Motivace pro slabé řešení

We have already seen that the Laplace equation models real world in many situations

$$-\Delta u = f \text{ in } \Omega,$$

and we learnt the basic properties of its classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ like explicit formulas in particular domains, maximum principles, uniqueness, regularity.

Let us revisit once again its derivation in case of heat equation.

[Evans 2.2] [Feynman, II, 12] Motivation 1

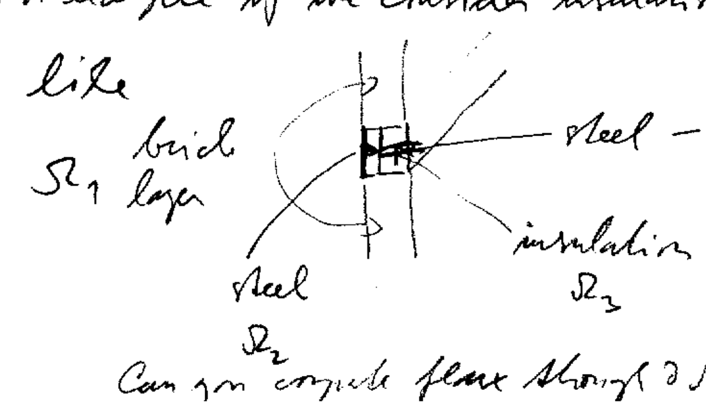
We assume that F $: \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is unknown heat flux density, f $: \Omega \to \mathbb{R}$ is density of heat sources. The principle of conservation of energy gives for any smooth $G \subset \Omega$:

$$\int_{\partial G} F \cdot \nu \, dS = \int_G f \qquad (*)$$

Since we are not interested in heat flux but in temperature u we need to know how to determine F as a function of u. This depends on material under consideration. Fourier's law of heat conduction sounds

$$F = -a \nabla u,$$

where a is thermal conductivity. It can be discontinuous function. For example if we consider insulation of a house the situation looks like



$$a = \begin{cases} a_1 & \text{in } \Omega_1 \\ a_2 & \Omega_2 \\ a_3 & \Omega_3. \end{cases}$$

Can you compute flux through $\partial\Omega$?

If a was smooth, e.g. $a=1$ in $\Omega_1 \cup \Omega_2 \cup \Omega_3$, and we would assume that the searched solution is smooth we would get

$$-\Delta u = f \text{ in } \Omega \quad (\text{compare UPDR}).$$

This is not possible in our case, if a is discontinuous instead we use co-area formula

to get from $(*)$: $\forall \varphi \in \mathcal{D}(\Omega): \int_\Omega a \nabla u \nabla \varphi = \int_\Omega f \varphi.$

(weak formulation of PDE). If everything smooth $-\operatorname{div}(a \nabla u) = f$

Advantages: • It is primary notion of solution - close to conservation laws basic principles of physics.

• It requires only mild assumptions on regularity of sols.
e.g. $a \in L^\infty(\Omega)$, $\nabla u \in L^2(\Omega)$, $\nabla \varphi \in L^2(\Omega)$, $f \in L^{\frac{2n}{n+2}}(\Omega)$
(will be shown later)

$$\frac{\frac{2n}{n-2}}{\frac{2n}{n-2}-1} = \frac{2n}{n+2}$$

• if Ω is a complex domain we do not know how to find an explicit formula for (a (classical) solution - it even need not exist, however it is relatively easy to show existence of the weak solution in correct setting and its uniqueness and other properties by means of functional analysis

Motivation 2: Calculus of variations

Let $L : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$, $\forall u \in C^1(\bar{\Omega})$ we define

$$F(u) = \int_a^b L(x, u(x), \nabla u(x))\,dx$$

and search for local minima or maxima of F in $X = \{u \in C^1(\bar\Omega),\ u = 0 \ldots\}$

$\boxed{\text{Def}}$ We say that $u_0 \in X$ is local minimum of F in X if $\exists \delta > 0$:

$$\forall u \in X,\ \|u - u_0\|_{C^1(\bar\Omega)} < \delta \Rightarrow F(u_0) \le F(u)$$

Lemma 1 (Necessary condition of minimum) Let $L \in C^1(\mathbb{R}^{2n+1})$ and $u_0 \in X$ be a local minimizer, $h \in \mathcal{D}(\Omega)$, $h \not\equiv 0$. Define $g(t) = F(u_0 + th)$.

Then $g'(0) = 0$.

Pf: $\left.\begin{array}{l}\bullet\ g \text{ has local minimum in } 0\\ \bullet\ g'(0) \text{ exists}\end{array}\right\} \Rightarrow g'(0) = 0$

Example: $L(x, u, p) := \frac{1}{2}|p|^2 - f(x)u$; $F(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - fu$

Compute $g'(0) = \lim_{t\to 0} \frac{1}{t} \int_\Omega \frac{1}{2}\left(|\nabla(u+th)|^2 - |\nabla u|^2\right) - f(u+th) + fu$

$= \lim_{t\to 0} \frac{1}{t} \int_\Omega \frac{2t}{2}\nabla u \nabla h + \frac{1}{2}t^2|\nabla h|^2 - tf(h) = \int_\Omega \nabla u \nabla h - fh \quad \forall h \in \mathcal{D}(\Omega)$

If everything smooth: $-\operatorname{div}(a\nabla u) = f$ in Ω weak formulation

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We will study weak formulation of the problem: [Evans 6.1.2]

$a^{ij}, b^i, c \in L^\infty(\Omega)$ $(i,j = 1,\ldots,n)$, $f \in L^2(\Omega)$:

$$-\operatorname{div}(a\nabla u) + b\nabla u + cu = f \text{ in } \Omega$$

i.e. $\forall \varphi \in \mathcal{D}(\Omega)$ $\displaystyle\int_\Omega a\nabla u \nabla\varphi + b\nabla u\,\varphi + cu\varphi = \int_\Omega f\varphi \qquad (+)$

We want that (+) can be extended so that we can set $\varphi = u$ to get

$$\int_\Omega a \nabla u \nabla u + b \nabla u \cdot u + c u^2 = \int_\Omega f u$$

The natural assumptions for u is that $\nabla u \in L^2(\Omega)$,

but what ∇u we mean? It will be explained in the next section.

Bědětrásta

Sobolevovy prostory - definice a základní vlastnosti

Motivace viz výše.

V celé kapitole bude $\Omega \subset \mathbb{R}^d$, $d \in \{2,3,\dots\}$ otevřená množina.

Repetition. UFA

Definition: Let $u \in L^1_{loc}(\Omega)$, $\alpha \in (\mathbb{N}_0)^d$ multi-index. A function $v \in L^1_{loc}(\Omega)$ is called the α^{th} weak derivative of u if

$$\forall \varphi \in \mathcal{D}(\Omega): \quad \int_\Omega \varphi v = \int_\Omega (-1)^{|\alpha|} u D^\alpha \varphi .$$

In this lecture all differential operators are understood in the weak sense.

Example. If $f(x) = |x|$, $g_i(x) = \frac{x_i}{|x|}$, then $\frac{\partial f}{\partial x_i} = g_i$, $i \in \{1,\dots,d\}$
(*)

Definition (Sobolev space). For $p \in [1,+\infty]$, $k \in \mathbb{N}$ we define Sobolev space

$$W^{k,p}(\Omega) = \{ u \in L^p(\Omega),\ \forall \alpha \in (\mathbb{N}_0)^d, |\alpha| \le k : D^\alpha u \in L^p(\Omega) \},$$

(obecně skalární či $\mathbb{C}$)

Sobolev norm

$$\|u\|_{k,p,\Omega} = \Big(\int_\Omega \sum_{|\alpha| \le k} |D^\alpha u|^p \Big)^{1/p} \qquad p \in [1,+\infty)$$

$$= \max_{|\alpha| \le k} \| D^\alpha u \|_\infty \qquad p = +\infty.$$

$u \in W^{k,p}_{loc}(\Omega)$ means, $\forall V \subset \overline{V} \subset \Omega$, V open holds $W^{k,p}(V) \ni u$.

• For $p = 2$ def $((u,\varphi))_{k,p} = \int_\Omega u\varphi + \dots + \nabla^k u : \nabla^k \varphi$

Remark: • Function in $W^{k,p}$ are determined up to a nullset. Similarly as in the case of Lebesgue spaces we factorize them according to the equivalence "equal a.e.".

• if we say $u \in W^{k,p}(\Omega)$ has some property (continuous, ...) we mean: there is a representation with this property.

• For $q \in [1,+\infty)$: $|||u|||_{k,p,\Omega} := \Big(\sum_{|\alpha| \le k} \| D^\alpha u \|^q_{p,\Omega} \Big)^{1/q}$ is an equivalent norm
$q = +\infty$ similar

Evans Section 5.2.2 Example 3 (2)

Example: $f(x) := |x|^{\alpha}$, $\alpha \in \mathbb{R}$

bodová derivace $\nabla f(x) = \alpha |x|^{\alpha-2} x = g(x)$ $\forall x \neq 0$ pro $\alpha \le 1$

kdy je to slabá? $\forall x \in \mathbb{R}^d$ pro $\alpha > 1$

Když platí $\forall \varphi \in \mathcal{D}(\mathbb{R}^d)$: $\displaystyle\int_{\mathbb{R}^d} |x|^{\alpha} \partial_{x_i} \varphi(x)\,dx = -\int_{\mathbb{R}^d} \alpha |x|^{\alpha-2} x_i \varphi(x)\,dx$

Použijeme integraci per partes na $\mathcal{O} = \mathcal{U}_R(0) \setminus \overline{\mathcal{U}_\rho(0)}$, R velké, ρ malé, $\operatorname{spt}\varphi \subset \mathcal{U}_R$

$$I_0 = \int_{\mathcal{O}} |x|^{\alpha} \partial_{x_i} \varphi(x)\,dx = -\int_{\mathcal{O}} \alpha |x|^{\alpha-2} x_i \varphi(x)\,dx + \int_{\partial \mathcal{U}_R(0)} |x|^{\alpha} \frac{x_i}{x} \varphi(x)\,dx$$

$$- \int_{\partial \mathcal{U}_\rho(0)} |x|^{\alpha} \frac{x_i}{x} \varphi(x)\,dx = I_1 + I_2 + I_3$$

$I_0 \longrightarrow \displaystyle\int_{\mathbb{R}^d} |x|^{\alpha} \partial_{x_i} \varphi(x)\,dx$ pokud $\alpha > -d$

$I_1 \longrightarrow -\displaystyle\int_{\mathbb{R}^d} \alpha |x|^{\alpha-2} x_i \varphi(x)\,dx$ pokud $\alpha > 1-d$

$I_2 = 0$ pro dost velké R

$I_3 \longrightarrow 0$ pro $\alpha + d > 0$ tj. $\alpha > -d$

Dohromady: $\displaystyle\int_{\mathbb{R}^d} |x|^{\alpha} \partial_{x_i} \varphi(x)\,dx = -\alpha \int_{\mathbb{R}^d} |x|^{\alpha-2} x_i \varphi(x)\,dx$

pro $\alpha > 1-d$

Proč. $g(x) := \alpha |x|^{\alpha-2} x_i$, $g \in L^1_{loc}(\mathbb{R}^d) \iff \alpha > 1-d$

"$\Leftarrow$" hir

"$\Rightarrow$" $\displaystyle\int_{\mathcal{U}_R(0)} |x|^{\alpha-2} |x_i|\,dx = \int_0^R \rho^{\alpha-2} \int_{\partial \mathcal{U}_\rho(0)} |x_i|\,dS(x)\,d\rho \ge \int_0^R \rho^{\alpha-2} \int_{\partial \mathcal{U}_\rho(0) \cap |x_i| > \rho} \rho\,dS(x)\,d\rho \overset{(?)}{\ge} c \int_0^R \rho^{\alpha-1+d-1}\,d\rho$

$\ge +\infty$ pro $\alpha - 1 + d - 1 \le -1$

Podobně $g_\alpha \in L^p(\mathbb{R}^d) \iff (\alpha-1)p > -d$ tj. $\alpha > 1-\frac{d}{p}$

so $f_\alpha \in W^{1,p}_{loc}(\mathbb{R}^d)$ if $\alpha > \max(1-\frac{d}{p}, 1-d) = 1-\frac{d}{p}$, $p \geq 1$

$\phi(x) := \frac{|x|^{2-d}}{(d-2)\gamma_d}$ fundamentální harmonická fce $d>2$

$2-d > 1-\frac{d}{p}$; $1 > d(1-\frac{1}{p})$; $1-d > -\frac{d}{p}$; $p < \frac{d}{d-1}$

spec. v lib. dimenzi $\phi \notin W^{1,2}_{loc}(\mathbb{R}^d)$

Př: Ať $\{q_k\}$ je hustá v $\mathbb{R}^d$, pak $\sum_{k=1}^{+\infty} 2^{-k}|x-q_k|^\alpha = f(x)$ splňuje

$f \in W^{1,p}_{loc}(\mathbb{R}^d)$ pro $\alpha > 1-\frac{d}{p}$, přitom

f is unbounded on any open subset of $\mathbb{R}^d$ if $\alpha < 0$

(both cases may happen if $0 > 1-\frac{d}{p}$, i.e., $p < d$)

Pozn: $\beta \le \alpha \iff \forall i \in \{1, \dots, d\}: \beta_i \le \alpha_i$

$\binom{\alpha}{\beta} = \prod_{i=1}^{d} \binom{\alpha_i}{\beta_i}$

Pozn: Podobné věty by šly formulovat pro prostory $W^{k,p}$. Přednáška: do 11.10.

<u>Examples</u>: 1) $f(x) = \operatorname{sgn} x \Rightarrow f \notin W^{1,p}(-1,1) \quad \forall p \ge 1$

$f' = \delta$!

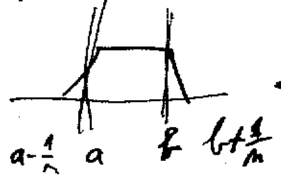
2) $f(x) = |x| \Rightarrow f \in W^{1,p}(-1,1) \quad \forall p \ge 1$, $f'(x) = \operatorname{sgn} x$

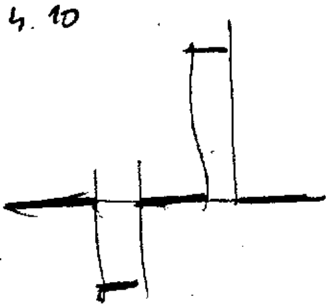
3) Recollection of AC functions – next page + recollection of mollification p. 4

F AC on $[a,b] \Rightarrow F \in W^{1,1}([a,b])$ Pick 14.4.9 Theorem 14.4.10

$F \in W^{1,1}([a,b]) \Rightarrow F$ AC on $[a,b]$

$\int F \varphi_n' = -\int F' \varphi_n$ Vol φ_n =





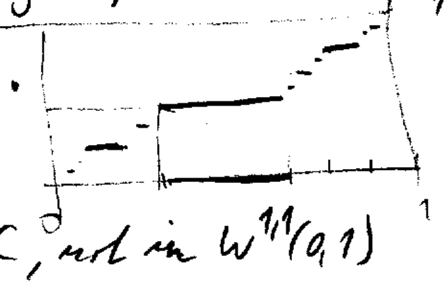
$\lim_{n \to +\infty} \left(\fint_{a-\frac{1}{n}}^{a} F - \fint_{b}^{b+\frac{1}{n}} F \right) = -\int_a^b F'$

Pick V 14.5.12

def. li $\tilde{F}(x) = \lim_{n \to +\infty} \fint_{x-\frac{1}{n}}^{x+1/n} F$ platí, že $\tilde{F} = F$ s.v. a $\tilde{F}$ je AC (je to integrál L^1 fce) Pick V 14.4.4

φ_n is not an allowed test function, mollify

$(\varphi_n)_\varepsilon \to \varphi_n$ a.e. and stay bd, limit passage by Lebesgue dominated theorem.



4) Cantor function u

is continuous function, but not AC, not in $W^{1,1}(0,1)$

even though $u' = 0$ s.v.

5) Example of func in $W^{1,p}$ in many D. [Ondráček]

Evans 5.2.3

Tvrzení 2 (Vlastnosti slabé derivace): Nechť $u, v \in W^{k,p}(\Omega)$, $k \in \mathbb{N}$, $p \in [1,+\infty]$ a $\alpha \in (\mathbb{N}_0)^d$, $|\alpha| \leq k$. Pak

i) $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ a $D^\alpha(D^\beta u) = D^\beta(D^\alpha u) = D^{\alpha+\beta} u$ pro $|\alpha|+|\beta| \leq k$

ii) $\lambda, \mu \in \mathbb{R} \Rightarrow \lambda u + \mu v \in W^{k,p}(\Omega)$ a $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$

iii) je-li $\tilde{\Omega} \subset \Omega$ otevřená, je $u \in W^{k,p}(\tilde{\Omega})$

iv) je-li $\zeta \in \mathcal{D}(\Omega)$, pak $\zeta u \in W^{k,p}(\Omega)$ a

$$D^\alpha(\zeta u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \zeta \, D^{\alpha-\beta} u .$$

Dk: i) Buď $\beta \in (\mathbb{N}_0)^d$, $|\beta| \leq k - |\alpha|$. Pak $\forall \varphi \in \mathcal{D}(\Omega)$

$$(-1)^{|\beta|} \int_\Omega D^\alpha u \, D^\beta \varphi = \int_\Omega (-1)^{|\alpha|+|\beta|} u \, D^{\alpha+\beta} \varphi = \int_\Omega D^{\alpha+\beta} u \, \varphi$$

a tedy $D^{\alpha+\beta} u = D^\beta(D^\alpha u) \underset{\text{podobně}}{=} D^\alpha(D^\beta u)$

Více $D^{\alpha+\beta} u \in L^p$ protože $|\alpha|+|\beta| \leq k$, tedy $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$.

ii) $$(-1)^{|\alpha|} \int_\Omega (\lambda u + \mu v) D^\alpha \varphi = (-1)^{|\alpha|} \lambda \int_\Omega u D^\alpha \varphi + (-1)^{|\alpha|} \mu \int_\Omega v D^\alpha \varphi = \int_\Omega (\lambda D^\alpha u + \mu D^\alpha v) \varphi \quad \perp$$

iii) V def. slabé derivace se omezíme na $\varphi \in \mathcal{D}(\tilde{\Omega})$, navíc když patří do L^p, patří i do L^p

Fix $j \in \{1,\dots,d\}$: $\forall i \neq j$

iv) je-li $\alpha_i = \beta_i$, $\alpha_j = \beta_j + 1$

Indukce: $$(-1)^{|\alpha|} \int_\Omega \zeta u \, D^\alpha \varphi = (-1)^{|\beta|+1} \int_\Omega D^\beta(\zeta u) \, \partial_{x_j} \varphi \overset{\text{ind}}{=} (-1)^1 \int_\Omega \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} D^\gamma \zeta \, D^{\beta-\gamma} u \, \partial_j \varphi =$$

$$\overset{\text{ind}}{=} \int_\Omega \partial_{x_j} \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} D^\gamma \zeta \, D^{\beta-\gamma} u \, \varphi = \int_\Omega \Big(\sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} D^\gamma \zeta \, D^{\alpha-\gamma} u \Big) \varphi$$

Krok 1: $\alpha = (0,\dots,\overset{j}{1},0,\dots)$: $$-\int_\Omega \zeta u \, \partial_{x_j} \varphi = -\int_\Omega u \, \partial_{x_j}(\zeta \varphi) + \int_\Omega u \, \partial_{x_j} \zeta \, \varphi = \int_\Omega (\partial_{x_j} u \, \zeta + u \, \partial_{x_j} \zeta) \varphi$$

$\perp$

Recall: Properties of convolution and approximative unit. (UFA IV,4)

Choose $h \in \mathcal{D}(\mathbb{R}^d)$, $\operatorname{spt} h \subset U(0,1)$, $\int_{\mathbb{R}^d} h = 1$, define

$$h^j(x) = j^d h(jx) \quad x \in \mathbb{R}^d$$

UFA V20: a) $f \in \mathcal{C}(\mathbb{R}^d)$: $f * h^j \overset{loc}{\rightrightarrows} f$ on $\mathbb{R}^d$

b) $f \in L^1_{loc}(\mathbb{R}^d)$: $f * h^j \to f$ in $L^1_{loc}(\mathbb{R}^d)$

c) $f \in L^p(\mathbb{R}^d)$ $f * h^j \to f$ in $L^p(\mathbb{R}^d)$ $p \in [1,+\infty)$

Def: For $u \in W^{k,p}(\Omega)$, we denote $u^j = u * h^j$.

Lemma 3 (Ziemer L 2.1.3): Let $u \in W^{k,p}(\Omega)$, $p \in [1,+\infty)$, then $\forall \alpha \in (\mathbb{N}_0)^d, |\alpha| \le k$:
$(D^\alpha u)^j = D^\alpha(u^j)$ & $u^j \to u$ in $W^{k,p}_{loc}(\Omega)$ (we extend u by 0 out of Ω)

If moreover $\Omega = \mathbb{R}^d$, then $u^j \to u$ in $W^{k,p}(\mathbb{R}^d)$

Pf: $\left.\begin{array}{l}\Omega' \subset \overline{\Omega'} \subset \Omega,\ \overline{\Omega'} \text{ compact} \\ \alpha \in (\mathbb{N}_0)^d,\ |\alpha| \le k\end{array}\right\}$: $D^\alpha(u^j) = \underbrace{D^\alpha u * h^j}_{=(D^\alpha u)^j}$ in Ω' because

$$\forall \varphi \in \mathcal{D}(\Omega'): (-1)^{|\alpha|}\int u^j D^\alpha \varphi = (-1)^{|\alpha|}\int_{\Omega'}\int_{\mathbb{R}^d} u(x-y)\, h^j(y)\, dy\, D^\alpha\varphi(x)\, dx =$$

$$= (-1)^{|\alpha|}\int_{\mathbb{R}^d} h^j(y)\left(\int_{\Omega'} u(x-y)\, D^\alpha\varphi(x)\, dx\right) dy = \int_{\mathbb{R}^d} h^j(y) \int_{\Omega'} (D^\alpha u)(x-y)\, \varphi(x)\, dx\, dy$$

$$= \int_{\Omega'} (D^\alpha u)^j(x)\, \varphi(x)\, dx.$$ It is needed, that $\operatorname{spt}\varphi + \operatorname{spt} h^j \subset \Omega$

$\Rightarrow D^\alpha(u^j) = (D^\alpha u)^j$ a.e. Ω' + smoothness $\Rightarrow$ claim

Consequently, we see $\forall \alpha \in (\mathbb{N}_0)^d, |\alpha| \le k$: $D^\alpha(u^j) \to D^\alpha u$ in $L^p_{loc}(\Omega)$

i.e. $u^j \to u$ in $W^{k,p}_{loc}(\Omega)$.

Clearly, if $\Omega = \mathbb{R}^d$ we can choose $\Omega' = \mathbb{R}^d$. $\perp$

<u>Absolutely continuous functions</u> (MA4) (např. Pick a spol.: Kap 14.4)

Reálná fce $f : [a,b] \to \mathbb{R}$ ($[a,b]$ bdd interval) se nazývá absolutně spojitá, pokud platí:

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall \{(x_i, y_i)\}_{i=1}^{n},\ (x_i, y_i) \text{ po dvou disjunktní intervaly } \subset [a,b]:$$

$$\sum_{i=1}^{n} |y_i - x_i| < \delta \Rightarrow \sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon$$

Ekvivalentní (Pick V14.4.9)

existuje f' vlastní s.v. v $[a,b]$, $f' \in L^1(a,b)$ a

$$f(x) - f(y) = \int_y^x f'(s)\,ds \quad \forall x, y \in [a,b],\ y < x$$

Arzela-Ascoli theorem (UFA nebo FA1)

$\boxed{\text{Theorem}}$ 4 Let $u \in L^p(\Omega)$. Then $u \in W^{1,p}(\Omega)$, $p \geq 1$ iff u has representative $\bar{u}$ that is absolutely continuous on $\mathcal{L}^{d-1}$ a.e. line segment in Ω parallel to the coordinate axes and whose (classical) partial derivatives belong to $L^p(\Omega)$.

(Ziemer Thm 2.1.4)

Dk: "$\Leftarrow$" Stačí ukázat, že $\left[\frac{\partial u}{\partial x_j}\right]$ [illegible] $(j \in \{1, \dots, d\})$

Fix $\varphi \in \mathcal{D}(\Omega)$, then φu is AC on a.e. line parallel with axe e_j

so a.e. $\tilde{x} \in \mathbb{R}^{d-1}$ $\quad 0 = \int_{\mathbb{R}} \partial_{x_j}(\varphi u)(\tilde{x}, x_j)\, dx_j = \int_{\mathbb{R}} (\partial_{x_j}\varphi \cdot u + \varphi\, \partial_{x_j} u)(\tilde{x}, x_j)\, dx_j$

Integrating over $\tilde{x} \in \mathbb{R}^{d-1}$ gives

$$\int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} [\partial_{x_j}\varphi] u + \varphi [\partial_{x_j} u] (\tilde{x}, x_j)\, dx_j \right) d\tilde{x} = 0$$

Since we know that $[\partial_{x_j}\varphi], u \in L^p(\Omega)$, $\varphi, \partial_{x_j}\varphi \in \mathcal{D}(\Omega)$, we can use Fubini theorem to get claim $[\partial_{x_j} u] = \partial_{x_j} u$ s.v. $\mathbb{R}^d$.

"$\Rightarrow$" Fix $R = \overbrace{[a_1, b_1] \times \dots \times [a_d, b_d]}^{Q} \subset \Omega$ (compact subset). Consider direction x_1.

We know $\displaystyle\int_R |u^j - u| + |\nabla(u^j - u)| \to 0 \quad j \to +\infty$

$\Downarrow$ Fubini

$$\int_Q \int_{a_1}^{b_1} |u^j - u| + |\nabla u^j - \nabla u| \to 0$$

$\Downarrow \leftarrow$ We need to choose subsequence. $\leftarrow$

(*) For a.e. $\tilde{x} \in Q$: $\displaystyle\int_{a_1}^{b_1} |u^j - u| + |\nabla u^j - \nabla u| \to 0$, $\exists\, \tilde{a}_1 \in [a_1, b_1]$: $u^j \to u$ in $\tilde{a}_1$

Fix such $\tilde{x} \in Q$, $u^j(t, \tilde{x})$ is AC on (a_1, b_1), so $\quad$ Henceforth [illegible] $\tilde{a}_1 = a_1$

$$|u^j(t_1, \tilde{x}) - u^j(t_2, \tilde{x})| = \left| \int_{t_2}^{t_1} \partial_1 u^j(s, \tilde{x})\, ds \right| \leq \int_{t_2}^{t_1} |\partial_1 u^j(s, \tilde{x})|\, ds$$

$$\forall \varepsilon > 0, \exists J \in \mathbb{N}, \forall j > J : \ |u^j(t_1, \tilde{x}) - u^j(t_2, \tilde{x})| \leq \int_{t_2}^{t_1} |\nabla u(s, \tilde{x})|\, ds + \varepsilon$$
$$\forall t_1, t_2 \in [a_1, b_1]$$

as $\int_{a_1}^{b_1} |\nabla u(t,\tilde{x})|\,dt < +\infty$, we obtain uniform boundedness of

$u^j(\cdot,\tilde{x})$ w.r.t. $j \geq J$; $j \in \mathbb{N}$

Similarly, $u^j(\cdot,\tilde{x})$ are absolutely continuous uniformly w.r.t. j.

[Indeed for $\varepsilon > 0$, find $J \in \mathbb{N}$ so that for $j > J$ $\int_{a_1}^{b_1} |\nabla u^j - \nabla u|(s,\tilde{x})\,ds < \varepsilon/2$

Since $\nabla u(\cdot,\tilde{x}) \in L^1(a_1,b_1)$ there is $\delta > 0$, $\forall E \subset (a_1,b_1)$, $|E| < \delta$: $\int_E |\nabla u(s,\tilde{x})| < \varepsilon/2$

Let $a_1 \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 < \dots \leq \alpha_L < \beta_L \leq b_1$ and $\sum_{\ell=1}^{L} |\alpha_\ell - \beta_\ell| < \delta$, then

$$\sum_{\ell=1}^{L} |u^j(\alpha_\ell,\tilde{x}) - u^j(\beta_\ell,\tilde{x})| \leq \sum_{\ell=1}^{L} \int_{\alpha_\ell}^{\beta_\ell} |\nabla u^j(s,\tilde{x})|\,ds \leq \int_{a_1}^{b_1} |\nabla u^j - \nabla u|(s,\tilde{x})\,ds +$$

$$\sum_{\ell=1}^{L} \int_{\alpha_\ell}^{\beta_\ell} |\nabla u| < \varepsilon$$
]

So $u^j(\cdot,\tilde{x})$ are uniformly continuous uniformly with respect to j.

Arzelà-Ascoli theorem $u^j(\cdot,\tilde{x}) \overset{\text{up to subseq.}}{\rightrightarrows} v$ on $[a_1,b_1]$, $v \in AC$ on $[a_1,b_1]$

$v = u(\cdot,\tilde{x})$ a.e. λ^1

[For $\varepsilon > 0$ find δ so that if $\sum |\alpha_\ell - \beta_\ell| < \delta$ then

$\forall j \in \mathbb{N}$: $\sum |u^j(\alpha_\ell,\tilde{x}) - u^j(\beta_\ell,\tilde{x})| < \frac{\varepsilon}{2}$

Fix $\alpha_\ell, \beta_\ell, L$. Find J so that $\|u^J(\cdot,\tilde{x}) - v\|_{L^\infty(a_1,b_1)} < \frac{\varepsilon}{20L}$

Then $\sum_{\ell=1}^{L} |v(\alpha_\ell) - v(\beta_\ell)| \leq \sum_{\ell=1}^{L} |v(\alpha_\ell) - u^J(\alpha_\ell)| + |v(\beta_\ell) \dots$

$+ \sum_{\ell=1}^{L} |u^J(\alpha_\ell) - u^J(\beta_\ell)| \leq 2L \frac{\varepsilon}{20L} + \frac{\varepsilon}{2} < \varepsilon$
]

Consider $\{R^k\}$ ~~sequence of all~~ rectangles in Ω with rational vertices. For R^1 choose subseq. $\{j^1_\ell\}$ such that (*) holds for a.e. $\tilde{x}$ in Q^1, denote E^1 the exceptional set $\lambda^{d-1}(E^1) = 0$.

Continue by induction and create sequence $\{j^\ell_\ell\}_\ell$; $E = \bigcup E^\ell$; $\lambda^{d-1}(E) = 0$

By previous considerations the sequence $u^{j^\ell_\ell}$ converge a.e. to the function u and moreover on ~~the~~ every segment parallel to x_1 axes such that its projection to $\{x_2, \dots, x_d\}$ plane does not belong to E converges uniformly to a AC function on this segment. We define $\bar{u} = \lim_{\ell \to +\infty} u^{j^\ell_\ell}$ in all points $\tilde{x}$ with $(x_2, \dots, x_d) \notin E$.

(7)

Corollary (Ziemer 2.1.11) Let $f:\mathbb{R}\to\mathbb{R}$ be a Lipschitz function and $u\in W^{1,p}(\Omega)$, $p\geq 1$. If $f\circ u\in L^p(\Omega)$, then $f\circ u\in W^{1,p}(\Omega)$ and for a.e. $x\in\Omega$

$$D(f\circ u)(x) = f'(u(x))\cdot Du(x).$$

Pf: Under additional assumptions $f\in C^1(\mathbb{R})$, $f'\in L^\infty(\mathbb{R})$ (G.T. Lemma 7.5)

We know $u^j \longrightarrow u$ in $W^{1,p}_{loc}(\Omega)$. Fix $\Omega'\subset\subset\Omega$.

It holds $\nabla(f\circ u^j) = f'\circ u^j\,\nabla u^j$. Fix $\varphi\in\mathcal{D}(\Omega')$.

$$-\int f\circ u\,\nabla\varphi = -\int (f\circ u - f\circ u^j)\nabla\varphi - \int f\circ u^j\nabla\varphi = -\int (f\circ u - f\circ u^j)\nabla\varphi +$$

$$\int f'\circ u^j\,\nabla u^j\cdot\varphi = -\int (f\circ u - f\circ u^j)\nabla\varphi + \int f'\circ u\,\nabla u\cdot\varphi + \int (f'\circ u^j\nabla u^j - f'\circ u\,\nabla u)\varphi$$

$$\int (f\circ u - f\circ u^j)\nabla\varphi \leq c\int_{\Omega'}|u-u^j| \longrightarrow 0 \text{ as } j\to+\infty$$

$$\left|\int (f'\circ u^j\,\nabla u^j - f'\circ u\,\nabla u)\varphi\right| \leq c\underbrace{\int_{\Omega'}|\nabla u^j - \nabla u|\,\varphi}_{\xrightarrow[j\to+\infty]{}0} + \int \underbrace{|f'\circ u^j - f'\circ u|}_{\substack{\text{a.e.}\to 0 \text{ up to subseq.}\\ \text{bdd}}}\,\underbrace{|\nabla u|}_{\in L^p}\,\varphi$$

$\xrightarrow[j\to+\infty]{} 0$ by Lebesgue's dom. thm

$$\Rightarrow \nabla(f\circ u) = f'\circ u\,\nabla u \in L^p(\Omega)$$

i.e. $f\circ u\in W^{1,p}(\Omega)$.

$\perp$

Definition: For a measurable function $u: \Omega \to \mathbb{R}$, let

$$u^+ = \max(u,0), \quad u^- = \min(u,0).$$

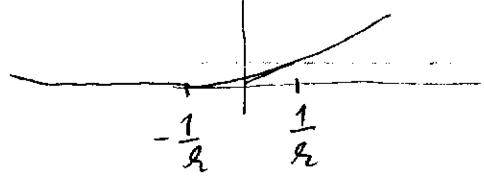
Corollary (Ziemer 2.1.8): Let $u \in W^{1,p}$, $p \geq 1$. Then $u^+, u^- \in W^{1,p}(\Omega)$ and

$$\left.\begin{aligned} Du^+ &= \begin{cases} Du & \text{if } u > 0 \\ 0 & u \leq 0 \end{cases} \\ Du^- &= \begin{cases} Du & u < 0 \\ 0 & u \geq 0 \end{cases} \end{aligned}\right\} \text{ a.e. } \Omega.$$

Pf: Immediate consequence of the previous Thm.

Independent proof: Find approximation of $\chi_{(0,+\infty)}(x)|x| = f(x)$

$f_k \ldots$



$; f_k \in C^\infty(\mathbb{R})$; $f_k \rightrightarrows f$ on $\mathbb{R}$

We know $\forall k \in \mathbb{N}$: $f_k(u) \in W^{1,p}(\Omega)$ Trosen p.3

$$f_\varepsilon(t) = \begin{cases} (t^2+\varepsilon^2)^{1/2} - \varepsilon & t > 0 \\ 0 & t \leq 0 \end{cases}, \quad f_\varepsilon \in C^1(\mathbb{R}) \; \& \; f_\varepsilon' \in L^\infty(\mathbb{R})$$

We know $-\int_{\Omega} [(u^2+\varepsilon^2)^{1/2} - \varepsilon] \nabla\varphi = \int_{\Omega} \frac{u \cdot \nabla u}{(u^2+\varepsilon^2)^{1/2}} \varphi$

$\{u>0\}$ $\quad \Big\downarrow \varepsilon \to 0+ \quad$ $\{u>0\}$ pointwise convergence + Lebesgue dominated thm

$$-\int_{\{u>0\}\cap\Omega} u \cdot \nabla\varphi = \int_{\Omega \cap \{u>0\}} \nabla u \cdot \varphi \qquad \perp$$

Change of variables for Sobolev functions

Theorem 7 (Ziemer 2.2.2) Let $T: \mathbb{R}^d \to \mathbb{R}^d$ be a bi-Lipschitzian mapping such that

$$T: \Omega' \to \Omega$$

$$\exists M > 0, \forall x, y \in \Omega, \forall x', y' \in \Omega': \quad \begin{array}{l} |T(x') - T(y')| \le M|x' - y'| \\ |T^{-1}(x) - T^{-1}(y)| \le M|x - y| \end{array}$$

If $u \in W^{1,p}(\Omega)$, $p \ge 1$, then $v = u \circ T \in W^{1,p}(V)$, $V = T^{-1}(\Omega)$ and

$$\left.\begin{array}{l} Du(T(x))\, dT(x,\xi) = Dv(x) \cdot \xi \quad \Big\} \text{ a.e. } x \in \Omega', \ \forall \xi \in \mathbb{R}^n \\ \text{note} \\ \nabla u(T(x)) \cdot \nabla T(x) \cdot \xi = \nabla v(x) \cdot \xi \end{array}\right\} (**)$$

Rem 8: V open, $u \in W^{1,p}(\Omega) \iff v \in W^{1,p}(V)$ and $\|u\|_{1,p,\Omega} \approx \|v\|_{1,p,V}$ constants depend on M

Pf: Assume additionally $T \in C^1$, $T^{-1} \in C^1$.

Consider set $\Omega_n := \{x \in \Omega,\ \operatorname{dist}(x, \partial\Omega) > \frac{1}{n}\} \cap U(0, n)$.

Def $u_\delta := u * \varrho^\delta$

We know: $u_\delta \to u$ in $W^{1,p}(\Omega_n)$ as $\varepsilon \to 0+$, $u_\varepsilon \in C^\infty(\overline{\Omega}_n)$, ε large

and assume $u_\delta \to u$ a.e. in Ω

Define $v_\delta = u_\delta \circ T$ on V_n, $v_\varepsilon \in C^1(V)$ and

$\boxed{\varepsilon = \delta}$ $\boxed{j \to +\infty}$

$$\nabla v_\varepsilon(x) = \nabla u_\varepsilon(T(x)) \cdot \nabla T(x) \qquad \forall x \in V_n \qquad (*)$$

$$(\partial_1 u_\varepsilon, \dots, \partial_n u_\varepsilon) \begin{pmatrix} \frac{\partial T_1}{\partial x_1} & \cdots & \frac{\partial T_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial T_n}{\partial x_1} & & \frac{\partial T_n}{\partial x_n} \end{pmatrix}$$

$$(\det \nabla T \det \nabla T^{-1} = 1 \le |\det \nabla T|\, c(M, n))$$

$$\Rightarrow \quad |\nabla v_\varepsilon(x)|^p \le |\nabla u_\varepsilon(T(x))|^p \cdot |\det \nabla T|\ c(n, M)$$

$$\int_V |\nabla v_\varepsilon(x)|^p \le c \int_V |\nabla u_\varepsilon(T(x))|^p |\det \nabla T(x)| = c \int_\Omega |\nabla u_\varepsilon|^p \le c(\|u\|^p_{1,p} + 1)$$

Similarly: $\displaystyle\int_V |v_\varepsilon|^p \le c \int_V |u_\varepsilon|^p \le c(\|u\|^p_{1,p} + 1)$

Similarly: $\forall_{\frac{1}{n} > \varepsilon, \varepsilon' > 0}$:

$$\|v_\varepsilon - v_{\varepsilon'}\|^p_{1,p} \le C \|u_\varepsilon - u_{\varepsilon'}\|^p_{1,p}$$

So $\{v_\varepsilon\}$ is Cauchy in $W^{1,p}(V_n)$. Denote v_n its limit point.

Diagonal argument gives us function v such that

$$v_\varepsilon \longrightarrow v \text{ in } W^{1,p}(V_n) \quad \forall n \in \mathbb{N}$$

With the estimate

$$\|v\|_{W^{1,p}(V_n)} \leq C(n, M)\, \|u\|_{W^{1,p}(\Omega)}.$$

Consequently: (Levi theorem) $\quad \|v\|_{W^{1,p}(V)} \leq C(M_1, M)\, \|u\|_{W^{1,p}(\Omega)}$.

Since $u_\varepsilon \longrightarrow u$ a.e. in Ω, we have $v = u \circ T$ a.e. in V.

(T is injective) $\qquad = \lim_{\varepsilon \to 0} u_\varepsilon \circ T$, denote $M = \{x,\ u_\varepsilon \to u \text{ not in } x\}$

$|M| = 0 \Rightarrow |T^{-1}(M)| = 0$

From (*) we get $\forall \varphi \in \mathcal{D}(V)$, $k \in \{1, \dots, d\}$

$$-\int_V v_\varepsilon \partial_k \varphi = \int_V \partial_k v_\varepsilon \varphi = \int_V \nabla u_\varepsilon(T(x))\, \partial_k T(x)\, \varphi(x)\, dx$$

$$\downarrow_{\varepsilon \to 0+}$$

$$-\int_V v\, \partial_k \varphi = \int_V \underbrace{\nabla u(T(x))\, \partial_k T(x)}_{\partial_k v(x)}\, \varphi(x)\, dx, \quad \text{i.e. } (\ast\ast)$$

$\perp$.

Věta 8 (Základní vlastnosti Sobolev. prostorů). Nechť $k \in \mathbb{N}$.

a) Je-li $p \in [1,+\infty]$, je $W^{k,p}(\Omega)$ Banachův prostor (úplný NLP) $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$

b) Je-li $p = 2$, je $W^{k,2}(\Omega)$ Hilbertův prostor $(W^{k,2}(\Omega), ((\cdot,\cdot))_{k,2})$

c) Je-li $p \in [1,+\infty)$, je $W^{k,p}(\Omega)$ separabilní.

d) Je-li $p \in (1,+\infty)$, je $W^{k,p}(\Omega)$ reflexivní.

Dk: a) NLP jasné, úplnost: Ať u_k ~~je~~ Cauchy v $W^{k,p}(\Omega)$. Pak $\forall \alpha \in (\mathbb{N}_0)^n, |\alpha| \le k$ je $D^\alpha u_k$ Cauchy v $L^p(\Omega)$ a tedy ex. v^α tak, že $D^\alpha u_k \to v^\alpha$ v $L^p(\Omega)$.

Je potřeba ukázat, že $D^\alpha v^0 = v^\alpha$. Z def. slabé derivace

$$\forall \varphi \in \mathcal{D}(\Omega): (-1)^{|\alpha|}\int_\Omega u_k D^\alpha \varphi = \int_\Omega D^\alpha u_k\, \varphi \quad \text{přechod } k \to +\infty$$

$$(-1)^{|\alpha|}\int_\Omega v^0 D^\alpha \varphi = \int_\Omega v^\alpha \varphi, \text{ tedy } v^\alpha = D^\alpha v^0.$$

b) Vlastnosti skalárního součinu jsou jasné.

Pozn: skalární součin $((\cdot,\cdot)): X \times X \to R$
1) lineární v 1. sl.
2) stejně symetrický
3) $\forall x \in X: (x,x) \ge 0$
$(x,x) = 0 \iff x = 0$

c) Definujme zobrazení $J: W^{k,p}(\Omega) \to (L^p(\Omega))^N$, N dost velké

$Ju = (u, \nabla u, \dots, \nabla^k u)$ $\qquad N = 1 + n + n^2 + \dots + n^k$

Tvrdíme: • $J(W^{k,p}(\Omega))$ je uzavřený podprostor prostoru $(L^p(\Omega))^N$.

(důkaz je obdobný a))

• $(L^p(\Omega))^N$ je separabilní pro $p \in [1,+\infty)$

$\Rightarrow J(W^{k,p}(\Omega))$ je separabilní $\Rightarrow W^{k,p}(\Omega)$ je separabilní.

d) J is isometric isomorphism and $(L^p(\Omega))^N$, $|||\cdot||| = \left(\sum \|\cdot\|_p^p\right)^{1/p}$ is reflexive (and uniformly convex). Consequently, $W^{k,p}$ is reflexive for $p \in (1,+\infty)$. (UFA V17) ~~uniformly convex for $p \in (1,+\infty)$~~.

Approximation of Sobolev functions by smooth functions

Remark: We already know that if $u \in W^{1,p}(\Omega)$ (extended by 0 outside of Ω), $p \in [1,+\infty)$, then $u^\delta = u * \eta^\delta \longrightarrow u$ in $W^{1,p}_{loc}(\Omega)$.

Clearly, $u^\delta \in C^\infty(\Omega)$. Natural question is, "Can one get approximation in $W^{1,p}(\Omega)$?"

Answer: "Yes!" First

Errors 257 (251, 253, 256, 260, 290)

Lemma 9 (Partition of unity ~~[illegible]~~) (Ziemer Lemma 2.3.1):

Let $E \subset \mathbb{R}^d$, $\mathcal{G}$ be a collection of open sets U such that $E \subset \bigcup_{U \in \mathcal{G}} U$.

Then there is family $\mathcal{F}$ of nonnegative functions $f \in \mathcal{D}(\mathbb{R}^d)$ such that $0 \le f \le 1$

and i) $\forall f \in \mathcal{F}, \exists U \in \mathcal{G}$: $\operatorname{spt} f \in U$

ii) $\forall K \subset E$ compact then $\operatorname{spt} f \cap K \neq \emptyset$ for only finitely many $f \in \mathcal{F}$

iii) $\sum_{f \in \mathcal{F}} f(x) = 1$ for every $x \in E$. (Finite sum)

iv) if E is compact then family $\mathcal{F}$ is finite

v) family $\mathcal{F}$ is at most countable

Pf: 1) Suppose E compact: $\exists N \in \mathbb{N}$: $E \in \bigcup_{i=1}^N U_i$, $U_i \in \mathcal{G}$

Claim $\exists K_i \subset U_i$, K_i compact : $E \subset \bigcup_{i=1}^N K_i$

It suffices to show that we can replace U_1 with $\tilde{U}_1^\varepsilon = \{x \in U_1, \operatorname{dist}(x, \partial U_1) > \varepsilon$

for suitable small $\varepsilon > 0$ so that $E \subset \tilde{U}_1^\varepsilon \cup \bigcup_{i=2}^N U_i$.

By contradiction: $\forall \varepsilon > 0, \exists x_\varepsilon \in E \setminus (\tilde{U}_1^\varepsilon \cup \bigcup_{i=1}^N U_i)$, since E compact we may assume $x_\varepsilon \to x_0 \in E \setminus \underbrace{\bigcup_{\varepsilon>0} \tilde{U}_1^\varepsilon}_{= U_1 \text{ (it is open)}} \cup \bigcup_{i=1}^N U_i = \emptyset$ ↯

$\Rightarrow \exists \varepsilon > 0$: $E \subset \bigcup_{i=1}^N U_i^\varepsilon$, set $K_i := \overline{U_i^\varepsilon} \cap K$ compact $\perp$

Define $g_i = (\chi_{K_i})^\varepsilon * \eta_i^\varepsilon \in \mathcal{D}(U_i)$, $g_i > 0$ on K_i. Then $g = \sum_{i=1}^N g_i > 0$ on some neighborhood of E (ε neigh). Take $(\chi_{B_{\varepsilon/2}(E)})^{\varepsilon/4} =: \varphi$ and define

$h = \varphi \cdot g + (1-\varphi) > 0$ on $\mathbb{R}^d$

We define: $\mathcal{F} = \{ f_i ; f_i = \frac{g_i}{h}, 1 \le i \le N \}$

2) E open

Define $E_i = E \cap \overline{\mathcal{U}(0,i)} \cap \{ x \in E ; \operatorname{dist}(x, \partial E) \ge \frac{1}{i} \}$ compact

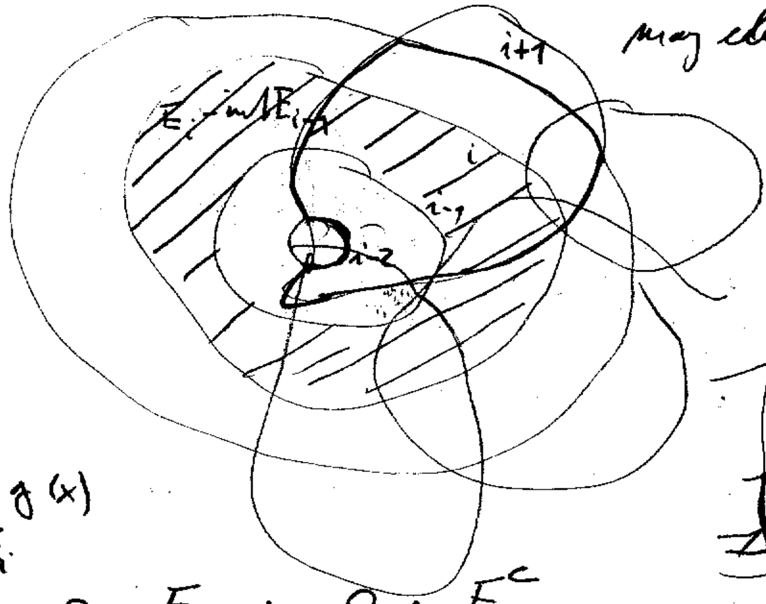
and $E = \bigcup_{i=1}^{+\infty} E_i$

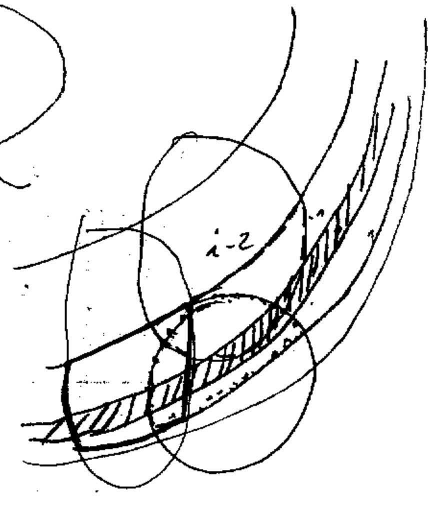
Let $\mathcal{G}_i$ be a collection of all open sets of the form $U \cap \{ \operatorname{int} E_{i+1} \setminus E_{i-2} \}$

where $U \in \mathcal{G}$ (Take $E_0, E_{-1} = \emptyset$)

clearly $\mathcal{G}_i$ ... open cover for $E_i - \operatorname{int} E_{i-1}$ ... partition of unity $\mathcal{F}_i$ with finitely many elements

~~Def~~ s





$$s(x) = \sum_{i=1}^{\infty} \sum_{g \in \mathcal{F}_i} g(x)$$

finite sum, $s > 0$ on E, $s = 0$ on E^c.

We define $\mathcal{F} = \left\{ f : f(x) = \begin{cases} \frac{g(x)}{s(x)} & \text{for some } g \in \mathcal{F}_i \text{ if } x \in E_i \\ 0 & x \notin E \end{cases} \right\}$

3) If E arbitrary apply step 2 to $\bigcup_{U \in \mathcal{G}} U$ that is open

$\perp$

Theorem 10 (Ziemer Thm 2.3.2) The set $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$. Actually, the set $W^{k,p}(\Omega) \cap \{u \in C^\infty(\Omega), \exists R>0 : \operatorname{spt} u \subset U_R(0)\}$ is dense in $W^{k,p}(\Omega)$.

Pf: Let $\Omega_i = \Omega \cap B(0,i) \cap \{x \in \mathbb{R}^d ; \operatorname{dist}(x, \partial\Omega) > \frac{1}{i}\}$.

Then $\{\Omega_i\}_{i=1}^{+\infty}$ is an open cover of Ω. Let $\mathcal{F}$ be a partition of unity subordinate to $\{\Omega_{i+1} \setminus \overline{\Omega_{i-1}}\}$, We set $\Omega_0, \Omega_{-1} = \emptyset$.

We assume that there is only 1 fnc ξ_i with $\operatorname{spt} \xi_i \subset \Omega_{i+1} \setminus \overline{\Omega_{i-1}}$ (otherwise add)

Fix $u \in W^{k,p}(\Omega)$ arbitrary, $\varepsilon > 0$.

Then $\forall i \in \mathbb{N}$; $\xi_i u \in W^{k,p}(\Omega)$, $\operatorname{spt}(\xi_i u) \subset \Omega_{i+1} \setminus \overline{\Omega_{i-1}}$.

Choose $\varepsilon_i > 0$, so that
- $\| h^{\varepsilon_i} * (\xi_i u) - \xi_i u \|_{W^{k,p}(\Omega)} \le \frac{\varepsilon}{2^{i+1}}$
- $\operatorname{spt} h^{\varepsilon} * (\xi_i u) \subset \Omega_{i+2} \setminus \overline{\Omega_{i-2}}$

It is possible by Lemma 3 (local approx.)

Define $v(x) = \sum_{i=1}^{+\infty} h^{\varepsilon_i} * (\xi_i u)(x)$ $\quad \forall x$ we sum only finitely many terms

$\forall U$ open, bdd $U \subset \overline{U} \subset \Omega$ —||—

$\Rightarrow v \in C^\infty(\Omega)$, $\|u - v\|_{k,p} = \cancel{\left[\| \sum_{i=1}^{\infty} \xi_i u - h^{\varepsilon_i} * (\xi_i u) \|_{k,p}^p = \right]}$

$$\cancel{\left[\left(\sum_{\substack{\alpha \in (\mathbb{N}_0)^d \\ |\alpha| \le k}} \int_\Omega \left| \sum_{i=1}^{+\infty} \xi_i u - h^{\varepsilon_i} * (\xi_i u) \right|^p \right) = \sum_{j=1}^{+\infty} \left(\sum_{\substack{\alpha \in (\mathbb{N}_0)^d \\ |\alpha| \le k}} \int_{\Omega_{j+1} \setminus \overline{\Omega_j}} \left| \sum_{i=1}^{+\infty} D(\xi_i u - h^{\varepsilon_i} * \xi_i u) \right|^p \right) \right]}$$

$$\|u - v\|_{W^{k,p}(\Omega_i)} \underset{\uparrow}{=} \left\| \sum_{j=1}^{N_i} \xi_j u - h^{\varepsilon_j} * (\xi_j u) \right\|_{W^{k,p}(\Omega_i)} \le \sum_{j=1}^{N_i} \| \quad \|_{W^{k,p}(\Omega_i)} \le$$

For N_i sufficiently large.

$$\sum_{j=1}^{\infty} \frac{\varepsilon}{2^{j+1}} < \varepsilon \qquad \text{Levi} \quad \|u - v\|_{W^{k,p}(\Omega)} < \varepsilon \Rightarrow v \in W^{k,p}(\Omega).$$

$\perp$.

Remark: The approximation functions can have jumps along $\partial\Omega$, they are not in $C^\infty(\overline{\Omega})$.

Set $\omega \in \mathcal{D}(\mathcal{U}(0,2))$, $\omega \equiv 1$ $\mathcal{U}(0,1)$, define $\omega_R(x) = \omega\left(\frac{x}{R}\right)$

$\forall \alpha \in (\mathbb{N}_0)^d$: $\quad \nabla^\alpha \omega_R(x) = \left|\frac{1}{R}\right|^{|\alpha|} (\nabla^\alpha \omega)\left(\frac{x}{R}\right)$

Fix $u \in W^{k,p}(\Omega)$, compute $\|u - u\cdot\omega_R\|_{k,p}^p = \sum\limits_{\substack{\alpha \\ |\alpha|\le k}} \int\limits_{\mathcal{U}(0,R)^c} |\nabla^\alpha(u - u\cdot\omega_R)|^p$

$$\le C \sum_{|\alpha|\le k} \int_{\mathcal{U}(0,R)^c} \sum_{\beta\le\alpha} \binom{\alpha}{\beta} |\nabla^\beta u \, \nabla^{\alpha-\beta}(1-\omega_R)|^p$$

$$\underset{R>1}{\le} c(d,p) \sum_{|\alpha|\le k} \int_{\mathcal{U}(0,R)^c \cap \Omega} \sum_{\beta\le\alpha} \binom{\alpha}{\beta} |\nabla^\beta u|^p \longrightarrow 0 \text{ by Lebesgue.}$$

Apply previous procedure to $u\cdot\omega_R$ with R large enough.

Pr: • Věta výše neplatí pro $p=+\infty$. ■ $u \in \begin{cases} 0 & x<0, \\ x & x\in[0,1], \\ 1 & x>1, \end{cases}$

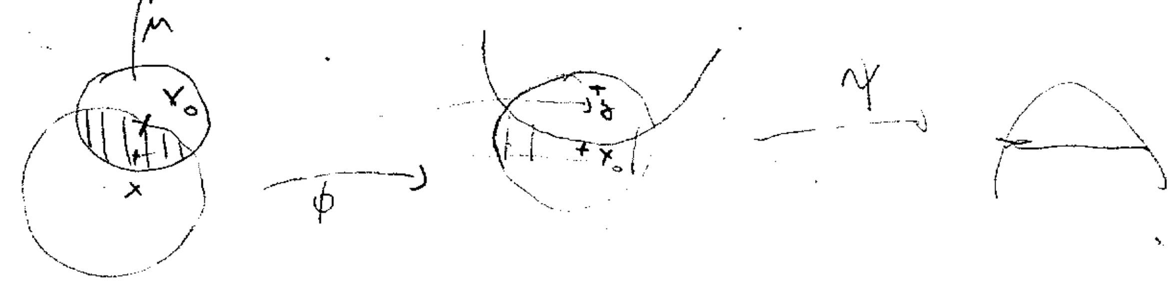
• $u' \in \chi_{(0,1)}$ nelze aprox. pomocí hladkých v L^∞ normě.

<u>Sets with smooth boundaries</u> – according Evans PDE's, Appendix C1

Definition: Let $\Omega \subset \mathbb{R}^d$ be open, bdd, $k \in \mathbb{N}_0$. We say that $\partial\Omega$ is C^k if

$\forall x_0 \in \partial\Omega$, $\exists r>0$, $\gamma: \mathbb{R}^{n-1} \to \mathbb{R}$, $\gamma \in C^k(\mathbb{R}^{n-1})$, $\phi: \mathbb{R}^n \to \mathbb{R}^n$ $\phi(x) = \cancel{x_0} + A(x-x_0)$ (Azotation (unitary matrix))

such that $\phi(\mathcal{U}(x_0,r) \cap \Omega) = \{ y \in \mathbb{R}^n : y_n > \gamma(y_1, \dots, y_{n-1}) \}$.
$\mathcal{U}(x_0,r)$



Flattening of the boundary near the point $x_0 \in \partial\Omega$.

Recall: Let $\Omega \subset \mathbb{R}^d$ (open, bdd) be with C^ℓ boundary, $\ell \in \mathbb{N}$, $x_0 \in \partial\Omega$. Take $r>0$ from previous definition and define $\Psi : y \in \mathbb{R}^d \to (y_1, \dots, y_{d-1}, y_d - f(y_1, \dots, y_{d-1}))$

We call $\psi \circ \phi$ flattening of the boundary in neighborhood of x_0.

It has following properties:
- $\psi \circ \phi \in C^k(\mathcal{U}(x_0, r))$
- $\det(\Psi \circ \Phi) = 1$ in $\mathcal{U}(x_0, r)$
- $\Psi \circ \Phi(\partial\Omega \cap \mathcal{U}(x_0, r)) = \partial(\psi \circ \phi(\partial\Omega \cap \mathcal{U}(x_0, r)) \cap \{y_d = 0\}$
- $\psi \circ \phi : \Omega \cap \mathcal{U}(x_0, r) \xrightarrow{1-1} \underbrace{\psi \circ \phi(\Omega \cap \mathcal{U}(x_0, r))}_{\text{open set}}$

- In this situation the outer normal is defined at any $x_0 \in \partial\Omega$, $n(x_0) = (+\partial_1 f, \dots, +\partial_{d-1} f, -1)$
- $\psi \circ \phi$ is bilipschitz mapping of $\Omega \cap \mathcal{U}(x_0, r)$ onto $\Psi \circ \Phi(\Omega \cap \mathcal{U}(x_0, r))$ so

$$u \in W^{1,p}(\Omega \cap \mathcal{U}(x_0, r)) \Leftrightarrow u \circ (\psi \circ \phi)^{-1} =: v \in W^{1,p}(\psi \circ \phi(\Omega \cap \mathcal{U}(x_0, r)))$$

$\exists C > 0$; $\quad \frac{1}{C}\|v\|_{1,p} \leq \|u\|_{1,p} \leq C\|v\|_{1,p}$.

Lemma 11 Let $u \in L^p(\mathbb{R}^d)$, $p \in [1, +\infty)$. For $h \in \mathbb{R}^d$, $h \neq 0$ define $u_h(x) := u(x+h)$ $\forall x \in \mathbb{R}^d$. Then $u_h \to u$ in $L^p(\mathbb{R}^d)$ as $h \to 0$.

Pf: Fix $\varepsilon > 0$. Find $v \in \mathcal{D}(\mathbb{R}^d)$ such that $\|u - v\|_p \leq \varepsilon$.

Then $\|u - u_h\|_p \leq \|u - v\|_p + \|v - v_h\|_p + \|v_h - u_h\|_p \leq 2\varepsilon + \|v - v_h\|_p$.

We know that there is a compact K such that $\forall h, |h| < 1$: $\operatorname{spt} v_h, \operatorname{spt} v \subset K$, moreover v is uniformly continuous on $\mathbb{R}^d$ so there is $\delta > 0$ s.t. $|v(x) - v_h(x)| < \frac{\varepsilon}{|K|}$ ($\forall h, |h| < \delta$) for $|h| < \min(\delta, 1)$

So $\|v_h - v\|_p^p \leq \int_{\mathbb{R}^d} |v(x) - v(x+h)|^p\,dx = \int_K |v(x) - v(x+h)|^p\,dx \leq \frac{\varepsilon}{|K|}|K| = \varepsilon$

Finally, for all $h, |h| < \delta$: $\|u - u_h\|_p < 3\delta$ $\quad \perp$

(+) **Lemma:** 12 Let $\varepsilon > 0$, $V := \mathcal{U}(0,R) \cap \{x_d > 0\}$, $u \in W^{k,p}(\{x_d > 0\})$ with supp $u \subset V$

Then there is a function $v \in C^{\infty}(\{x_d \geq 0\})$ such that supp $v \subset \mathcal{U}(0,2R) \cap \{x_d \geq 0\}$

and $\|v - u\|_{W^{k,p}(V)} < \varepsilon$.

Pf: We find $\lambda > 0$ and define $h = (0,0,\dots,0,-\lambda)$ so that $\|u - u_h\|_p < \varepsilon/2$ by Lemma 11. Let ω_k be approximation unit with supp $\omega_k \subset \mathcal{U}(0,\frac{1}{k})$

For $k > \max\left(\frac{1}{\lambda}, \frac{1}{R}\right)$: $u_h * \omega_k$ is well defined and we know that $u_h * \omega_k \xrightarrow{k \to +\infty} u_h$ in $W^{k,p}(\{x_d > 0\})$

There is $\bar{k}$ such that $\|u_h * \omega_{\bar{k}} - u_h\|_{W^{k,p}(\mathbb{R}^d_+)} \leq \varepsilon$. We define $v = u_h * \omega_{\bar{k}}$

Then $\|u - v\|_{k,p,\mathbb{R}^d_+} \leq \|u - u_h\|_{k,p,\mathbb{R}^d_+} + \|u_h - v\|_{k,p,\mathbb{R}^d_+} \leq \varepsilon$.

Since $\bar{k} > \frac{1}{R}$, clearly supp $v \subset \mathcal{U}(0,2R) \cap \{x_d \geq 0\}$

$\perp$

(+) Lemma

Theorem 13 (Approximation up to the boundary) [Evans, 5.3.3 Theorem 3]

Let $k \in \mathbb{N}$, $p \in [1, +\infty)$, $\Omega \subset \mathbb{R}^d$ be bdd with C^1 bdry. Then $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$.

We give proof for $k = 1$.

Pf: Since $\partial\Omega$ is C^1 there is $\mathcal{U}(x_0)$ for $\forall x_0 \in \partial\Omega$ from its def.* Clearly

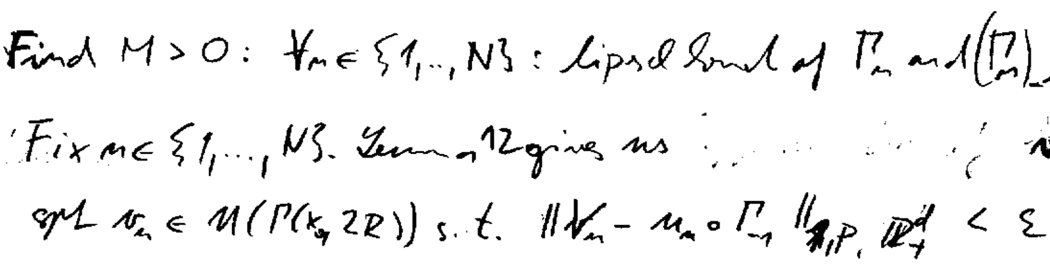
$\Omega \cup \bigcup_{x_0 \in \partial\Omega} V(x_0)$ is an open covering of $\overline{\Omega}$. Since $\overline{\Omega}$ is compact there is a finite subcovering $\Omega \cup \bigcup_{n=1}^{N} V(x_n)$. We denote $\{\xi_n\}_{n=0}^{N}$ partition of unity subordinate to this subcovering. Fix $u \in W^{k,p}(\Omega)$, $\varepsilon > 0$. Then

$u = \sum_{n=0}^{N} u\xi_n$ and denote $u\xi_n =: u_n$, it holds spt $u_n \subset \overline{\Omega} \cap \mathcal{U}(x_n)$, $n \in \{1, \dots, N\}$, spt $u_0 \subset\subset \Omega$

Case "0": By Lemma 3 there is $v_0 \in W^{k,p}(\Omega)$ such that $\|v_0 - u_0\|_{k,p} \le \varepsilon$

Case "> 0": $\forall n \in \{1, \dots, N\}$ there is a bilipschitz flattening of the boundary Γ_n

* For every x_0 there is a bilipschitz mapping Γ (flattening of the boundary) and $r > 0$: $\Gamma : \mathcal{U}(x_0, r) \to \Gamma(\mathcal{U}(x_0, r))$. There is $R > 0$ such that $\mathcal{U}(\Gamma(x_0), 2R) \subset \Gamma(\mathcal{U}(x_0, r))$. Denote $V(x_0) := \Gamma_{-1}(\mathcal{U}(\Gamma(x_0)), R)$



Find $M > 0$: $\forall n \in \{1, \dots, N\}$: lipschitz bound of Γ_n and $(\Gamma_n)_{-1}$ is bdd by M.

Fix $n \in \{1, \dots, N\}$. Lemma 12 gives us $v_n \in C^\infty(\{x_d \ge 0\})$ with spt $v_n \in \mathcal{U}(\Gamma(x_0), 2R)$ s.t. $\|V_n - u_n \circ \Gamma_{-1}\|_{k,p,\mathbb{R}^d_+} < \varepsilon$. Then $v_n := V_n \circ \Gamma$ satisfies

$\|v_n - u_n\|_{k,p,\Omega \cap \mathcal{U}(x_0, r)} \le C(M)\varepsilon$ or after extending 0: $\|v_n - u_n\|_{k,p,\Omega} \le C(M)\varepsilon$.

We define $v = \sum_{n=0}^{N} v_n$ and get $\|v - u\|_{k,p} \le \sum_{n=0}^{N} \|v_n - u_n\|_{k,p} \le \varepsilon + C(M)\varepsilon N$ $\perp$.

Remark: It is enough to assume that $\partial\Omega$ has so-called segment property [Adams, Theorem 3.18] p. 54. Much weaker assumption, works with arbitrary $k \in \mathbb{N}$

Extensions [Evans, Section 5.4, Thm 1]

Theorem: Assume $\Omega \subset \mathbb{R}^d$ open, bdd and $\partial\Omega$ is C^1. Fix a $V \subset \mathbb{R}^d$ open such that $\Omega \subset\subset V$. Then there is a bdd linear operator

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d) \text{ such that for all } u \in W^{1,p}(\Omega)$$

i) $Eu = u$ a.e. in Ω

ii) $\operatorname{spt} Eu \subset V$

iii) $\|E\| \leq C$, with $C = C(p, \Omega, V)$

Lemma: Let $u \in C^1(\overline{\mathbb{R}^d_+})$, $\operatorname{spt} u \subset U(0,R)$. There is $\tilde{E}u \in C^1(\mathbb{R}^d)$ such that $\tilde{E}$ linear; [illegible], $\|\tilde{E}\|_{\mathcal{L}(W^{1,p}(U(0,R)), U(0,2R))_{W^{1,p}}} \leq C(p,R)$

Pf:

We define $\tilde{E}u(x) = \begin{cases} u(x) & x \in U(0,R) \\ -3u(x_1, \dots, x_{d-1}, -x_d) + 4u(x_1, \dots, x_{d-1}, -\frac{x_d}{2}) & \end{cases}$

$\perp$

Proof of Thm: Find sets $U(x_0)$ $\forall x_0 \in \partial\Omega$, $U(x_0) \subset V$

$\forall x_0$ find $\Gamma : U(x_0) \longrightarrow \Gamma(U(x_0))$; find $R_{x_0} > 0$; $U(\Gamma(x_0), R_x) \subset \Gamma(U(x_0))$.

Denote $\Omega_x = \Gamma^{-1}(U(\Gamma(x_0)), R_{x_0})$, $\Omega \cup \bigcup_{x \in \partial\Omega} \Omega_x$ is open covering and there is finite subcovering, with $\{\eta_0, \eta_1, \dots, \eta_N\}$ subordinate partition of unity.

Set $u_0 = u$

Fix $u \in C^1(\overline{\Omega})$. And find extension V_m to $\eta_m u \circ (\Gamma_m)^{-1}$, $m \in \{1, \dots, N\}$ by Lemma.

Then $v_m := V_m \circ \Gamma_m$ is extension of $u\eta_m$ such that $\|v_m\|_{1,p,\Omega_m} \leq C(p, \Omega, V) \|u\eta_m\|_{1,p,\Omega}$

We define $Eu := \sum_{m=0}^{N} v_m$ then clearly $E \in \mathcal{L}(C^1, \|\ \|_{W^{1,p}(\Omega)}; C^1, \|\ \|_{W^{1,p}(V)})$

$\exists C = C(p, \Omega, V) : \|E\| < C$; $\forall u \in W^{1,p}(\Omega)$ $\operatorname{spt} Eu \subset V$.

For $x \in \Omega$: $v_m(x) = V_m(\Gamma_m(x)) = (\eta_m u) \circ \Gamma_m^{-1} \circ \Gamma_m(x) = \eta_m u(x)$, so $Eu(x) = \sum_{m=0}^{N} u\eta_m = u(x)$

since $\{\eta_m\}$ is partition of unity

Since $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$ we can uniquely extend E to mapping

$$E : W^{1,p}(\Omega) \to W^{1,p}(V)$$

$\perp$