Script of lecture nmma405

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1 Motivation for weak solution

Many principles of physics can be written in the form of a partial differential equation, see (1).

1.1 Heat flow through a nonhomogeneous material

If data are not smooth, we cannot expect regularity of solutions. This situation happens for example if we are interested in heat flow through a real wall built of several material with different heat conductivity. If we are interested in stationary flow we need to solve an equation $-\operatorname{div}(A\nabla u) = 0$ in $\Omega \subset \mathbb{R}^d$ with a boundary condition $u = u_0$ on $\partial\Omega$. The unknown temperature is $u : \Omega \to \mathbb{R}$. The set Ω, the function $u_0 : \partial\Omega \to \mathbb{R}$ and the matrix function $A : \Omega \to \mathbb{R}^{d\times d}$ are given. The function A is influenced by the heat conductivity and can be discontinuous.

1.2 Calculus of variations

Let $L : \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}$, $L = L(p, z, x)$. For $u \in C^1(\overline{\Omega})$ we define

$$I(u) = \int_\Omega L(\nabla u(x), u(x), x)\,\mathrm{d}\,x.$$

We search for a local minimum or maximum of I in $X = \{u \in C^1(\overline{\Omega}), u = 0 \text{on } \partial\Omega$.

Definition 1. *We say that $u_0 \in X$ is a local minimizer of I in X if*

$$\exists\delta > 0, \forall u \in X : \|u - u_0\|_{C^1(X)} < \delta \implies I(u_0) \le I(u).$$

Lemma 1 (1-necessary condition of minima)**.** *Let $L \in C^1(\mathbb{R}^{2d+1})$, $u_0 \in X$ be a local minimizer of I in X, $h \in \mathcal{D}(\Omega)$, $h \neq 0. Define for t \in \mathbb{R}$ $g(t) = I(u_0 + th)$. Then $g'(0) = 0$, i.e.*

$$\forall h \in \mathcal{D}(\Omega) : \int_\Omega \partial_p L(\nabla u(x), u(x), x)\cdot\nabla h(x)\,\mathrm{d}\,x + \partial_z L(\nabla u(x), u(x), x)h(x)\,\mathrm{d}\,x = 0. \tag{1}$$

The equation (1) is a weak formulation of the PDE

$$\operatorname{div} \nabla_p L(\nabla u(x), u(x), x) + \partial_z L(\nabla u(x), u(x), x) = 0$$

for an unknown function u.

2 Sobolev spaces

In the whole section $\Omega \subset \mathbb{R}^d$ is an open set.

Definition 2. *Let $u \in L^1_{loc}(\Omega)$, $\alpha \in \mathbb{N}_0^d$ be a multi-index. A function $v \in L^1_{loc}(\Omega)$ is called the α^{th} weak derivative of u if*

$$\forall \varphi \in \mathcal{D}(\Omega) : \int_\Omega \varphi v = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi.$$

We denote it by $D^\alpha u$.

In the rest all derivatives will be understand in the weak sense if not explicitely differently.

Definition 3 (Sobolev space)**.** *For $p \in [1, +\infty]$, $k \in \mathbb{N}$ we define Sobolev space*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) | \forall \alpha \in \mathbb{N}_0^d : |\alpha| \leq k \implies D^\alpha u \in L^p(\Omega)\}.$$

For $u \in W^{k,p}(\Omega)$ we define

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\int_\Omega \sum_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq k} |D^\alpha u|^p \right)^{\frac{1}{p}} & \text{if } p \in [1, +\infty), \\ \max_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)} & \text{if } p = +\infty. \end{cases}$$

We denote $V \Subset \Omega$ if V is open and bounded subset of Ω such that $\overline{V} \subset \Omega$.
We say that $u \in W^{k,p}_{loc}(\Omega)$ if for any $V \Subset \Omega$, $u \in W^{k,p}(V)$.
For $u, v \in W^{k,2}(\Omega)$ we define

$$\langle u, v \rangle_{W^{k,2}(\Omega)} = \int_\Omega \sum_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq k} D^\alpha u D^\alpha v.$$

Remark 1.

- *Functions in $W^{k,p}(\Omega)$ are determined up to a set of Lebesgue measure zero.*
- *If we say that $u \in W^{k,p}(\Omega)$ has some property, e.g. u is continuous, we mean that there is a representative with this property.*

- *If $p \in [1, +\infty)$ let us define for $u \in W^{k,p}(\Omega)$*

$$|||u||| = \left(\sum_{\alpha \in \mathbb{N}_0^d, |\alpha| \le k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}.$$

Then $|||\cdot|||$ is an equivalent norm on $W^{k,p}(\Omega)$ to $\|\cdot\|_{W^{k,p}(\Omega)}$.

Example 1. *Function $f_\alpha(x) = |x|^\alpha$ for $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^d$ belongs to $W^{1,p}_{loc}(\mathbb{R}^d)$, $p > 1$ if $\alpha > 1 - \frac{d}{p}$.*

2.1 Basic properties of Sobolev spaces

Theorem 1 (2)**.** *(Properties of the weak derivative) (4, Section 5.2.3) Let $u, v \in W^{k,p}(\Omega)$, $k \in \mathbb{N}$, $p \in [1, +\infty]$ and $\alpha \in (\mathbb{N}_0)^d$, $|\alpha| < k$. Then*

1. *$D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ and $D^\alpha(D^\beta u)) = D^\beta(D^\alpha u))$ for $|\alpha| + |\beta| \le k$*
2. *$\lambda, \mu \in \mathbb{R} \implies \lambda u + \mu v \in W^{k,p}(\Omega)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$.*
3. *if $\tilde{\Omega} \subset \Omega$ open, then $u \in W^{k,p}(\tilde{\Omega})$*
4. *if $\eta \in \mathcal{D}(\Omega)$, then $\eta u \in W^{k,p}(\Omega)$ and*

$$D^\alpha(\eta u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \eta D^{\alpha - \beta} u.$$

Remark 2. *For $\alpha, \beta \in \mathbb{N}_0^d$, $\alpha! = \Pi_{j=1}^d \alpha_j!$ and the number $\binom{\alpha}{\beta}$ is defined by $\alpha!/((\alpha - \beta)!\beta!)$.*

Example 2.

1. *If $d = 1$ and $f(x) = \operatorname{sgn}(x)$ then for any $p \in [1, +\infty]$, $f \notin W^{1,p}(-1, 1)$.*
2. *If $d = 1$ and $f(x) = |x|$ then for any $p \in [1, +\infty]$, $f \in W^{1,p}(-1, 1)$.*
3. *$W^{1,1}(-1, 1) = AC(-1, 1)$*
4. *Cantor function c is continuous on $(0, 1)$, with $c' = 0$ a.e. in $(0, 1)$, but for all $p \ge 1$, $c \notin W^{1,p}(0, 1)$. The function c is not absolutely continuous.*

Let $h \in \mathcal{D}(\mathbb{R}^d)$, $\operatorname{spt} h \subset U(0, 1)$, $\int_{\mathbb{R}^d} h = 1$. We define $h^j(x) = j^d h(jx)$ for $x \in \mathbb{R}^d$.

Definition 4. *For $u \in W^{k,p}(\Omega)$ we denote $u^j = u \star h^j$ where the expression on the right hand side is well defined.*

Lemma 2 (3)**.** *(3, Lemma 2.1.3) Let $u \in W^{k,p}(\Omega)$, $p \in [1, +\infty)$, then for all $\alpha \in \mathbb{N}_0^d$, $|\alpha| \le k$ there holds $(D^\alpha u)^j = D^\alpha(u^j)$ and $u^j \to u$ in $W^{k,p}_{loc}(\Omega)$.*

Theorem 2 (4)**.** *(3, Theorem 2.1.4)*
Let $u \in L^p(\Omega)$, $p \geq 1$. Then $u \in W^{1,p}(\Omega)$ if and only if u has a representative $\tilde{u}$ that is absolutely continuous on λ^{d-1} a.e. line segments in Ω parallel to the coordinate axis and whose classical partial derivatives (that exits almost everywhere) belong to $L^p(\Omega)$.

Proof was not presented.

Corollary 1 (5)**.** *(3, 2.1.11) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function and $u \in W^{1,p}(\Omega)$, $p \geq 1$. If $f \circ u \in L^p(\Omega)$ then $f \circ u \in W^{1,p}(\Omega)$ and for a. e. $x \in \mathbb{R}$ $\nabla(f \circ u)(x) = f'(u(x))\nabla u(x)$.*

Definition 5. *For a function $u : \Omega \to \mathbb{R}$ let $u^+ = \max(u, 0)$, $u^- = \min(u, 0)$.*

Corollary 2 (6)**.** *(3, 2.1.8) Let $u \in W^{1,p}(\Omega)$, $p \geq 1$. Then $u^+, u^- \in W^{1,p}(\Omega)$ and*

$$Du^+ = \begin{cases} Du & \text{if } u > 0 \\ 0 & \text{if } u \leq 0 \end{cases} \qquad Du^- = \begin{cases} Du & \text{if } u < 0 \\ 0 & \text{if } u \geq 0 \end{cases}$$

a.e. in Ω.

Theorem 3 (7)**.** *(3, 2.2.2) Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a bi-Lipschitzian mapping such that $T : \Omega' \to \Omega$ and*

$$\exists M > 0, \forall x, y \in \Omega, \forall x', y' \in \Omega' : \begin{array}{l} |T(x') - T(y')| \leq M|x' - y'| \\ |T^{-1}(x) - T^{-1}(y)| \leq M|x - y|. \end{array}$$

If $u \in W^{1,p}(\Omega)$, $p \geq 1$, then $v = u \circ T \in W^{1,p}(V)$ where $V = T^{-1}(\Omega)$ and for a. e. $x \in \Omega'$ and any $\xi \in \mathbb{R}^d$

$$\nabla u(T(x))\nabla T(x)\xi = \nabla u(x)\xi$$

Remark 3 (8)**.** *In the situation of the previous theorem there is $C > 0$ such that for any $U \subset \Omega$, $V = T^{-1}U$ open sets, $\|u\|_{W^{1,p}(U)} \leq C\|v\|_{W^{1,p}(V)} \leq C^2\|u\|_{W^{1,p}(U)}$.*

Theorem 4 (8)**.** *(Basic properties of Sobolev spaces) Let $k \in \mathbb{N}$.*

1. *If $p \in [1, +\infty]$, $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$ is a Banach space.*
2. *$(W^{k,2}(\Omega), \langle\cdot,\cdot\rangle_{k,2})$ is a Hilbert space.*
3. *If $p \in [1, +\infty)$, $W^{k,p}(\Omega)$ is separable.*
4. *If $p \in (1, +\infty)$, $W^{k,p}(\Omega)$ is reflexive.*

Theorem 1 (9,10)**.** *(2, Theorem 3.8) Let $p \in [1, +\infty)$, $N \in \mathbb{N}$ be a number of multiindices $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| \leq m$. For every $L \in W^{m,p}(\Omega)^*$ there exists an element $(v \in L^{p'}(\Omega))^N$ such that, writing the vector v in the form $(v)_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq N}$ we have for all $u \in W^{m,p}(\Omega)$*

$$L(u) = \sum_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq N} \langle D^\alpha u, v \rangle. \qquad (2)$$

Moreover $\|L\|_{W^{m,p}(\Omega)^*} = \inf \|v\|_{L^{p'}(\Omega)^N} = \min \|v\|_{L^{p'}(\Omega)^N}$*, the infimum being taken over, and attained on the set of all* $v \in L^{p'}(\Omega)^N$ *for which* (2) *holds for every* $u \in W^{m,p}(\Omega)$.

2.2 Approximation and extension of Sobolev functions

Lemma 3 (11)**.** *(Partition of unity) (3, Lemma 2.3.1) Let* $E \subset \mathbb{R}^d$, $\mathcal{G}$ *be a collection of open sets such that* $E \subset \bigcup_{U\in\mathcal{G}} U$. *Then there is a family* $\mathcal{F}$ *of nonnegative functions* $f \in \mathcal{D}(\mathbb{R}^d)$ *such that* $0 \le f \le 1$ *and*

1. $\forall f \in \mathcal{F}, \exists U \in \mathcal{G} : \operatorname{spt} f \subset U$
2. $\forall K \subset E, K$ *compact* : $\operatorname{spt} f \cap K \neq \emptyset$ *for only finitely many* $f \in \mathcal{F}$
3. $\sum_{f\in\mathcal{F}} f(x) = 1$ *for every* $x \in E$
4. *if* E *is compact, the family* $\mathcal{F}$ *is finite*
5. *family* $\mathcal{F}$ *is at most countable*

Theorem 5 (12)**.** *(3, Theorem 2.3.2) The set* $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ *is dense in* $W^{k,p}(\Omega)$. *The set* $\{f \in C^\infty(\Omega), \exists R > 0 : \operatorname{spt} f \subset U(0,R)\} \cap W^{k,p}(\Omega)$ *is dense in* $W^{k,p}(\Omega)$.

Lemma 4 (13)**.** *Let* $u \in L^p(\mathbb{R}^d)$, $p \in [1,+\infty)$. *For* $h \in \mathbb{R}^d$, $h \neq 0$ *and* $x \in \mathbb{R}^d$ *define* $u_h(x) = u(x+h)$. *Then* $u_h \to u$ *in* $L^p(\mathbb{R}^d)$ *as* $h \to 0$.

Lemma 5 (14)**.** *Let* $V = U(0,R) \cap \{x \in \mathbb{R}^d; x_d > 0\}$, $\epsilon > 0$, $u \in W^{k,p}(\{x \in \mathbb{R}^d; x_d > 0\})$ *with* $\operatorname{spt} u \subset V$. *Then there is a function* $v \in C^\infty(\{x \in \mathbb{R}^d; x_d \ge 0\})$ *such that* $\operatorname{spt} v \subset U(0,2R) \cap \{x \in \mathbb{R}^d; x_d \ge 0\}$ *and* $\|u - v\|_{W^{k,p}(V)} < \epsilon$.

Theorem 6 (15)**.** *(4, Section 5.3.3, Theorem 3),(2, Theorem 3.18) Let* $k \in \mathbb{N}$, $p \in [1,+\infty)$, $\Omega \subset \mathbb{R}^d$ *be bounded with* C^1 *boundary. Then* $C^\infty(\overline{\Omega})$ *is dense in* $W^{k,p}(\Omega)$.

Lemma 6 (16)**.** *Let us equip* $X = \{U \in C^1(\{x \in \mathbb{R}^d | x_d \ge 0\}) | \operatorname{spt} U \subset U(0,R)\}$ *with a norm* $\|\cdot\|_X = \|\cdot\|_{W^{1,p}(U(0,R))\cap\{x\in\mathbb{R}^d|x_d\ge 0\}}$ *and* $Y = \{U \in C^1(\mathbb{R}^d) | \operatorname{spt} U \subset U(0,2R))\}$ *with a norm* $\|\cdot\|_Y = \|\cdot\|_{W^{1,p}(U(0,2R))}$. *Then there is a linear mapping* $\tilde{E} : X \to Y$ *such that*

$$\|\tilde{E}\|_{\mathcal{L}((X,\|\cdot\|_X),(Y,\|\cdot\|_Y))} < C(p,R).$$

and $\tilde{E}u = u$ *on* $\{x \in \mathbb{R}^d | x_d \ge 0\}$ *for any* $u \in X$.

Theorem 7 (17)**.** *(4, Section 5.4, Theorem 1) Assume* $\Omega \subset \mathbb{R}^d$ *open, bounded and with* C^1 *boundary. Fix* $V \subset \mathbb{R}^d$ *open such that* $\Omega \Subset V$. *Then there is a bounded linear operator* $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ *such that for all* $u \in W^{1,p}(\Omega)$

1. $Eu = u$ *a.e. in* Ω
2. $\operatorname{spt} Eu \subset V$
3. $\|E\| \le C$ *with* $C = C(p,\Omega,V)$

2.3 Embeddings of Sobolev spaces

We introduce a notation

$$\int_{\mathbb{R}^{d-1}} f \widehat{\mathrm{d}\, x_i} = \int_{\mathbb{R}^{d-1}} f \,\mathrm{d}\, x_i \ldots \mathrm{d}\, x_{i-1} \,\mathrm{d}\, x_{i+1} \ldots \mathrm{d}\, x_d.$$

Lemma 7 (18)**.** *(4, Section 5.6, Theorem 1) Let $d \geq 2$, for $i \in \{1, \ldots, d\}$, $u_i \in C_c^1(\mathbb{R}^{d-1})$ and u_i be independent of x_i. Then*

$$\int_{\mathbb{R}^d} \prod_{i=1}^{d} |u_i| \leq (\prod_{i=1}^{d} \int_{\mathbb{R}^{d-1}} |u_i|^{d-1} \widehat{\mathrm{d}\, x_i})^{\frac{1}{d-1}}.$$

Lemma 8 (19)**.** *(4, Section 5.6, Theorem 1) Let $d > 2$, $u \in C_c^1(\mathbb{R}^d)$. Then for $p \in [1, d)$, $p^* = \frac{dp}{d-p}$, i.e. $-\frac{d}{p^*} = 1 - \frac{d}{p}$*

$$\|u\|_{L^{p^*}(\Omega)} \leq p\frac{d-1}{d-p} \|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

Theorem 8 (20)**.** *Let $p \in [1, d)$, $d > 2$. Then $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^{p^*}(\mathbb{R}^d)$.*

Definition 6. *For $p \in [1, +\infty]$ we define $W_0^{k,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{k,p}}$.*

Theorem 9 (21)**.** *Let $p \in [1, d)$, $d > 2$, Ω bounded. Then for all $q \in [1, p^*]$ exists $C > 0$ such that for all $u \in W_0^{1,p}(\Omega)$ there holds $\|u\|_{L^q(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}$.*

Remark 4. *$\|\cdot\|_{1,p}$ and $\|\nabla \cdot\|_p$ are equivalent norms on $W_0^{1,p}(\Omega)$ if Ω is bounded.*

Theorem 10 (22)**.** *Let $p \in [1, d)$, $d > 2$, $\Omega \subset \mathbb{R}^d$ bounded with C^1 boundary. Then*

$$\exists C_p > 0, \forall u \in W^{1,p}(\Omega) : \|u\|_{L^{p^*}(\Omega)} \leq C_p \|u\|_{W^{1,p}(\Omega)}.$$

Lemma 9 (24)**.** *(5, Lemma 7.16) Let $u \in C^1(\mathbb{R}^d)$, $\Omega \subset \mathbb{R}^d$ bounded convex, $x \in \Omega$. Then*

$$|u(x) - \fint_{\Omega} u| \leq \frac{R^d}{d|\Omega|} \int_{\Omega} |\nabla u(y)||y - x|^{1-d} \,\mathrm{d}\, y.$$

Theorem 11 (25-Sobolev-Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be bounded and convex. Then*

$$\forall q < p^*, \exists C > 0, \forall u \in W^{1,p}(\Omega) : \|u - \fint_{\Omega} u\|_{L^q(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}.$$

Remark 5. *(3, Corollary 4.2.3) Previous theorem holds also if $p \geq 1$ and $q = p^*$.*

Lemma 10 (26)**.** *Let $u \in C_c^1(\mathbb{R}^d)$, $\alpha = 1 - \frac{d}{p}$. Then*

$$\forall x, y \in \mathbb{R}^d : \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \leq C(p, d)\|\nabla u\|_{L^p(\mathbb{R}^d)}, \quad |u(x)| \leq C(p, d)\|u\|_{W^{1,p}(\mathbb{R}^d)}.$$

Definition 7. *We define for* $\alpha \in (0,1]$ *and* $f : \Omega \to \mathbb{R}$ *a*

$$[f]_{C^{0,\alpha}(\overline{\Omega})} := \sup\{\frac{|f(x)-f(y)|}{|x-y|^\alpha}; x,y \in \Omega, x \neq y\},$$
$$\|f\|_{C^{0,\alpha}(\Omega)} = \|f\|_{L^\infty(\Omega)} + [f]_{C^{0,\alpha}(\overline{\Omega})}.$$

We define $C^{0,\alpha}(\overline{\Omega}) = \{f : \Omega \to \mathbb{R}; \|f\|_{C^{0,\alpha}(\overline{\Omega})} < +\infty\}$.

Theorem 12 (27)**.** *(6, Theorem 1.3.3) Let* $\alpha \in (0,1]$*. The space* $(C^{0,\alpha}(\overline{\Omega}), \|\cdot\|_{0,\alpha})$ *is a Banach space.*

Theorem 13 (28)**.** *Let* $p \in (d,+\infty]$, $\alpha = 1 - \frac{d}{p}$*, then* $W^{1,p}(\mathbb{R}^d) \hookrightarrow C^{0,\alpha}(\mathbb{R}^d)$.

Theorem 14 (29)**.** *Let* $p \in (d,+\infty]$, $\Omega \subset \mathbb{R}^d$ *bounded with* C^1 *boundary. Then* $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$.

Theorem 15 (30)**.** *(4, Theorem 5.5.1) Let* $d \in \{2,\dots\}$, $\Omega \subset \mathbb{R}^d$ *be bounded with* C^1 *boundary,* $p \in [1,+\infty)$, $p^\# = \frac{(d-1)p}{d-p}$ *if* $p < d$*. Let*

$$q \in \begin{cases} [1,p^\#] & \text{if } p < d, \\ [1,+\infty) & \text{if } p = d, \\ [1,+\infty] & \text{if } p > d. \end{cases}$$

Then there is a bounded linear operator $\mathrm{Tr} : W^{1,p}(\Omega) \to L^q(\partial\Omega)$ *such that for* $f \in C^\infty(\overline{\Omega})$ *the equality* $\mathrm{Tr}\, f = f|_{\partial\Omega}$ *holds on* $\partial\Omega$.

Theorem 16 (31)**.** *(2, Theorem 6.2), (2, Theorem 5.4) Let* $d \in \{2,\dots\}$, $\Omega \subset \mathbb{R}^d$ *be bounded with* C^1 *boundary,* $p \in [1,+\infty)$.

case $p < d$
- *If* $q \in [1,p^*)$ *the embedding* $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ *is compact.*
- *If* $q \in [1,p^\#)$ *the embedding* $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ *is compact.*

case $p = d$
- *If* $q \in [1,+\infty)$ *the embeddings* $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ *and* $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ *are compact.*

case $p > d$
- *If* $\alpha \in [0, 1-\frac{d}{p})$ *the embedding* $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ *is compact.*
- *If* $\alpha \in [0, 1-\frac{d}{p})$ *the embeddings* $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\partial\Omega)$ *is compact.*

This theorem was presented in a different form without proof.

Theorem 17 (32)**.** *Let* Ω *be bounded with* C^1 *boundary,* $p \in [1,+\infty)$*. Then*

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) | \mathrm{Tr}\, u = 0 \text{ on } \partial\Omega.$$

2.4 Difference quotients and weak derivatives

Definition 8. *Let $u \in L^1_{loc}(\Omega)$, $i \in \{1,\dots,d\}$. The i-th difference quotient of size $h \in \mathbb{R} \setminus \{0\}$ is $D_i^h u(x) = \frac{1}{h}(u(x+he_i) - u(x))$ for $x \in \Omega$ s.t. $x + he_i \in \Omega$.*

Theorem 18 (32)**.** *i) Let $p \in [1,+\infty)$, $u \in W^{1,p}(\Omega)$. Then there is $C > 0$ such that for all $V \Subset \Omega$, $i \in \{1,\dots,d\}$, $|h| < \frac{1}{2}(\operatorname{dist}(V,\partial\Omega))$ there holds $\|D_i^h u\|_{L^p(V)} \le C\|\partial_i u\|_{L^p(\Omega)}$.*

ii) Let $p \in (1,+\infty)$, $u \in L^p(\Omega)$ and there is $C > 0$, $V \Subset \Omega$, $i \in \{1,\dots,d\}$ such that for all $|h| < \frac{1}{2}(\operatorname{dist}(V,\partial\Omega))$ there holds $\|D_i^h u\|_{L^p(V)} \le C$. Then the weak derivative $\partial_i u$ exists and $\|\partial_i u\|_{L^p(V)} \le C$.

3 Linear elliptic PDE's of second order

In this section we will assume

Assumption 1 (33)**.** *The set Ω and functions $A = (a_{ij})_{i,j=1}^d : \Omega \to \mathbb{R}^{d\times d}$, $b = (b_i)_{i=1}^d : \Omega \to \mathbb{R}^d$, $c, f : \Omega \to \mathbb{R}$, $g, u_0 : \partial\Omega \to \mathbb{R}$ are given with the following properties.*

- *$\Omega \subset \mathbb{R}^d$ with C^1 boundary, a bounded domain*
- *there is $\alpha > 0$ such that for all $\xi \in \mathbb{R}^d$ and a.e. $x \in \Omega$ there holds $\alpha|\xi|^2 \le A\xi \cdot \xi$*
- *for all $i, j \in \{1,\dots,d\}$ there holds $a_{ij}, b_i, c \in L^\infty(\Omega)$*
- *$f \in L^2(\Omega)$*
- *$g \in L^2(\partial\Omega)$*
- *u_0 is a trace of a function from $W^{1,2}(\Omega)$, we denote it again $u_0 \in W^{1,2}(\Omega)$*

We will study the equation

$$- \operatorname{div}(A\nabla u) + b \cdot \nabla u + cu = f \quad \text{in } \Omega \tag{3}$$

with two types of boundary conditions. We will prescribe either Dirichlet boundary condition

$$u = u_0 \quad \text{on } \partial\Omega \tag{4}$$

or Neumann boundary condition

$$A\nabla u \cdot \nu = g \quad \text{on } \partial\Omega, \text{ here } \nu \text{ denotes the normal unit vector to } \Omega. \tag{5}$$

Definition 9. *We say that $u : \Omega \to \mathbb{R}$ is a weak solution to the problem* (3) *with the boundary condition* (4) *if $u \in W^{1,2}(\Omega)$, $u - u_0 \in W_0^{1,2}(\Omega)$, i.e.* $\operatorname{Tr} u = u_0$, *and*

$$\forall \varphi \in W_0^{1,2}(\Omega) : \int_\Omega A\nabla u \cdot \nabla\varphi + b \cdot \nabla u\varphi + cu\varphi = \int_\Omega f\varphi. \tag{6}$$

We say that $u : \Omega \to \mathbb{R}$ is a weak solution to the problem (3) *with the boundary condition* (5) *if $u \in W^{1,2}(\Omega)$ and*

$$\forall \varphi \in W^{1,2}(\Omega) : \int_\Omega A\nabla u \cdot \nabla \varphi + b \cdot \nabla u \varphi + cu\varphi = \int_\Omega f\varphi + \int_{\partial\Omega} g \operatorname{Tr}(\varphi). \qquad (7)$$

3.1 Existence of a weak solution by Riesz Theorem

Theorem 19. *(7, Theorem 19) Let H be a real Hilbert space. Define for $y \in H$, $f_y \in H^*$ by $f_y(x) = \langle x, y \rangle$ for all $x \in H$. The mapping $I : H \to H^*$, $I(y) = f_y$ is linear isometry of H onto H^*.*

Theorem 20 (35)**.** *Let Assumption 1 hold. Moreover let for all $i, j \in \{1, \dots, d\}$ and a.e. $x \in \Omega$ $a_{ij}(x) = a_{ji}(x)$, $b(x) = 0$.*

1. *Then there is $\gamma < 0$ such that if $c > \gamma$ on Ω then a weak solution of* (3) *and* (4) *exists. It satisfies $\|u\|_{W^{1,2}(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|u_0\|_{W^{1,2}(\Omega)})$ for a suitable $C > 0$ independent of f and u_0.*
2. *If $c > 0$ on Ω then there is a weak solution of* (3) *and* (5)*. It satisfies $\|u\|_{W^{1,2}(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}$ for a suitable $C > 0$ independent of f and g.*

The solutions are unique.

Lemma 11 (36 Lax Milgram)**.** *(4) Let H be a real Hilbert space with a scalar product $\langle \cdot, \cdot \rangle_H$ and an induced norm $\| \cdot \|_H$. Let $B : H \times H \to \mathbb{R}$ be a bilinear mapping that is*

- *(elliptic) $\exists m > 0, \forall u \in H : m\|u\|_H^2 \le B(u, u)$*
- *(bounded) $\exists M > 0, \forall u, v \in H : B(u, v) \le M\|u\|_H\|v\|_H$*

Then for every $F \in H^*$ there is a unique $u \in H$ such that $\forall v \in H : B(u, v) = F(v)$. Moreover, $\|u\|_H \le \frac{1}{m}\|F\|_{H^*}$.

Theorem 21 (37)**.** *Let Assumption 1 hold. Then there is $\gamma \in \mathbb{R}$ such that if $c > \gamma$ on Ω then there is a weak solution u of* (3) *and* (4) *or* (5)*. The solution is unique and satisfies $\|u\|_{W^{1,2}(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|u_0\|_{W^{1,2}(\Omega)})$, resp. $\|u\|_{W^{1,2}(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)})$.*

Theorem 22 (38)**.** *Let*

- *$A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $b : \mathbb{R}^d \to \mathbb{R}^d$, $c, f : \mathbb{R}^d \to \mathbb{R}$*
- *$A, b, c \in L^\infty(\mathbb{R}^d)$, $f \in L^2$*

There is $\gamma \in \mathbb{R}$ such that $c > \gamma$ implies existence of $u \in W^{1,2}(\mathbb{R}^d)$ such that

$$\forall \varphi \in W^{1,2}(\mathbb{R}^d) : \int_{\mathbb{R}^d} A\nabla u \cdot \nabla \varphi + b \cdot \nabla u \varphi + cu\varphi = \int_{\mathbb{R}^d} f\varphi.$$

The solution is unique and $\|u\|_{W^{1,2}(\mathbb{R}^d)} \le C\|f\|_{L^2(\mathbb{R}^d)}$.

3.2 Application of Fredholm Theorems

We introduce the differential operator

$$Lu = -\operatorname{div}(A\nabla u) + b \cdot \nabla u + cu - \operatorname{div}(du) \tag{8}$$

and its formal adjoint

$$L^* u = -\operatorname{div}(A^T \nabla u) + d \cdot \nabla u + cu - \operatorname{div}(bu) \tag{9}$$

We consider here only homogeneous Dirichlet boundary condition $u = 0$ on $\partial\Omega$.

If we assume sufficient regularity of functions c and d we may apply the theory developed in the previous section to get existence of a weak solutions to the problem $Lu = f$ in Ω and $u = 0$ on $\partial\Omega$. The statement $u \in W_0^{1,2}(\Omega)$ solves the problem $Lu = f$ in Ω with the boundary condition $u = 0$ on $\partial\Omega$ is understood in the weak sense in what follows.

We will assume that Assumption 1 hold and moreover for simplicity $b, d, \in W^{1,\infty}(\Omega)$.

Theorem 23 (39-Fredholm alternative)**.** *1. (a) Either for all $f \in L^2(\Omega)$ there exists a unique $u \in W_0^{1,2}(\Omega)$ a weak solution of $Lu = f$ in Ω, $u = 0$ on $\partial\Omega$*

(b) or there is $u \in W_0^{1,2}(\Omega) \setminus \{0\}$ a weak solution of $Lu = 0$ in Ω, $u = 0$ on $\partial\Omega$.

2. *In case 1b) denote $\operatorname{Ker} L = \{u \in W_0^{1,2}(\Omega); Lu = 0\} \neq \emptyset$, $\operatorname{Ker} L^* = \{u \in W_0^{1,2}(\Omega); L^*u = 0\}$. Then $\dim \operatorname{Ker} L = \dim \operatorname{Ker} L^*$.*

3. *In case 1b) there is a weak solution to $Lu = f$ in Ω, $u = 0$ on $\partial\Omega$ if $f \in L^2(\Omega)$ and for all $\varphi \in \operatorname{Ker} L^*$, $\int_\Omega f\varphi = 0$.*

Theorem 24 (40)**.** *(4, Section 6.2, Theorem 5) Let Ω be a bounded domain. There is at most countable set $\Sigma \subset \mathbb{R}$ such that the following is equivalent:*

1. *$\lambda \notin \Sigma$*

2. *$\forall f \in L^2(\Omega), \exists! u \in W_0^{1,2}(\Omega)$ a weak solution of the problem $Lu = \lambda u + f$ in Ω, $u = 0$ on $\partial\Omega$.*

If Σ is not finite, then $+\infty$ is its only cluster point.

Remark 6. *The set Σ is called (real) spectrum of L.*

Theorem 25 (41)**.** *Let the operator L satisfy: A be symmetric ($\forall i, j \in \{1, \ldots, d\}$: $a_{ij} = a_{ji}$), $\forall j \in \{1, \ldots, d\} : b_j = d_j$. Let Σ be the set from Theorem 24. Then*

1. *Σ is infinite. If we denote $\Sigma = \{\lambda_k\}_{k=1}^{+\infty}$ then $\lambda_k \to +\infty$ as $k \to +\infty$.*

2. *There exists an orthonormal basis $\{w_k\}_{k=1}^{+\infty}$ of $L^2(\Omega)$ such that $w_k \in W^{1,2}(\Omega)$ and it solves $Lw_k = \lambda w_k$ in Ω, $w_k = 0$ on $\partial\Omega$ for some $\lambda \in \Sigma$.*

3. *If $b = d = 0$ and $c \geq 0$ on Ω, then $\Sigma \subset (0, +\infty)$.*

Theorem 26 (43-maximum principle)**.** *Let $u_0 \in L^\infty(\partial\Omega) \cap \mathrm{Tr}(W^{1,2}(\Omega))$, $c \geq 0$ on Ω and $u \in W^{1,2}(\Omega)$ is a weak solution to $-\operatorname{div}(A\nabla u) + cu = 0$ in Ω, $u = u_0$ on $\partial\Omega$. Then $u \in L^\infty(\Omega)$ and $\|u\|_{L^\infty} \leq \|u_0\|_{L^\infty(\partial\Omega)}$.*

Theorem 27 (44)**.** *Let $a_{ij} \in C^1(\overline{\Omega})$, $b_i, c \in L^\infty(\Omega)$ for all $i, j \in \{1, \ldots, d\}$, $f \in L^2(\Omega)$, $u \in W^{1,2}(\Omega)$ be a weak solution of $Lu = f$ in Ω, $u = 0$ on $\partial\Omega$. Then $u \in W^{1,2}(\Omega)$ and $\|u\|_{W^{2,2}(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})$. The constant $C > 0$ is independent of f and u.*

4 Nonlinear elliptic PDE's of second order

4.1 Basics of Calculus of Variations

Setting:

1. $\Omega \subset \mathbb{R}^d$ open bounded set with smooth boundary
2. $L : \mathbb{R}^d \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ a function called Lagrangian, $L = L(p, z, x)$, $p \in \mathbb{R}^d$, $z \in \mathbb{R}$, $x \in \Omega$.
3. $g : \partial\Omega \to \mathbb{R}$

We are looking for a minimizer of

$$I(w) = \int_\Omega L(\nabla w(x), w(x), x) \,\mathrm{d}\, x$$

on the set of functions $X = \{w; w = g \text{ on } \partial\Omega\}$.

We will assume coercivity of L

$$\exists q \in (1, +\infty), \exists \alpha > 0, \beta \geq 0, \forall p \in \mathbb{R}^d, z \in \mathbb{R}, x \in \Omega : L(p, z, x) \geq \alpha |p|^q - \beta. \tag{10}$$

Remark 7.

- *If L is coercive then $I(w) \to +\infty$ as $\|\nabla w\|_{L^q(\Omega)} \to +\infty$.*
- $$\inf_{w \in X} I(w) = \inf\{I(w); w \in X, \|\nabla w\|_q \leq \left(\frac{2L(w_0) + \beta'}{\alpha'}\right)^{\frac{1}{q}}\}$$

  for any $w_0 \in X$ and suitable α' and β'.

Definition 10. $X = \{w \in W^{1,q}(\Omega);\ \mathrm{Tr}\, w = g$ *on* $\partial\Omega\}$.

Lemma 12 (45)**.** *Let $R > 0$, $A = \{w \in X; \|\nabla w\|_{L^q(\Omega)} < R\}$, then there is $R' > 0$ such that $A \subset U(0, R') \subset W^{1,q}(\Omega)$.*

Corollary 3 (46)**.** *Choose $w_k \subset X$ such that $I(w_k) \to \inf_{w \in X} I(w)$, then $\exists R' > 0, \forall k \in$: $\|w_k\|_{1,q} \leq R'$, i.e. minimizing sequences are bounded.*

Definition 11. *We say that* I *is weakly sequentially lower semicontinuous on* $W^{1,q}(\Omega)$ *if* $I(u) \leq \liminf_{k\to+\infty} I(w_k)$*, whenever* $w_k \rightharpoonup u$ *in* $W^{1,q}(\Omega)$*.*

Theorem 28 (47)**.** *Assume that* L *is smooth (*C^2 *is definitely enough/too much), bounded below and in addition*

$$\text{the mapping } p \to L(p,z,x) \text{ is convex for any } z \in \mathbb{R},\ x \in \Omega. \tag{11}$$

Then I *is weakly sequentially lower semicontinuous on* $W^{1,q}(\Omega)$*.*

Theorem 29 (48)**.** *Assume that* L *satisfies the coercivity condition* (10)*, and is convex with respect to the variable* p*, see* (11)*, and* X *is not empty. Then there is (at least one) function* $u \in X$ *solving* $I(u) = \inf_{w\in X} I(w)$*.*

Theorem 30 (49)**.** *Suppose that* L *is smooth and independent of* z *and*

$$\exists q > 1, \theta > 0, \forall p \in \mathbb{R}^d, \xi \in \mathbb{R}^d, x \in \Omega : \sum_{i,j=1}^{d} \partial_{p_i}\partial_{p_j} L(p,x)\xi_i\xi_j \geq \theta|\xi|^q.$$

Then there is at most one minimizer of I*.*

Proof. Theorem was presented in a student's presentation. □

Definition 12. *We say that* $u \in X$ *is a weak solution to the boundary value problem*

$$-\operatorname{div} \nabla_p L(\nabla u, u, x) + \partial_z L(\nabla u, u, x) = 0 \quad \text{in } \Omega, \tag{12}$$

with boundary condition $u = g$ *on* $\partial\Omega$ *for the Euler Lagrange equation provided*

$$\forall v \in W_0^{1,q}(\Omega) : \int_\Omega \nabla_p L(\nabla u, u, x) \cdot \nabla v + \partial_z L(\nabla u, u, x) v = 0.$$

Theorem 31 (50)**.** *Assume* L *verifies the growth conditions*

$$\exists C > 0, \forall p \in \mathbb{R}^d, z \in \mathbb{R}, x \in \Omega : |L(p,z,x)| \leq C(|p|^q + |z|^q + 1)$$
$$\exists C > 0, \forall p \in \mathbb{R}^d, z \in \mathbb{R}, x \in \Omega : |\nabla_p L(p,z,x)| + |\nabla_z L(p,z,x)| \leq C(|p|^{q-1} + |z|^{q-1} + 1)$$

and $u \in X$ *satisfies* $I(u) = \inf_{w\in X} I(w)$*. Then* u *is a weak solution of* (12)*.*

Proof. Just a sketch of a proof. Computation was shown in a presentation but without precise reasoning for interchange of limit passage and integration. □

4.2 Existence of a weak solution by method of Brower and Minty

Assumption 2 (51)**.** *Let* $a : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$*,* $q > 1$ *satisfy*

- *a is a Caratheodory function, i.e. for a.e.* $x \in \Omega$ *the mapping* $(z,p) \to a(x,z,p)$ *is continuous and for all* $z \in \mathbb{R}, p \in \mathbb{R}^d$ *the mapping* $x \to a(x,z,p)$ *is measurable*

- *(boundedness)* $\exists C > 0, \forall x \in \Omega, z \in \mathbb{R}, p \in \mathbb{R}^d : |a(x,z,p)| \leq C(1+|p|)^{q-1}$
- *(coercivity)* $\exists C_1, C_2 > 0, \forall x \in \Omega, z \in \mathbb{R}, p \in \mathbb{R}^d : C_1|p|^q - c_2 \leq a(x,z,p) \cdot p$.
- *(monotony)* $\forall x \in \Omega, z \in \mathbb{R}, p_1, p_2 \in \mathbb{R}^d : (a(x,z,p_1) - a(x,z,p_2)) \cdot (p_1 - p_2) \geq 0$

Remark 8.
- *Monotony is an assumption of a similar type as convexity in variational techniques.*
- *Coercivity was needed also for variational techniques.*
- *Boundedness was not needed for variational techniques.*

We consider the next problem: for a given a, f and u_0 find a solution u to the partial differential equation

$$-\operatorname{div} a(x,u,\nabla u) = f \quad \text{in } \Omega \tag{13}$$

with Dirichlet boundary condition $u = u_0$ on $\partial\Omega$.

Definition 13 (weak formulation of (13))**.** *Let $f \in W_0^{1,q}(\Omega)^*$, $u_0 : \partial\Omega \to \mathbb{R}$ and $a : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. We call $u \in W^{1,q}(\Omega)$ a weak solution of the problem* (13) *with boundary condition $u = u_0$ on $\partial\Omega$ if $\operatorname{Tr} u = u_0$ on $\partial\Omega$ and*

$$\forall \varphi \in W_0^{1,q}(\Omega) : \int_\Omega a(x,u,\nabla u) \cdot \nabla\varphi = \rangle f, \varphi \langle.$$

Remark 9. *Under Assumption 2 all terms in the definition are well defined.*

Theorem 32 (52)**.** *If $f \in (W_0^{1,q})^*$, Assumption 2 holds and $u_0 \in W^{1,q}(\Omega)$, then there is a weak solution of the problem* (13) *with boundary condition $u = u_0$ on $\partial\Omega$.*

Lemma 13. *Let $R > 0$, $m \in \mathbb{N}$, $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ be continuous such that for all $c \in \partial U(0,R) : \Phi(c) \cdot c \geq 0$. Then there is a $c_0 \in \overline{U(0,R)}$ such that $\Phi(c_0) = 0$.*

Proof. The proof rests on Brower fixed point theorem but was not presented. □

Theorem 33 (53)**.** *Let assumptions of Theorem 32 hold. Let a be independent of z, i.e. $a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, $a = a(x,p)$, and strictly monotone in p, i.e.*

$$\forall p_1, p_2 \in \mathbb{R}^d, p_1 \neq p_2, \text{a.e. } x \in \Omega : (a(x,p_1) - a(x,p_2)) \cdot (p_1 - p_2) > 0.$$

Then the weak solution to the problem (13) *with the boundary condition $u = u_0$ in $\partial\Omega$ is unique.*

Proof. Will be proved in presentation. □

5 Did not fit into schedule

Theorem 34 (54-Maximum principle)**.** *Let Assumption 2 hold, a be strictly monotone in p, for all $z \in \mathbb{R}$ and a.e. $x \in \Omega$ $a(x, z, 0) = 0$, $f = 0$ and $u_0 \in L^\infty(\partial\Omega) \cap \operatorname{Tr} W^{1,q}(\Omega)$. Let $u \in W^{1,q}(\Omega)$ be a weak solution to* (13) *with the boundary condition $u = u_0$ on $\partial\Omega$. Then $u \in L^\infty(\Omega)$ and $\|u\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\partial\Omega)}$.*

Proof. The theorem was not presented. □

Theorem 35 (55-local regularity)**.** *Let Assumption 2 hold, a be independent of z and x, $f = 0$ and*

$$\exists\theta > 0, \forall p_1, p_2 \in \mathbb{R}^d : (a(p_1) - a(p_2)) \cdot (p_1 - p_2) \geq \theta(|p_1| + |p_2|)^{q-2}|p_1 - p_2|^2$$
$$\exists C > 0, \forall p_1, p_2 \in \mathbb{R}^d : |a(p_1) - a(p_2)| \leq C(|p_1| + |p_2|)^{q-2}|p_1 - p_2|.$$

Let $u \in W^{1,q}(\Omega)$ be a weak solution to (13) *with the boundary condition $u = u_0$ on $\partial\Omega$ and B be a ball of radius $R > 0$ such that $B \subset 2B \subset \Omega$. Then $|\nabla u|^{\frac{q}{2}} \in W^{1,2}(B)$ and*

$$\int_B |\nabla|\nabla u|^{\frac{q}{2}}|^2 \leq \frac{C}{R^2}\int_{2B} |\nabla u|^q.$$

Proof. Theorem was not presented. □

5.1 Existence of a weak solution by Banach fixed point theorem

Theorem 36 (56-nonlinear Lax Milgram)**.** *Let X be a real Hilbert space, $T : X \to X$ Lipschitz continuous, i.e.*

$$\exists M > 0, u, v \in X : \|Tu - Tv\|_X \leq M\|u - v\|_X$$

and strongly monotone, i.e.

$$\exists m > 0, \forall u, v \in X : (Tu - Tv, u - v)_X \geq m\|u - v\|_X^2.$$

Then for any $F \in X$ exists a unique $u \in X$ such that $Tu = F$.

Proof. The theorem was not presented. □

Example 3. *For any $f \in L^2(\Omega)$ there is a weak solution to the problem $-\operatorname{div}\left(\operatorname{arctg}(1 + |\nabla u|^2)\nabla u\right) = -\operatorname{div} f$ in Ω with homogeneous Dirichlet boundary condition $u = 0$ on $\partial\Omega$.*

Proof. The example was not presented. □

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