Weak Stegall spaces
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Remark on references. The unspecified references, and the spaces \( K_B \) as well, are from the paper O.Kalenda, Stegall compact spaces which are not fragmentable, Topol. Appl. 96 (1999), no.2, 121–132.

**Proposition W1.** Let \( X \) be a topological space. Then the following assertions are equivalent.

(i) Any minimal usco mapping of any complete metric space \( M \) into \( X \) is singlevalued at least at one point of \( M \).

(ii) Any minimal usco mapping of any complete metric space \( M \) into \( X \) is singlevalued at points of a dense subset of \( M \).

(iii) Any minimal usco mapping of any complete metric space \( M \) into \( X \) is singlevalued at points of a second category subset of \( M \).

(iv) Any minimal usco mapping of any complete metric space \( M \) into \( X \) is singlevalued at points of a dense Baire subspace of \( M \).

**Proof.** The implications \((iv) \Rightarrow (iii) \Rightarrow (i) \) and \((iv) \Rightarrow (ii) \Rightarrow (i) \) are obvious. It remains to prove \((i) \Rightarrow (iv) \). Let \( M \) be a complete metric space, \( \varphi : M \to X \) a minimal usco mapping such that \( A = \{m \in M \mid \varphi(x) \text{ is not a singleton}\} \) is not a dense Baire subspace of \( M \). Then there is \( U \subset M \) nonempty open such that \( U \cap A \) is meager in \( U \), and a dense \( G_\delta \) subset \( G \) of \( U \) such that \( G \cap A = \emptyset \). We apply twice Lemma 2 to get that \( \varphi \upharpoonright G \) is a minimal usco mapping. Moreover, \( G \) is completely metrizable, and \( \varphi \upharpoonright G \) is not singlevalued at any point of \( G \), which completes the proof. \( \square \)

A space \( X \) satisfying one of the equivalent conditions of the above proposition we will call a **weakly Stegall** space, or we will write \( X \in w-S \).

**Proposition W2.** (a) Let \( X \in w-S \) and \( f : Y \to X \) be continuous one-to-one. Then \( Y \in w-S \).

(b) If \( X = \bigcup_{n \in \mathbb{N}} X_n \) with each \( X_n \) closed in \( X \), and if \( X_n \in w-S \) for every \( n \), then \( X \in w-S \).

(c) If \( X \in w-S \) and \( Y \) is a perfect image of \( X \) then \( Y \in w-S \). In particular, continuous image of a compact space lying in \( w-S \) lies in \( w-S \) too.

(d) If \( X \in w-S \) and \( Y \in S \) then \( X \times Y \in w-S \).

**Proof.** (a) If \( M \) is a complete metric space and \( \varphi : M \to Y \) is a minimal usco, then, by Lemma 1, \( f \circ \varphi \) is also a minimal usco. Since \( X \in w-S \), there is \( m \in M \) such that \( f(\varphi(m)) \) is a singleton. Now, since \( f \) is one-to-one, \( \varphi(m) \) is a singleton too.

(b) Let \( M \) be a complete metric space and \( \varphi : M \to X \) a minimal usco. Put \( M_n = \varphi^{-1}(X_n) \). Then \( M_n \) is a sequence of closed sets covering \( M \), hence there is some \( n \) such that \( M_n \) has nonempty interior in \( M \). Let \( U \subset M_n \) be nonempty open. By Lemma 1(c) we get \( \varphi(U) \subset X_n \). By Lemma 2 the restriction \( \varphi \upharpoonright U \) is minimal usco. Since \( X_n \in w-S \), there is \( m \in U \) such that \( \varphi(m) \) is a singleton.

(c) Let \( f : X \to Y \) be a perfect mapping of \( X \) onto \( Y \). Then \( f^{-1} \) is an usco mapping. Let \( \varphi : M \to Y \) be a minimal usco, where \( M \) is a complete metric space. Then \( f^{-1} \circ \varphi \) is usco. Let \( \psi \subset f^{-1} \circ \varphi \) be a minimal usco. Then there is \( m \in M \) such that \( \psi(m) \) is a singleton. Clearly we have \( f \circ \psi \subset \varphi \), hence, by minimality of \( \varphi \), \( f \circ \psi = \varphi \). Therefore \( \varphi(m) = f(\psi(m)) \) is a singleton.

(d) Let $M$ be a complete metric space and $\varphi : M \to X \times Y$ be a minimal usco. Then $\pi_X \circ \varphi$ is a minimal usco $M \to X$, so there is $A \subset M$ of second category such that $\pi_X \circ \varphi$ is singlevalued at all points of $A$. Similarly $\pi_Y \circ \varphi$ is singlevalued at points of a residual set $B \subset M$ (since $Y \in S$). Then $\varphi$ is singlevalued at points of $A \cap B$, which is a nonempty set. □

**Lemma W1.** Let $M$ be a complete metric space and $f : M \to X$ a continuous map such that for every $U \subset M$ open $f(U)$ has no isolated points. Then there is a nonempty compact perfect set $P \subset M$ such that $f \mid P$ is one-to-one.

**Proof.** Let $\rho$ be a complete metric on $M$ such that $\rho \leq 1$. We can construct by induction nonempty open sets $U_s \subset M$ indexed by finite sequences of 0 and 1 satisfying

(i) $\overline{\bigcup_{s \in 0} U_{s \uparrow 1}} \subset U_s$,

(ii) $f(\overline{U_{s \uparrow 0}}) \cap f(\overline{U_{s \uparrow 1}}) = \emptyset$,

(iii) $\text{diam} U_s \leq 2^{-|s|}$.

Put $U_\emptyset = M$. If we have constructed $U_s$ then by the assumption on $f$ we get that $f(U_s)$ has no isolated points and hence we can choose two distinct points $x_0, x_1 \in f(U_s)$. Choose $V_0, V_1$ two disjoint open neighborhoods of $x_0, x_1$ and $U_{s \uparrow i}$ of sufficiently small diameter such that $\overline{U_{s \uparrow i}} \subset U_s \cap f^{-1}(V_i)$ for $i = 0, 1$. This completes the construction.

Now put $K = \bigcup_{\alpha \in 2^\omega \cap \mathbb{N}} U_{\alpha \uparrow n}$. Then $K$ is a compact perfect set and $f \mid K$ is one-to-one by the construction. □

**Remark.** By a similar method one can prove that whenever $M$ is Čech complete and $f : M \to X$ is in the lemma, there is a compact set $K \subset M$ such that $f(K)$ is uncountable.

**Proposition W3.** Let $K \subset \mathbb{R}$ be a compact perfect set, $B \subset K^d$ arbitrary. Then $K_B \in w$-$S$ if and only of $B$ does not contain any perfect subset.

**Proof.** Let $F : K_B \to K$ be the natural surjection. If $B$ contains a perfect set $P$ then $F^{-1} : P^d \to K_B$ is, by Proposition 6(6), a minimal usco. Moreover, $P^d$ is completely metrizable and $F^{-1}$ is not singlevalued at any point of $P^d$.

Now suppose that $B$ contains no perfect set. Let $M$ be a complete metric space and $\varphi : M \to K_B$ a minimal usco, nowhere singlevalued. By Proposition 6(5) there is $G \subset M$ dense $G_\delta$ such that for $m \in G$ we have $\varphi(m) \subset \{x\} \times \{0, 1\}$ for some $x \in K$. So $\varphi \mid G$ is a minimal usco (Lemma 2) which is exactly 2-valued. By Proposition 6(6) we get that $F \circ \varphi : G \to B$ satisfies the assumptions of Lemma W1. Hence $B$ contains a perfect set, a contradiction. □

**Lemma W2.** Let $\varphi_a : M_a \to X_a$ be an usco mapping for each $a \in A$. Put $M = \prod_{a \in A} M_a$, $X = \prod_{a \in A} X_a$ and let $\varphi : M \to X$ be defined by the formula $\varphi((m_a)_{a \in A}) = \prod_{a \in A} \varphi_a(m_a)$. Then $\varphi$ is an usco mapping. Moreover, if each $\varphi_a$ is minimal so is $\varphi$.

**Proof.** We denote by $\pi_a$ the projection of $X$ (or $M$) onto the $a$-th coordinate. Similarly for any $F \subset A$ the projection onto $\prod_{a \in F} X_a$ (or $\prod_{a \in F} M_a$) is denoted by $\pi_F$.

Clearly the values of $\varphi$ are compact. Let $m \in M$ and $U \subset X$ be open with $\varphi(m) \subset U$. By the definition of the product topology we get for every $x \in \varphi(m)$ a finite set $F_x \subset A$ and an open set $V_x$ in $\prod_{a \in F_x} X_a$ such that $x \in \pi_{F_x}^{-1}(V_x) \subset U$.
By compactness of \( \varphi(m) \) there is \( H \subset \varphi(m) \) with \( \varphi(x) \subset \bigcup_{x \in H} \pi_{F_x}^{-1}(V_x) \subset U \). Put \( F = \bigcup_{x \in H} F_x \). Then there is an open set \( V \) in \( \prod_{a \in F} X_a \) such that \( \bigcup_{x \in H} \pi_{F_x}^{-1}(V_x) = \pi_F^{-1}(V) \). Hence \( \varphi(m) \subset \pi_F^{-1}(V) \subset U \). Now, if there is no neighborhood \( W \) of \( m \) with \( \varphi(W) \subset \pi_F^{-1}(V) \) then there is a net \( m^\tau \in M \) converging to \( m \) and \( x^\tau \in \varphi(M^\tau) \setminus \pi_F^{-1}(V) \). Since each \( \varphi_a \) is usco, there is a subnet of \( x_a^\tau \) converging to some point of \( \varphi_a(m_a) \). And since \( F \) is finite we can without loss of generality suppose that for each \( a \in F \) the net \( x_a^\tau \) converges to some \( x_a \in \varphi_a(m_a) \). So there is \( \tau_0 \) such that for \( \tau \geq \tau_0 \) we have \( (x_a^\tau)_{a \in F} \in V \), \( x^\tau \in \pi_F^{-1}(V) \), a contradiction. Hence \( \varphi \) is usco.

Next suppose that each \( \varphi_a \) is minimal. Let \( U \subset M \) and \( W \subset X \) be open with \( \varphi(U) \cap W \neq \emptyset \). Again by the definition of product topology there is \( F \subset A \) finite and open sets \( U_a \subset M_a \) and \( W_a \subset X_a \) such that \( \bigcap_{a \in F} \pi_a^{-1}(U_a) \subset U \),

\[
\bigcap_{a \in F} \pi_a^{-1}(W_a) \subset W \quad \text{and} \quad \varphi \left( \bigcap_{a \in F} \pi_a^{-1}(U_a) \right) \cap \left( \bigcap_{a \in F} \pi_a^{-1}(U_a) \right) \neq \emptyset.
\]

It follows, by definition of \( \varphi \), that \( \varphi_a(U_a) \cap W_a \neq \emptyset \) for every \( a \in F \). Since \( \varphi \) is minimal, by Lemma 1, we get a nonempty open \( V_a \subset U_a \) with \( \varphi_a(V_a) \subset W_a \). So \( \varphi \left( \bigcap_{a \in F} \pi_a^{-1}(V_a) \right) \subset \left( \bigcap_{a \in F} \pi_a^{-1}(U_a) \right) \), hence \( \varphi \) is minimal by Lemma 1. \( \square \)

**Example W1.** Let \( K = [0,1] \). There is \( B \subset (0,1) \) such that \( K_B \in w\cdot S \) but \( K_B \times K_B \notin w\cdot S \).

**Proof.** By [J.Oxtoby, Measure and category, Springer-Verlag 1971] there is \( D \subset \mathbb{R} \) such that neither \( D \) nor its complement contain a perfect compact set. Put \( B = \left( D \cap \left( 0, \frac{1}{2} \right) \right) \cup \left( \frac{1}{2} + \left( (0, \frac{1}{2}) \setminus D \right) \right) \). Then clearly \( B \) contains no perfect compact set, so by Proposition W3 we get that \( K_B \in w\cdot S \). We will show that the product \( K_B \times K_B \) contain a homeomorphic copy of \( K_{(0,1)} \) and hence it is not weakly Stegall (by Propositions W2 and W3). Let us define \( f : K_{(0,1)} \rightarrow K_B \times K_B \) by the formula \( f((t, \varepsilon)) = (f_1((t, \varepsilon)), f_2((t, \varepsilon))) \), where

\[
\begin{align*}
f_1((t, \varepsilon)) &= \begin{cases} \left( \frac{1}{2}, \varepsilon \right), & t \in B \\ \left( \frac{1}{2}, 0 \right), & t \notin B \end{cases}, \\
f_2((t, \varepsilon)) &= \begin{cases} \left( \frac{1}{2} + \varepsilon, 0 \right), & t \in B \\ \left( \frac{1}{2} + \frac{1}{2}, \varepsilon \right), & t \notin B \end{cases}.
\end{align*}
\]

It is easy to see (by Proposition 6(1)) that \( f_1 \) and \( f_2 \) are continuous, so \( f \) is countinuous too. And it follows easily from the definition of \( B \) that \( f \) is one-to-one. \( \square \)