain positive functionals: We can write $\phi = \phi_1 - \phi_2 + i(\phi_3 - \phi_4)$ with positive $\phi_k \in W_*$ ($k = 1, 2, 3, 4$) such that $\|\phi_k - \phi_{k+1}\| = \|\phi_k\| + \|\phi_{k+1}\| \le \|\phi\|$, $k = 1, 3$ (cf. [47, Theorem III.4.2]). Then we set

$$[\phi] = \frac{1}{2} \sum_{k=1}^{4} \phi_k = \frac{1}{2}(|\phi_1 - \phi_2| + |\phi_3 - \phi_4|)$$

and observe that $[\phi] \in W_*$ is positive, $\|[\phi]\| \le \|\phi\|$ and $|\phi(a)| \le 2[\phi](a)$ for all positive $a \in W$.

Finally, let us remark that if A is a C*-algebra, then A^{**} is a von Neumann algebra and $A^* = (A^{**})_*$, thus $|\phi|$ and $[\phi]$ make sense also for continuous functionals on a C*-algebra.

LEMMA 5.3. *Let W be von Neumann algebra, let $w \in W$ be a unitary element and $\delta \in (0, 1)$. Let $\phi \in W_*$ be a norm-one functional such that $\phi(w) > 1 - \delta$ (in particular, $\phi(w) \in \mathbb{R}$). Then $\psi := w^* |\phi|$ is a norm-one element of W_* satisfying $\psi(w) = 1$ and $\|\phi - \psi\| < \sqrt{2\delta}$.*

Proof. On the one hand, we have $\|\psi\| \leq \||\phi\|| = \|\phi\| = 1$. On the other hand, since $\psi(w) = (w^*|\phi|)(w) = |\phi|(ww^*) = |\phi|(1) = \||\phi\|| = 1$, we deduce that $\|\psi\| = 1$. Applying (12) of Lemma 5.2 we obtain

$$\||\phi - w^*|\phi|\|| \leq \sqrt{2}|1 - \phi(w)|^{1/2} \leq \sqrt{2\delta}. \blacksquare$$

We continue by extending the previous lemma to JW*-algebras.

LEMMA 5.4. *Let M be a JW*-algebra, $w \in M$ a unitary element and $\delta \in (0,1)$. Let $\phi \in M_*$ be a norm-one functional such that $\phi(w) > 1 - \delta$ (in particular, $\phi(w) \in \mathbb{R}$). Then there exists a norm-one functional $\psi \in M_*$ satisfying $\psi(w) = 1$ and $\|\phi - \psi\| < \sqrt{2\delta}$.*

Proof. Let us assume that M is a JW*-subalgebra of a von Neumann algebra W. Let 1 denote the unit of M. Then 1 is a projection in W, thus, up to replacing W by $1W1$, we may assume that M contains the unit of W.

We observe that w, being a unitary element in M, is unitary in W. Let $\tilde{\phi} \in W_*$ be a norm-preserving extension of ϕ provided by [7, Theorem]. By hypothesis, $1 - \delta < \phi(w) = \tilde{\phi}(w) \leq \|\phi\| = \|\tilde{\phi}\| = 1$. Now, applying Lemma 5.3 to W, $\tilde{\phi} \in W_*$ and the unitary w, we find a norm-one functional $\tilde{\psi} \in W_*$ satisfying $\tilde{\psi}(w) = 1$ and $\|\tilde{\phi} - \tilde{\psi}\| < \sqrt{2\delta}$. Since $w \in M$ and $1 = \tilde{\psi}(w)$, the functional $\psi = \tilde{\psi}|_M$ has norm one, $\psi(w) = 1$ and clearly $\|\phi - \psi\| < \sqrt{2\delta}$. $\blacksquare$

LEMMA 5.5. *Let M be a JW*-algebra, let $\phi \in M_*$ and $\delta > 0$. Suppose a_1, a_2 are norm-one elements in M such that*

$$\big|\|\phi\| - \phi(a_k)\big| < \delta\|\phi\| \quad \text{for } k = 1, 2.$$

Then there is a positive functional $\omega \in M_*$ satisfying $\|\omega\| \leq 2\sqrt{2\delta}\,\|\phi\|$ and

$$|\phi\{x, x, a_1 - a_2\}| \leq 4\|x\|_\omega^2 \quad \text{for all } x \in M.$$

Proof. Similarly to the proof of Lemma 5.4 we may assume that M is a JW*-subalgebra of a von Neumann algebra W containing the unit of W.

Let $\tilde{\phi} \in W_*$ be a norm-preserving normal extension of ϕ (see [7, Theorem]). Working in W_* we set $\tilde{\psi}_l = a_1\tilde{\phi} - a_2\tilde{\phi}$ and $\tilde{\psi}_r = \tilde{\phi}a_1 - \tilde{\phi}a_2$. By (13) of Lemma 5.2 we have $\||\tilde{\phi}\| - a_k\tilde{\phi}\| \leq \sqrt{2\delta}\,\|\tilde{\phi}\|$ ($k = 1, 2$), hence $\|\tilde{\psi}_l\| \leq 2\sqrt{2\delta}\,\|\tilde{\phi}\|$. Likewise we get $\|\tilde{\psi}_r\| \leq 2\sqrt{2\delta}\,\|\tilde{\phi}\|$ with (14) of Lemma 5.2. Set $\tilde{\omega} = ([\tilde{\psi}_l] + [\tilde{\psi}_r])/2$. Then $\|\tilde{\omega}\| \leq 2\sqrt{2\delta}\,\|\tilde{\phi}\|$ and

$$|\tilde{\phi}\{x, x, a_1 - a_2\}| = \tfrac{1}{2}|\tilde{\psi}_l(xx^*) + \tilde{\psi}_r(x^*x)| \leq [\tilde{\psi}_l](xx^*) + [\tilde{\psi}_r](x^*x)$$

$$\leq ([\tilde{\psi}_l] + [\tilde{\psi}_r])(xx^* + x^*x) = 4\tilde{\omega}(\{x, x, 1\}) = 4\|x\|_{\tilde{\omega}}^2.$$

It remains to set $\omega = \tilde{\omega}|_M$. $\blacksquare$

LEMMA 5.6. *Let M be a JW*-algebra, $\phi \in M_*$ and let a be a norm-one element of M. Then there is a positive functional $\omega \in M_*$ such that*

$$\|\omega\| \leq \|\phi\| \quad \text{and} \quad \forall x \in W : |\phi\{x, x, a\}| \leq 4\|x\|_\omega^2.$$

Proof. The proof resembles that of Lemma 5.5. Assume that M is a JW*-subalgebra of a von Neumann algebra W and $1_W \in M$. Let $\tilde{\phi} \in W_*$ be a norm-preserving extension of ϕ (see [7, Theorem]). Set $\tilde{\psi}_l = a\tilde{\phi}$ and $\tilde{\psi}_r = \tilde{\phi}a$. Then $\|\tilde{\psi}_l\| \le \|a\| \|\tilde{\phi}\| = \|\tilde{\phi}\|$, and similarly $\|\tilde{\psi}_r\| \le \|\tilde{\phi}\|$. Set $\tilde{\omega} = ([\tilde{\psi}_l] + [\tilde{\psi}_r])/2$. Then $\|\tilde{\omega}\| \le \|\tilde{\phi}\|$ and

$$|\tilde{\phi}\{x,x,a\}| = \tfrac{1}{2}|\tilde{\psi}_l(xx^*) + \tilde{\psi}_r(x^*x)| \le [\tilde{\psi}_l](xx^*) + [\tilde{\psi}_r](x^*x)$$
$$\le ([\tilde{\psi}_l] + [\tilde{\psi}_r])(xx^* + x^*x) = 4\tilde{\omega}(\{x,x,1\}) = 4\|x\|_{\tilde{\omega}}^2.$$

Finally, we may set $\omega = \tilde{\omega}|_M$. ∎

Proof of Proposition 5.1. It follows from [23, Theorem 7.1] that any JBW*-algebra M can be represented as $M_1 \oplus^{\ell_\infty} M_2$ where M_1 is a finite JBW*-algebra and M_2 is a JW*-algebra. The validity of Proposition 5.1 for finite JBW*-algebras follows immediately from Observation 4.1. Since the validity of Proposition 5.1 is clearly preserved by ℓ_∞-sums, it remains to prove it for JW*-algebras.

So, assume that M is a JW*-algebra and $\varphi \in M_*$. By homogeneity we may assume $\|\varphi\| = 1$. Fix $\varepsilon > 0$. Choose $\delta > 0$ such that $12\sqrt{2\delta} < \varepsilon$. By the Wright–Youngson extension of the Russo–Dye theorem, the convex hull of all unitary elements in M is norm dense in the closed unit ball of M (see [50, Theorem 2.3] or [9, Fact 4.2.39]). We can therefore find a unitary element w such that $\varphi(w) > 1 - \delta$. By Lemma 5.4 there exists a norm-one functional $\psi \in M_*$ satisfying $\psi(w) = 1$ and $\|\varphi - \psi\| < \sqrt{2\delta}$. Set $u = s(\varphi)$.

For $x \in M$ we then have

$$\|x\|_\varphi^2 = \varphi\{x,x,u\} = \psi\{x,x,w\} + (\varphi - \psi)\{x,x,w\} + \varphi\{x,x,u-w\}.$$

Applying Lemma 5.6 to $\varphi - \psi$ and w we find a positive functional $\omega_1 \in M_*$ with $\|\omega_1\| \le \|\varphi - \psi\| < \sqrt{2\delta}$ such that

$$|(\varphi - \psi)\{x,x,w\}| \le 4\|x\|_{\omega_1}^2 \quad \text{for } x \in M.$$

Applying Lemma 5.5 to the functional φ and the pair $w, u \in M$ we get a positive functional $\omega_2 \in M_*$ with $\|\omega_2\| \le 2\sqrt{2\delta}$ such that

$$|\varphi\{x,x,u-w\}| \le 4\|x\|_{\omega_2}^2 \quad \text{for } x \in M.$$

Hence for each $x \in M$ we have

$$\|x\|_\varphi^2 \le \|x\|_\psi^2 + 4(\|x\|_{\omega_1}^2 + \|x\|_{\omega_2}^2) = \|x\|_\psi^2 + \|x\|_{4(\omega_1+\omega_2)}^2,$$

where we used the fact that ω_1 and ω_2 are positive functionals. As $s(\psi) \le w$ (just have in mind $\psi(w) = 1$ and (7)), w is unitary and

$$\|4(\omega_1 + \omega_2)\| < 12\sqrt{2\delta},$$

it is enough to set $\varphi_1 = \psi$ and $\varphi_2 = 4(\omega_1 + \omega_2)$. ∎

REMARK 5.7. (1) Note that by [23, Proposition 7.5] any finite tripotent in a JBW*-algebra is majorized by a unitary element, hence Proposition 4.2

is indeed a stronger version of Proposition 5.1 in the special case in which
the JBW*-algebra M is a direct sum of a finite JBW*-algebra and a type I
JBW*-algebra. (For (i) and (iv) of Proposition 5.1 see the remarks before
the statement of Proposition 4.2.) Further, as will be seen at the begin-
ning of the next section, Proposition 5.1 is the main ingredient for proving
Proposition 3.5.

(2) There is an alternative way of proving Proposition 5.1. It follows from
[23, Theorem 7.1] that any JBW*-algebra M can be represented by $M_1 \oplus^{\ell^\infty}$
$M_2 \oplus^{\ell^\infty} M_3$ where M_1 is a finite JBW*-algebra, M_2 is a type I JBW*-algebra
and M_3 is a von Neumann algebra. So, we can conclude using Proposition 4.2
and giving the above argument only for von Neumann algebras (which is
slightly easier).

6. Proofs of the main results

Proof of Proposition 3.5. Let M be a JBW*-algebra, $\varphi \in M_*$ and $\varepsilon > 0$.
By homogeneity we may assume that $\|\varphi\| = 1$. Let φ_1, φ_2 and w correspond
to φ and $\varepsilon/2$ by Proposition 5.1. Since w is unitary, we have $M_2(w) = M$,
hence we may apply Lemma 3.3 to get $\psi_2 \in M_*$ such that

$$s(\psi_2) \leq w, \quad \|\psi_2\| \leq \|\varphi_2\|, \quad \|\cdot\|_{\varphi_2} \leq \sqrt{2}\,\|\cdot\|_{\psi_2}.$$

Then

$$\begin{aligned} \|\cdot\|_\varphi^2 &\leq \|\cdot\|_{\varphi_1}^2 + \|\cdot\|_{\varphi_2}^2 \leq \|\cdot\|_{\varphi_1}^2 + 2\|\cdot\|_{\psi_2}^2 \\ &= \|\cdot\|_{\varphi_1 + 2\psi_2}^2 = (\|\varphi_1\| + 2\|\psi_2\|)\|\cdot\|_\psi^2, \end{aligned}$$

where

$$\psi = \frac{\varphi_1 + 2\psi_2}{\|\varphi_1\| + 2\|\psi_2\|}.$$

(Note that the first equality follows from the fact that the support tripotents
of both functionals are below w.) Since the functionals φ_1 and ψ_2 attain their
norms at w, we deduce that $\|\psi\| = 1$. It remains to observe that

$$\|\varphi_1\| + 2\|\psi_2\| \leq \|\varphi\| + 2\|\varphi_2\| \leq 1 + \varepsilon. \quad \blacksquare$$

Having proved Proposition 3.5, we know that Proposition 3.4 is valid as
well. Using it and Theorem 3.1 we get the following theorem.

THEOREM 6.1. *Let M be a JBW*-algebra, let H be a Hilbert space and
let $T : M \to H$ be a weak*-to-weak continuous linear operator. Given $\varepsilon > 0$,
there is a norm-one functional $\varphi \in M_*$ such that*

$$\|Tx\| \leq (\sqrt{2} + \varepsilon)\|T\|\,\|x\|_\varphi \quad \textit{for } x \in M.$$

Now we get the main result by the standard dualization.

Proof of Theorem 1.3. Let $T : B \to H$ be a bounded linear operator
from a JB*-algebra into a Hilbert space. Let $\varepsilon > 0$. Since Hilbert spaces are

reflexive, the second adjoint operator T^{**} maps B^{**} into H and is weak*-to-weak continuous. Further, B^{**} is a JBW*-algebra (cf. [26, Theorem 4.4.3] and [49] or [10, Proposition 5.7.10] and [9, Theorems 4.1.45 and 4.1.55]), so Theorem 6.1 provides the respective functional $\varphi \in (B^{**})_* = B^*$. ∎

We further note that for JB*-algebras we have two different forms of the Little Grothendieck Theorem: a triple version (the just proved Theorem 1.3) and an algebraic version (an analogue of Theorem C). The difference is that the first form provides just a norm-one functional while the second form provides a state, i.e., a positive norm-one functional. Let us now show that the algebraic version may be proved from the triple version.

THEOREM 6.2. *Let M be a JBW*-algebra, let H be a Hilbert space and let $T : M \to H$ be a weak*-to-weak continuous linear operator. Given $\varepsilon > 0$, there is a state $\varphi \in M_*$ such that*

$$\|Tx\| \le (2 + \varepsilon)\|T\|\varphi(x \circ x^*)^{1/2} \quad \textit{for } x \in M.$$

Proof. By Theorem 6.1 there is a norm-one functional $\psi \in M_*$ such that

$$\|Tx\| \le \left(\sqrt{2} + \frac{\varepsilon}{\sqrt{2}}\right)\|T\| \, \|x\|_\psi \quad \text{for } x \in M.$$

Since M is unital and $M_2(1) = M$, Lemma 3.3 yields a norm-one functional $\varphi \in M_*$ with $s(\varphi) \le 1$ and $\|\cdot\|_\psi \le \sqrt{2}\|\cdot\|_\varphi$. Then φ is a state (note that $\varphi(1) = 1$) and

$$\|Tx\| \le (2 + \varepsilon)\|T\| \, \|x\|_\varphi \quad \text{for } x \in M.$$

It remains to observe that

$$\|x\|_\varphi = \sqrt{\varphi\{x, x, 1\}} = \sqrt{\varphi(x \circ x^*)} \quad \text{for } x \in M. \quad \blacksquare$$

THEOREM 6.3. *Let B be a JB*-algebra, let H be a Hilbert space and let $T : B \to H$ be a bounded linear operator. Then there is a state $\varphi \in B^*$ with*

$$\|Tx\| \le 2\|T\|\varphi(x \circ x^*)^{1/2} \quad \textit{for } x \in B.$$

Proof. Since B^{**} is a JBW*-algebra, T^{**} maps B^{**} into H and T^{**} is weak*-to-weak continuous, by Theorem 6.2 we get a sequence (φ_n) of states on B such that

$$\|Tx\| \le \left(2 + \frac{1}{n}\right)\|T\|\varphi_n(x \circ x^*)^{1/2} \quad \text{for } x \in B \text{ and } n \in \mathbb{N}.$$

Let $\tilde{\varphi}$ be a weak* cluster point of the sequence (φ_n). Then $\tilde{\varphi}$ is positive, $\|\tilde{\varphi}\| \le 1$ and

$$\|Tx\| \le 2\|T\|\tilde{\varphi}(x \circ x^*)^{1/2} \quad \text{for } x \in B.$$

Now we can clearly replace $\tilde{\varphi}$ by a state. Indeed, if $\tilde{\varphi} \ne 0$, we take $\varphi = \tilde{\varphi}/\|\tilde{\varphi}\|$. If $\tilde{\varphi} = 0$, then $T = 0$ and hence φ may be any state. (Note that in case B is unital, $\tilde{\varphi}$ is already a state.) ∎

We finish this section by showing that our main result easily implies Theorem B.

Proof of Theorem B from Theorem 1.3. Let A be a C*-algebra, let H be a Hilbert space and let $T : A \to H$ be a bounded linear operator. By Theorem 1.3 there is a sequence (ψ_n) of norm-one functionals in A^* with

$$\|Tx\| \leq \left(\sqrt{2} + \frac{1}{n}\right)\|T\|\,\|x\|_{\psi_n} \quad \text{for } x \in A \text{ and } n \in \mathbb{N}.$$

Recall that A^{**} is a von Neumann algebra. Set $u_n = s(\psi_n) \in A^{**}$. Then

$$\|x\|_{\psi_n}^2 = \psi_n\{x, x, u_n\} = \tfrac{1}{2}(\psi_n(xx^*u_n) + \psi_n(u_n x^*x))$$
$$= \tfrac{1}{2}(u_n\psi_n(xx^*) + \psi_n u_n(x^*x))$$

for $x \in A$. Moreover, $\varphi_{1,n} = u_n\psi_n$ and $\varphi_{2,n} = \psi_n u_n$ are states on A (note that $\varphi_{1,n} = |\psi_n|$ and $\varphi_{2,n} = |\psi_n^*|$) such that

$$\|Tx\| \leq \left(\sqrt{2} + \frac{1}{n}\right)\|T\| \cdot \frac{1}{\sqrt{2}}(\varphi_{1,n}(xx^*) + \varphi_{2,n}(x^*x))^{1/2} \quad \text{for } x \in A,\ n \in \mathbb{N}.$$

Suppose (φ_1, φ_2) is a weak* cluster point of the sequence $((\varphi_{1,n}, \varphi_{2,n}))_n$ in $B_{A^*} \times B_{A^*}$. Then φ_1, φ_2 are positive functionals of norm at most one with

$$\|Tx\| \leq \|T\|(\varphi_1(xx^*) + \varphi_2(x^*x))^{1/2} \quad \text{for } x \in A.$$

Just as above we may replace φ_1 and φ_2 by states. ∎

7. Examples and problems

QUESTION 7.1. *Do Theorems* 1.3 *and* 6.1 *hold with the constant* $\sqrt{2}$ *instead of* $\sqrt{2} + \varepsilon$?

We remark that these theorems do not hold with a constant strictly smaller than $\sqrt{2}$. Indeed, assume that Theorem 1.3 holds with a constant K. Then Theorem B holds with constant $K/\sqrt{2}$ (see the proof of the relationship of these two theorems in Section 6). But the best constant for Theorem B is 1, due to [22].

Since the example in [22] uses a rather involved combinatorial construction, we provide an easier example showing that the constant in Theorem 1.3 has to be at least $\sqrt{2}$.

EXAMPLE 7.2. Let H be an infinite-dimensional Hilbert space. Let $A = K(H)$ be the C*-algebra of compact operators. Fix an arbitrary unit vector $\xi \in H$ and define $T : A \to H$ by $Tx = x\xi$ for $x \in A$. It is clear that $\|T\| = \|\xi\| = 1$. Fix an arbitrary norm-one functional $\varphi \in A^*$. We are going to prove that

$$(15) \qquad \sup\left\{\frac{\|Tx\|}{\|T\|\,\|x\|_\varphi} \;;\; x \in A,\ \|x\|_\varphi \neq 0\right\} \geq \sqrt{2}.$$

Recall that $K(H)^*$ is identified with $N(H)$, the space of nuclear operators on H equipped with the nuclear norm, and $K(H)^{**}$ is identified with $B(H)$, the von Neumann algebra of all bounded linear operators on H. Using the trace duality we deduce that there is a nuclear operator z on H such that $\operatorname{tr}(|z|) = \|z\|_N = 1$ and $\varphi(x) = \operatorname{tr}(zx)$ for $x \in A$. Consider the polar decomposition $z = u|z|$ in $B(H)$. Then $|z| = u^*z$, hence $s(\varphi) \le u^*$. (Note that $\varphi(u^*) = \operatorname{tr}(zu^*) = \operatorname{tr}(u^*z) = \operatorname{tr}(|z|) = 1$, hence $s(\varphi) \le u^*$ by (7). The converse inequality holds as well, but it is not important.) It follows that for each $x \in A$ we have

$$\|x\|_\varphi^2 = \varphi(\{x, x, u^*\}) = \tfrac{1}{2}\varphi(xx^*u^* + u^*x^*x) = \tfrac{1}{2}\operatorname{tr}(xx^*u^*z + u^*x^*xz)$$
$$= \tfrac{1}{2}(\operatorname{tr}(xx^*|z|) + \operatorname{tr}(u^*x^*xz)).$$

If $\eta \in H$ is a unit vector, we define the operator

$$y_\eta(\zeta) = \langle \zeta, \xi \rangle \eta, \quad \zeta \in H.$$

Then $y_\eta \in A$, $\|y_\eta\| = 1$ and $\|Ty_\eta\| = 1$. Moreover,

$$y_\eta^*(\zeta) = \langle \zeta, \eta \rangle \xi,$$

hence

$$y_\eta y_\eta^*(\zeta) = \langle \zeta, \eta \rangle \eta \quad \text{and} \quad y_\eta^* y_\eta(\zeta) = \langle \zeta, \xi \rangle \xi.$$

Thus

$$\|y_\eta\|_\varphi^2 = \tfrac{1}{2}(\operatorname{tr}(|z|y_\eta y_\eta^*) + \operatorname{tr}(zu^*y_\eta^*y_\eta)) = \tfrac{1}{2}(\langle |z|\eta, \eta \rangle + \langle zu^*\xi, \xi \rangle)$$
$$\le \tfrac{1}{2}(1 + \langle |z|\eta, \eta \rangle).$$

Consequently,

$$\inf \{\|x\|_\varphi^2 \,; \, x \in A, \|Tx\| = 1\} \le \tfrac{1}{2}\inf \{1 + \langle |z|\eta, \eta \rangle \,; \, \|\eta\| = 1\}$$
$$= \tfrac{1}{2} + \tfrac{1}{2}\min \sigma(|z|),$$

where the last equality follows from [15, Theorem 15.35]. Now, z is a nuclear operator of norm one. Thus $0 \in \sigma(|z|)$ as H has infinite dimension. Hence

$$\inf \{\|x\|_\varphi \,; \, x \in A, \|Tx\| = 1\} \le 1/\sqrt{2},$$

which yields inequality (15). $\qquad\square$

REMARK 7.3. If H is a finite-dimensional Hilbert space, the construction from Example 7.2 could be done as well. Now $A = K(H) = B(H)$ can be identified with the algebra of $n \times n$ matrices where $n = \dim H$. Now $\sigma(|z|)$ need not contain 0, but at least one of the eigenvalues of $|z|$ is at most $\frac{1}{n}$. So, we get a lower bound $\sqrt{\frac{2n}{n+2}}$ for the constant in Theorem 1.3.

Next we address the optimality of the algebraic version of the Little Grothendieck Theorem.

QUESTION 7.4. *What is the optimal constant in Theorems C, 6.2, 6.3? In particular, do these theorems hold with the constant $\sqrt{2}$?*

Note that the constant cannot be smaller than $\sqrt{2}$ due to Example 7.2. The following example shows that 7.2 cannot yield a greater lower bound.

EXAMPLE 7.5. Let H, A, ξ and T be as in Example 7.2. Let $u \in A^{**} = B(H)$ be any unitary element. Then

$$\varphi_u(x) = \langle x\xi, u\xi \rangle, \quad x \in A,$$

defines a norm-one functional in A^* such that $s(\varphi_u) \leq u$ and, moreover,

$$\|Tx\| \leq \sqrt{2}\,\|x\|_{\varphi_u} \quad \text{for } x \in A.$$

Indeed, it is clear that $\|\varphi_u\| \leq 1$. Since $\varphi_u(u) = 1$, necessarily $\|\varphi_u\| = 1$ and $s(\varphi) \leq u$. Moreover, for $x \in A$ we have

$$\|x\|_{\varphi_u}^2 = \varphi_u\{x, x, u\} = \tfrac{1}{2}\varphi_u(xx^*u + ux^*x) = \tfrac{1}{2}(\langle xx^*u\xi, u\xi \rangle + \langle ux^*x\xi, u\xi \rangle)$$
$$= \tfrac{1}{2}(\|x^*u\xi\|^2 + \|x\xi\|^2) \geq \tfrac{1}{2}\|x\xi\|^2 = \tfrac{1}{2}\|Tx\|^2. \quad \blacksquare$$

We continue by recalling the example of [22] showing optimality of Theorem B and explaining that it shows optimality neither of Theorem C nor of Theorem 6.3.

An important tool to investigate optimality of constants in Theorem B is the following characterization.

PROPOSITION 7.6 ([43, Proposition 23.5]). *Let A be a C^*-algebra, H a Hilbert space, $T : A \to H$ a bounded linear map and K a positive number. Then the following two assertions are equivalent:*

(i) *There are states φ_1, φ_2 on A such that*

(16) $$\|Tx\| \leq K\|T\|(\varphi_1(x^*x) + \varphi_2(xx^*))^{1/2} \quad \text{for } x \in A.$$

(ii) *For any finite sequence (x_j) in A we have*

(17) $$\left(\sum_j \|Tx_j\|^2\right)^{1/2} \leq K\|T\|\left(\left\|\sum_j x_j^* x_j\right\| + \left\|\sum_j x_j x_j^*\right\|\right)^{1/2}.$$

The following proposition is a complete analogue of the preceding one and can be used to study optimality of Theorem 6.3. We have not found it explicitly formulated in the literature, but its proof is completely analogous to that of Proposition 7.6, given in [43].

PROPOSITION 7.7. *Let A be a unital JB^*-algebra, H a Hilbert space, $T : A \to H$ a bounded linear map and K a positive number. Then the following two assertions are equivalent:*

(i) *There is a state φ on A such that*

(18) $$\|Tx\| \leq K\|T\|\varphi(x^* \circ x)^{1/2} \quad \text{for } x \in A.$$

(ii) *For any finite sequence (x_j) in A we have*

(19) $$\Big(\sum_j \|Tx_j\|^2\Big)^{1/2} \leq K\|T\| \Big\|\sum_j x_j^* \circ x_j\Big\|^{1/2}.$$

We recall the example originated in [22] and formulated and proved in this setting in [43].

EXAMPLE 7.8 ([43, Lemma 11.2]). Consider an integer $n \geq 1$. Let $N = 2n+1$ and $d = \binom{2n+1}{n} = \binom{2n+1}{n+1}$. Let τ_d denote the normalized trace on the space M_d of $d \times d$ (complex) matrices. There are $x_1, \ldots, x_N$ in M_d such that $\tau_d(x_i^* x_j) = 1$ if $i = j$ and $= 0$ otherwise, satisfying

(20) $$\sum_j x_j^* x_j = \sum_j x_j x_j^* = NI$$

and moreover such that, with $a_n = (n+1)/(2n+1)$,

(21) $$\forall \alpha = (\alpha_i) \in \mathbb{C}^N : \Big\|\sum_j \alpha_j x_j\Big\|_{(M_d)^*} = d\sqrt{a_n}\Big(\sum_j |\alpha_j|^2\Big)^{1/2}. \quad \blacksquare$$

In the following example we show that the previous one yields the optimality of Theorem B but does not help to find the optimal constant for Theorem C or Theorem 6.3. The first part is proved already in [22] (cf. [43, Section 11]) but we include the proof for completeness and in order to compare it with the second part.

EXAMPLE 7.9. Fix $n \geq 1$. With the notation of Example 7.8 define $T : M_d \to \ell_2^N$ by
$$T(x) = (\tau_d(x_j^* x))_{j=1}^N, \quad x \in M_d.$$
Let $(\eta_j)_{j=1}^N$ be the canonical orthonormal basis of ℓ_2^N. Then the dual mapping $T^* : \ell_2^N \to M_d^*$ satisfies

$$\langle T^*(\eta_j), x \rangle = \langle \eta_j, T(x) \rangle = \tau_d(x_j^* x) = \frac{1}{d} \operatorname{tr}(x_j^* x) \quad \text{for } x \in M_d,$$

thus $T^*(\eta_j) = \frac{1}{d} x_j^*$ (we use the trace duality). Then (21) shows that

$$\|T^*(\alpha)\| = \frac{1}{d}\Big\|\sum_{j=1}^N \alpha_j x_j^*\Big\|_{(M_d)^*} = \sqrt{a_n}\,\|\alpha\| \quad \text{for } \alpha \in \ell_2^N.$$

In particular, $\frac{1}{\sqrt{a_n}}T^*$ is an isometric embedding, thus $\frac{1}{\sqrt{a_n}}T$ is a quotient mapping. Hence, $\|T\| = \sqrt{a_n}$.

Further, $T(x_j) = \eta_j$ for $j = 1, \ldots, N$, so

$$\sum_{j=1}^N \|T(x_j)\|^2 = N \quad \text{and} \quad \Big\|\sum_{j=1}^N x_j^* x_j\Big\| + \Big\|\sum_{j=1}^N x_j x_j^*\Big\| = 2\|NI\| = 2N.$$

Thus due to Proposition 7.6 the optimal value of the constant in Theorem B is bounded below by

$$\frac{1}{\sqrt{2a_n}} = \sqrt{\frac{2n+1}{2n+2}} \to 1.$$

On the other hand,

$$\left\| \sum_{j=1}^{N} x_j^* \circ x_j \right\| = \|NI\| = N,$$

thus Proposition 7.7 implies that the optimal value of the constant in Theorem C is bounded below by

$$\frac{1}{\sqrt{a_n}} = \sqrt{\frac{2n+1}{n+1}} \to \sqrt{2},$$

so it gives nothing better than Example 7.2.

In fact, this operator T satisfies Theorem C with constant $\frac{1}{\sqrt{a_n}} \leq \sqrt{2}$. To see this observe that $(x_j)_{j=1}^{N}$ is an orthonormal system in M_d equipped with the normalized Hilbert–Schmidt inner product. Hence, any $x \in M_d$ can be expressed as

$$x = y + \sum_{j=1}^{N} \alpha_j x_j,$$

where α_j are scalars and $y \in \{x_1, \dots, x_N\}^{\perp_{\text{HS}}}$. Then $T(x) = (\alpha_j)_{j=1}^{N}$ and

$$\tau_d(x^* \circ x) = \tau_d(x^*x) = \tau_d(y^*y) + \sum_{j=1}^{N} |\alpha_j|^2 \geq \sum_{j=1}^{N} |\alpha_j|^2 = \|T(x)\|^2.$$

Hence

$$\|T(x)\| \leq \tau_d(x^* \circ x)^{1/2} = \frac{1}{\sqrt{a_n}} \|T\| \tau_d(x^* \circ x)^{1/2}.$$

Since τ_d is a state, the proof is complete. ∎

We continue by an example showing that there is a real difference between the triple and algebraic versions of the Little Grothendieck Theorem.

EXAMPLE 7.10.

(a) Let M be any JBW*-triple and let $\varphi \in M_*$ be a norm-one functional. Then
$$|\varphi(x)| \leq \|x\|_\varphi \quad \text{for all } x \in M,$$
hence $\varphi : M \to \mathbb{C}$ satisfies Theorem D(3) with constant 1.

(b) Let M_2 be the algebra of 2×2 matrices. Then there is a norm-one functional $\varphi : M_2 \to \mathbb{C}$ not satisfying Theorem C with constant smaller than $\sqrt{2}$.

(c) In particular, the constant $\sqrt{2}$ in Lemma 3.3 is optimal.

Proof. (a) The desired inequality was already stated in [4, comments before Definition 3.1]. Let us give some details. We set $e = s(\varphi)$. Then

$$|\varphi(x)| = |\varphi(P_2(e)x)| = |\varphi(\{P_2(e)x, e, e\})| \le \|P_2(e)x\|_\varphi \|e\|_\varphi = \|P_2(e)x\|_\varphi.$$

Moreover,

$$\begin{aligned} \|x\|_\varphi^2 &= \varphi(\{x, x, e\}) = \varphi(P_2(e)\{x, x, e\}) \\ &= \varphi(\{P_2(e)x, P_2(e)x, e\} + \{P_1(e)x, P_1(e)x, e\}) \\ &= \|P_2(e)x\|_\varphi^2 + \|P_1(e)x\|_\varphi^2 \ge \|P_2(e)x\|_\varphi^2. \end{aligned}$$

(b) Each $a \in M_2$ can be represented as $a = (a_{ij})_{i,j=1,2}$. Define $\varphi : M_2 \to \mathbb{C}$ by the formula

$$\varphi(a) = a_{12}, \quad a \in M_2.$$

It is clear that $\|\varphi\| = 1$ and that $\varphi(s) = 1$ where

$$s = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let ψ be any state on M_2. Then

$$\|s\|_\psi^2 = \psi(\{s, s, \mathbf{1}\}) = \tfrac{1}{2}\psi(ss^* + s^*s) = \tfrac{1}{2}\psi(\mathbf{1}) = \tfrac{1}{2}.$$

Thus $\varphi(s) = \sqrt{2}\,\|s\|_\psi$ for any state ψ on $A = M_2$.

(c) This follows from (b) (consider $p = \mathbf{1}$). ∎

8. Notes and problems on general JB*-triples. The main result, Theorem 1.3, is formulated and proved for JB*-algebras. The assumption that we deal with a JB*-algebra, not with a general JB*-triple, was strongly used in the proof. Indeed, the key step was to prove the dual version for JBW*-algebras, Theorem 6.1, and we substantially used the existence of unitary elements. So, the following problem remains open.

QUESTION 8.1. *Is Theorem* 1.3 *valid for general JB*-triples?*

We do not know how to attack this question. However, there are some easy partial results. Moreover, some of our achievements may be easily extended to JBW*-triples. In this section we collect such results.

The first example shows that for some JB*-triples the optimal constant in the Little Grothendieck Theorem is indeed $\sqrt{2}$. This is proved by completely elementary methods.

EXAMPLE 8.2. Let H be a Hilbert space considered as the triple $B(\mathbb{C}, H)$ (i.e., a type 1 Cartan factor). That is, the triple product is given by

$$\{x, y, z\} = \tfrac{1}{2}(\langle x, y \rangle z + \langle z, y \rangle x), \quad x, y, z \in H.$$

The dual coincides with the predual and it is isometric to H. Let $y \in H^*$ be a norm-one element, i.e., we consider it as the functional $\langle \cdot, y \rangle$. Then $s(y) = y$.

So, for $x \in H$ we have
$$\|x\|_y^2 = \langle \{x,x,y\}, y \rangle = \tfrac{1}{2}\langle \langle x,x\rangle y + \langle y,x\rangle x, y\rangle = \tfrac{1}{2}(\|x\|^2 + |\langle x,y\rangle|^2)$$
$$\geq \tfrac{1}{2}\|x\|^2.$$
Hence, if K is another Hilbert space and $T : H \to K$ is a bounded linear operator, then for any norm-one $y \in H^*$ we have
$$\|Tx\| \leq \|T\|\,\|x\| \leq \sqrt{2}\,\|T\|\,\|x\|_y,$$
so we have the Little Grothendieck Theorem with constant $\sqrt{2}$.

Moreover, the constant $\sqrt{2}$ is optimal in this case as soon as $\dim H \geq 2$. Indeed, let $T : H \to H$ be the identity. Given any norm-one element $y \in H$, we may find a norm-one element $x \in H$ with $x \perp y$. The above computation shows that $\|x\| = \sqrt{2}\,\|x\|_y$. ∎

Another case, non-trivial but well known, is covered by the following example.

EXAMPLE 8.3. Assume that E is a finite-dimensional JB*-triple. Then E is reflexive and, moreover, any bounded linear operator $T : E \to H$ (where H is a Hilbert space) attains its norm. Hence E satisfies the Little Grothendieck Theorem with constant $\sqrt{2}$ by Theorem D(1). ∎

We continue by checking which methods used in the present paper easily work for general triples.

OBSERVATION 8.4. *Proposition 4.2 holds for corresponding JBW*-triples as well.*

Proof. It is clear that it is enough to prove this separately for finite JBW*-triples and for type I JBW*-triples. The case of finite JBW*-triples is trivial (one can take $\varphi_2 = 0$). So, let M be a JBW*-triple of type I, $\varphi \in M_*$ and $\varepsilon > 0$. Set $e = s(\varphi)$. Then $M_2(e)$ is a type I JBW*-algebra (see [8, comments on pp. 61–62 or Theorem 4.2]) and $\varphi|_{M_2(e)} \in M_2(e)_*$. Apply Proposition 4.2 to $M_2(e)$ and $\varphi|_{M_2(e)}$ to get φ_1 and φ_2. The pair of functionals $\varphi_1 \circ P_2(e)$ and $\varphi_2 \circ P_2(e)$ completes the proof. ∎

Observe that the validity of Proposition 4.2 for finite JBW*-triples is trivial but useless if we have no unitary element. However, the 'type I part' may be used at least in some cases.

PROPOSITION 8.5. *Let $M = L^\infty(\mu) \,\overline{\otimes}\, B(H,K)$, where H and K are infinite-dimensional Hilbert spaces. Then Proposition 3.4 holds for M.*

Proof. Let us start by showing that Peirce-2 subspaces of tripotents in M are upwards directed by inclusion. To this end first observe that $M = pV$, where V is a von Neumann algebra and $p \in V$ is a properly infinite projection. This is explained for example in [24, p. 43]. Now assume that $u_1, u_2 \in pV$ are two tripotents (i.e., partial isometries in V with final

projections below p). By [24, Lemma 9.8(c)] there are projections $q_1, q_2 \in V$ such that $q_j \geq p_i(u_j)$ and $q_j \sim p$ for $j = 1, 2$. Further, by [24, Lemma 9.8(a)] we have $q_1 \vee q_2 \sim p$, so there is a partial isometry $u \in V$ with $p_i(u) = q_1 \vee q_2$ and $p_f(u) = p$. Then $u \in pV = M$ and $M_2(u) \supset M_2(u_1) \cup M_2(u_2)$.

Now we proceed with the proof of the statement itself. Let $\varphi_1, \varphi_2 \in M_*$ and $\varepsilon > 0$. Note that M is of type I, hence we may apply Observation 8.4 to get the respective decomposition $\varphi_1 = \varphi_{11} + \varphi_{12}$. Let $u \in M$ be a tripotent such that $M_2(u)$ contains $s(\varphi_{11}), s(\varphi_{12}), s(\varphi_2)$. Such a u exists as Peirce-2 subspaces of tripotents in M are upwards directed by inclusion as explained above. We can find a unitary $v \in M_2(u)$ with $s(\varphi_{11}) \leq v$ (recall that $s(\varphi_{11})$ is a finite tripotent and use [23, Proposition 7.5]). We conclude by applying Lemma 3.3. ∎

Combine Proposition 8.5 with Theorem 3.1 to get the following:

COROLLARY 8.6. *Let* $M = L^\infty(\mu) \overline{\otimes} B(H, K)$, *where* H *and* K *are infinite-dimensional Hilbert spaces. Then Theorem* 6.1 *holds for* M.

We finish by pointing out main problems concerning JBW*-triples.

QUESTION 8.7. *Assume that* M *is a JBW*-triple of one of the following forms:*

- $M = L^\infty(\mu, C)$, *where* μ *is a probability measure and* C *is a finite-dimensional JB*-triple without unitary element.*
- $M = pV$, *where* V *is a von Neumann algebra and* p *is a purely infinite projection.*
- $M = pV$, *where* V *is a von Neumann algebra and* p *is a finite projection.*

Is Theorem 6.1 *valid for* M?

Note that these three cases correspond to the three cases distinguished in [25]. We conjecture that the second case may be proved by adapting the results of Section 5 (but we do not see an easy way) and that the third case is the most difficult one (much as in [25]).

REMARK 8.8. Haagerup applied in [21] ultrapower techniques to relax some of the extra hypotheses assumed by Pisier in the first approach to a Grothendieck inequality for C*-algebras. We should include a few words justifying that Haagerup's techniques are not effective in the setting of JB*-triples. Indeed, while a cluster point (in a reasonable sense) of states of a unital C*-algebra is a state, a cluster point of norm-one functionals may be even zero. This is true for weak (or weak*) limits and also for ultrapowers. The ultrapower, $E_{\mathcal{U}}$, of a JB*-triple, E, with respect to an ultrafilter $\mathcal{U}$ is again a JB*-triple with respect to the natural extension of the triple product (see [13, Corollary 10]), and E can be regarded as a JB*-subtriple of $E_{\mathcal{U}}$ via the inclusion of elements as constant sequences. Given a norm-one functional

$\tilde{\varphi} \in E_{\mathcal{U}}^*$ the restriction $\varphi = \tilde{\varphi}|_E$ belongs to E^*, however we cannot guarantee that $\|x\|_{\tilde{\varphi}} = \|[x]_{\mathcal{U}}\|_{\tilde{\varphi}}$ is bounded by a multiple of $\|x\|_{\varphi}$. Let us observe that both prehilbertian seminorms coincide on elements of E when the latter is a unital C*-algebra and $\tilde{\varphi}$ is a state on E.

References

[1] E. M. Alfsen and F. W. Shultz, *State Spaces of Operator Algebras. Basic Theory, Orientations, and C*-Products*, Math. Theory Appl., Birkhäuser Boston, Boston, MA, 2001.

[2] E. M. Alfsen and F. W. Shultz, *Geometry of State Spaces of Operator Algebras*, Math. Theory Appl., Birkhäuser Boston, Boston, MA, 2003.

[3] T. Barton and Y. Friedman, *Grothendieck's inequality for JB*-triples and applications*, J. London Math. Soc. (2) 36 (1987), 513–523.

[4] T. Barton and Y. Friedman, *Bounded derivations of JB*-triples*, Q. J. Math. Oxford Ser. (2) 41 (1990), 255–268.

[5] T. Barton and R. M. Timoney, *Weak*-continuity of Jordan triple products and its applications*, Math. Scand. 59 (1986), 177–191.

[6] R. Braun, W. Kaup and H. Upmeier, *A holomorphic characterization of Jordan C*-algebras*, Math. Z. 161 (1978), 277–290.

[7] L. J. Bunce, *Norm preserving extensions in JBW*-triple preduals*, Q. J. Math. 52 (2001), 133–136.

[8] L. J. Bunce and A. M. Peralta, *Images of contractive projections on operator algebras*, J. Math. Anal. Appl. 272 (2002), 55–66.

[9] M. Cabrera García and Á. Rodríguez Palacios, *Non-Associative Normed Algebras. Vol. 1. The Vidav–Palmer and Gelfand–Naimark Theorems*, Encyclopedia Math. Appl. 154, Cambridge Univ. Press, Cambridge, 2014.

[10] M. Cabrera García and Á. Rodríguez Palacios, *Non-Associative Normed Algebras. Vol. 2. Representation Theory and the Zel'manov Approach*, Encyclopedia Math. Appl. 167, Cambridge Univ. Press, Cambridge, 2018.

[11] H. Choda, Y. Kijima and Y. Nakagami, *Some extremal properties in the unit ball of von Neumann algebras*, Kōdai Math. Sem. Rep. 21 (1969), 175–181.

[12] C.-H. Chu, *Jordan Structures in Geometry and Analysis*, Cambridge Tracts in Math. 190, Cambridge Univ. Press, Cambridge, 2012.

[13] S. Dineen, *Complete holomorphic vector fields on the second dual of a Banach space*, Math. Scand. 59 (1986), 131–142.

[14] C. M. Edwards and G. T. Rüttimann, *Orthogonal faces of the unit ball in a Banach space*, Atti Sem. Mat. Fis. Univ. Modena 49 (2001), 473–493.

[15] M. Fabian, P. Habala, P. Hájek, V. Montesinos and V. Zizler, *Banach Space Theory: The Basis for Linear and Nonlinear Analysis*, CMS Books Math./Ouvrages Math. SMC, Springer, New York, 2011.

[16] Y. Friedman and B. Russo, *Structure of the predual of a JBW*-triple*, J. Reine Angew. Math. 356 (1985), 67–89.

[17] Y. Friedman and B. Russo, *The Gel'fand–Naĭmark theorem for JB*-triples*, Duke Math. J. 53 (1986), 139–148.

[18] Y. Friedman and B. Russo, *Conditional expectation and bicontractive projections on Jordan C*-algebras and their generalizations*, Math. Z. 194 (1987), 227–236.

[19] I. Gohberg, S. Goldberg and M. A. Kaashoek, *Classes of Linear Operators. Vol. I*, Operator Theory Adv. Appl. 49, Birkhäuser, Basel, 1990.

[20] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. São Paulo 8 (1953), 1–79.

[21] U. Haagerup, *The Grothendieck inequality for bilinear forms on C*-algebras*, Adv. Math. 56 (1985), 93–116.

[22] U. Haagerup and T. Itoh, *Grothendieck type norms for bilinear forms on C*-algebras*, J. Operator Theory 34 (1995), 263–283.

[23] J. Hamhalter, O. F. K. Kalenda and A. M. Peralta, *Finite tripotents and finite JBW*-triples*, J. Math. Anal. Appl. 490 (2020), no. 1, art. 124217, 65 pp.

[24] J. Hamhalter, O. F. K. Kalenda, A. M. Peralta and H. Pfitzner, *Measures of weak non-compactness in preduals of von Neumann algebras and JBW*-triples*, J. Funct. Anal. 278 (2020), no. 1, art. 108300, 69 pp.

[25] J. Hamhalter, O. F. K. Kalenda, A. M. Peralta and H. Pfitzner, *Grothendieck's inequalities for JB*-triples: proof of the Barton–Friedman conjecture*, Trans. Amer. Math. Soc. 374 (2021), 1327–1350.

[26] H. Hanche-Olsen and E. Størmer, *Jordan Operator Algebras*, Monogr. Stud. Math. 21, Pitman, Boston, MA, 1984.

[27] L. A. Harris, *Bounded symmetric homogeneous domains in infinite dimensional spaces*, in: Proc. on Infinite Dimensional Holomorphy (Lexington, KY, 1973), Lecture Notes in Math. 364, Springer, Berlin, 1974, 13–40.

[28] G. Horn, *Characterization of the predual and ideal structure of a JBW*-triple*, Math. Scand. 61 (1987), 117–133.

[29] G. Horn, *Classification of JBW*-triples of type I*, Math. Z. 196 (1987), 271–291.

[30] G. Horn and E. Neher, *Classification of continuous JBW*-triples*, Trans. Amer. Math. Soc. 306 (1988), 553–578.

[31] W. Kaup, *Über die Klassifikation der symmetrischen hermiteschen Mannigfaltigkeiten unendlicher Dimension. I*, Math. Ann. 257 (1981), 463–486.

[32] W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z. 183 (1983), 503–529.

[33] W. Kaup, *On real Cartan factors*, Manuscripta Math. 92 (1997), 191–222.

[34] W. Kaup and H. Upmeier, *Jordan algebras and symmetric Siegel domains in Banach spaces*, Math. Z. 157 (1977), 179–200.

[35] H. H. Kuo, *Gaussian Measures in Banach Spaces*, Lecture Notes in Math. 463, Springer, Berlin, 1975.

[36] O. Loos, *Bounded symmetric domains and Jordan pairs*, lecture notes, Univ. of California at Irvine, 1977.

[37] J. Lukeš, J. Malý and L. Zajíček, *Fine Topology Methods in Real Analysis and Potential Theory*, Lecture Notes in Math. 1189, Springer, Berlin, 1986.

[38] A. A. Milyutin, *Isomorphism of the spaces of continuous functions over compact sets of the cardinality of the continuum*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. 2 (1966), 150–156 (in Russian).

[39] A. M. Peralta, *Little Grothendieck's theorem for real JB*-triples*, Math. Z. 237 (2001), 531–545.

[40] A. M. Peralta and A. Rodríguez Palacios, *Grothendieck's inequalities for real and complex JBW*-triples*, Proc. London Math. Soc. 83 (2001), 605–625.

[41] H. Pfitzner, *Perturbation of l^1-copies and measure convergence in preduals of von Neumann algebras*, J. Operator Theory 47 (2002), 145–167.

[42] G. Pisier, *Grothendieck's theorem for noncommutative C*-algebras, with an appendix on Grothendieck's constants*, J. Funct. Anal. 29 (1978), 397–415.

[43] G. Pisier, *Grothendieck's theorem, past and present*, Bull. Amer. Math. Soc. (N.S.) 49 (2012), 237–323.

[44] R. A. Poliquin and V. E. Zizler, *Optimization of convex functions on w^*-compact sets*, Manuscripta Math. 68 (1990), 249–270.

[45] R. A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer Monogr. Math., Springer, London, 2002.

[46] Ş. Strătilă and L. Zsidó, *Lectures on von Neumann Algebras*, Editura Academiei, Bucureşti, and Abacus Press, Tunbridge Wells, 1979.

[47] M. Takesaki, *Theory of Operator Algebras. I*, Springer, New York, 1979.

[48] D. M. Topping, *Jordan algebras of self-adjoint operators*, Mem. Amer. Math. Soc. 53 (1965), 48 pp.

[49] J. D. M. Wright, *Jordan C*-algebras*, Michigan Math. J. 24 (1977), 291–302.

[50] J. D. M. Wright and M. A. Youngson, *A Russo–Dye theorem for Jordan C*-algebras*, in: Functional Analysis: Surveys and Recent Results (Paderborn, 1976), North-Holland Math. Stud. 27; Notas Mat. 63, North-Holland, Amsterdam, 1977, 279–282.

[51] M. A. Youngson, *A Vidav theorem for Banach Jordan algebras*, Math. Proc. Cambridge Philos. Soc. 84 (1978), 263–272.

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