• current age $Z(t) = age + t$, where age is the age at the entry to the study

$$\lambda(t|Z) = \lambda_0(t) \cdot \exp\{\alpha(age+t) + \beta^T Z\} = \lambda_0(t)\cdot e^{\alpha t} \cdot \exp\{\alpha\, age + \beta^T Z\}$$

and now compare it with model where only age itself is in the model: $\tilde\lambda(t|Z) = \tilde\lambda_0(t)\cdot \exp\{\tilde\alpha\, age + \tilde\beta^T Z\}$

$age = 0, Z = 0 \Rightarrow \lambda_0(t)\cdot e^{\alpha t}$ vs. $\tilde\lambda_0(t)$, so interpretation of baseline hazard changes

↳ interpretation of λ_0 and α is tied together

$$\frac{\tilde\lambda(t|Z_1)}{\tilde\lambda(t|Z_2)} \overset{Z_1=Z_2}{=} \exp\{\tilde\alpha(age_1 - age_2)\} \rightsquigarrow e^{\tilde\alpha} = \tilde\lambda(t|age+1, Z)/\tilde\lambda(t|age, Z)$$

$$\frac{\lambda(t|Z_1)}{\lambda(t|Z_2)} \overset{Z_1=Z_2}{=} \exp\{\alpha(age_1 - age_2)\} \rightsquigarrow e^{\alpha} = \lambda(t|age+1, Z)/\lambda(t|age, Z)$$

the same interpretation and also $\boxed{\alpha = \tilde\alpha}$ because partial likelihood has the same shape

What if interaction? $\lambda(t|Z) = \lambda_0(t)\cdot\exp\{\alpha(age+t) + \beta\cdot Z + \gamma(age+t)\cdot Z\}$

$$\tilde\lambda(t|Z) = \tilde\lambda_0(t)\cdot\exp\{\tilde\alpha(age) + \tilde\beta\cdot Z + \tilde\gamma\cdot age\ Z\}$$

$age = 0 = Z \rightsquigarrow \lambda_0(t)\cdot e^{\alpha t}$ vs $\tilde\lambda_0(t)$ and only $age = 0 \rightsquigarrow \lambda_0(t)\cdot e^{\alpha t + \gamma t Z}\cdot e^{\beta Z}$ vs $\tilde\lambda_0(t)\cdot e^{\tilde\beta Z}$

under $Z = 0$ we are under the same conditions as above $\rightsquigarrow$ α and $\tilde\alpha$ have the same interpretation

$$\frac{\tilde\lambda(t|age=0, Z_1)}{\tilde\lambda(t|age=0, Z_2)} = \exp\{\tilde\beta(Z_1 - Z_2)\}$$

but

$$\frac{\lambda(t|age=0, Z_1)}{\lambda(t|age=0, Z_2)} = \frac{\lambda_0(t)\exp\{\alpha t + \beta Z_1 + \gamma t Z_1\}}{\lambda_0(t)\exp\{\alpha t + \beta Z_2 + \gamma t Z_2\}} = \exp\{\beta(Z_1 - Z_2) + \underline{\gamma t(Z_1 - Z_2)}\}$$

general unit increase in Z: $\frac{\tilde\lambda(t|age, Z+1)}{\tilde\lambda(t|age, Z)} = \exp\{\tilde\beta + \tilde\gamma\cdot age\}$ but: $\frac{\lambda(t|age, Z+1)}{\lambda(t|age, Z)} = \exp\{\beta + \gamma(age+t)\}$

combined:

$$\frac{\dfrac{\tilde\lambda(t|age+1, Z+1)}{\tilde\lambda(t|age+1, Z)}}{\dfrac{\tilde\lambda(t|age, Z+1)}{\tilde\lambda(t|age, Z)}} = e^{\tilde\gamma} \qquad \frac{\dfrac{\lambda(t|age+1, Z+1)}{\lambda(t|age+1, Z)}}{\dfrac{\lambda(t|age, Z+1)}{\lambda(t|age, Z)}} = \frac{e^{\beta+\gamma(age+1+t)}}{e^{\beta+\gamma(age+t)}} = e^{\gamma}$$

PL without interaction:

$$\prod_{i=1}^{d} \frac{\exp\{\tilde\alpha A_i + \tilde\beta^T Z_i\}}{\sum_{j=1}^{n} Y_j(t_i)\exp\{\tilde\alpha A_j + \tilde\beta^T Z_j\}} \quad \text{vs} \quad \prod_{i=1}^{d} \frac{\exp\{\cancel{\alpha t_i} + \alpha A_i + \beta^T Z_i\}}{\sum_{j=1}^{n} Y_j(t_i)\exp\{\cancel{\alpha t_i} + \alpha A_j + \beta^T Z_j\}} \rightsquigarrow \begin{matrix}\alpha = \tilde\alpha \\ \beta = \tilde\beta\end{matrix}$$

PL with interaction

$$\prod_{i=1}^{d} \frac{\exp\{\tilde\alpha A_i + \tilde\beta Z_i + \tilde\gamma A_i Z_i\}}{\sum_{j=1}^{n} Y_j(t_i)\exp\{\tilde\alpha A_j + \tilde\beta Z_j + \tilde\gamma A_j Z_j\}} \quad \text{vs} \quad \prod_{i=1}^{d} \frac{\exp\{\cancel{\alpha t_i} + \gamma t_i Z_i + \alpha A_i + \beta Z_i + \gamma A_i Z_i\}}{\sum_{j=1}^{n} Y_j(t_i)\exp\{\cancel{\alpha t_i} + \gamma t_i Z_j + \alpha A_j + \beta Z_j + \gamma A_j Z_j\}}$$

$\rightsquigarrow$ in general $(\alpha, \beta, \gamma) \neq (\tilde\alpha, \tilde\beta, \tilde\gamma)$

in R: using tt( ) function: `coxph(Surv(time, delta) ~ bili + tt(age), data = pbc)` (time transformation)

in this example yields the same coefficients as `~ bili + age`

however "`~bili * age`" and "`~bili * tt(age)`" do have similar, but not the same estimates of coefficients

Warning: time and age need to be in the same units! Ex: age [years], time [days] $\Rightarrow$ use `~bili*tt(age), tt = function(x, t, ...) {x + t/365.25}` within coxph function