Lecture 5 | 26.03.2024

Linear regression model
(with interactions)

Overview: Multiple regression model

- Mathematical relationship between a continuous dependent variable Y and a set of explanatory (independent) variables $X_1, \dots, X_p$ (may be continuous, binary, or categorical – or any combination)
- Typically expressed for some general function $f : \mathbb{R}^p \longrightarrow \mathbb{R}$ but for the linear regression model we use a more specific notation of the form

$$Y = \beta_0 + \beta_1 X_1 + \beta_{p-1} X_{p-1} + \varepsilon = \boldsymbol{X}^\top \boldsymbol{\beta} + \varepsilon$$

- The corresponding data (empirical) model assumed for a random sample $\{(Y_i, \boldsymbol{X}_i);\ i = 1, \dots, n\}$ drawn from some joint distribution function $F_{(Y, \boldsymbol{X})}$ takes the form

$$Y_i = \boldsymbol{X}_i^\top \boldsymbol{\beta} + \varepsilon_i$$

  for random vectors $\boldsymbol{X}_i = (1, X_{i1}, \dots, X_{i(p-1)})^\top$ where we assume (by default) the presence of the intercept parameter $\beta_0 \in \mathbb{R}$ in the model (in other words, $X_{i0} = 1$ almost surely)

Quantifying the effect of X on Y

- One of the main goals of the regression model (regression analysis in general) is to quantify the effect of some given explanatory variable on the dependent variable Y.
- Formally, the explanatory variable may have an effect on the whole (conditional) distribution of Y... however, we are rather focussing on some simple characteristics instead
- Typical characteristic related to the linear regression model is the conditional mean of Y given X. Therefore, the effect of X on Y is also typically interpreted in terms of the correponding change of the conditional expected value when the value of X changes
- The quantification of the effect may be numerical (in terms of the estimation of the corresponding parameter) or it can be statistical (stochasticin terms of evaluating how important/significant the estimated effect is (or both simultaneously))

Association vs. causality

$\hookrightarrow$ the regression model is typically a model that explains only an associattion (elationship) between two (or more) subpopulations that differ with respect to the value of the explanatory covariate(s)

Association vs. causality

$\hookrightarrow$ the regression model is typically a model that explains only an associattion (elationship) between two (or more) subpopulations that differ with respect to the value of the explanatory covariate(s)

- **Associative interpretation**
  - Comparing two sub-populations that differ wrt to X
  - Interpreting the effect of X in terms of the comparison of two subjects
- **Causal interpretation**
  - Comparing the same sub-population before and after the change
  - Interpreting the effect of X in terms of a change within the subject

$\hookrightarrow$ it is a very common mistake that the associative regression model is (unintentionally) interpreted as a causal model... however, for a causal interpretation we usually need much stricter assumptions (a randomized trial)

Correlation among explanatory variables

- **Ideal scenario**
  - balanced data
  - uncorrelated predictors
  - each coeffcient β_j can be estimated separately
  - interpretation of the estimated coefficients is relatively fixed

- **Typical real situations**
  - unbalanced data
  - correlated predictor variables (multicolinearity)
  - variance of the estimated parameters typically increases
  - the interpretation of the estimated coefficients become vague

$\hookrightarrow$ briefly saying, the estimated parameter β_j stands for a change in the expected (conditional) value of Y which comes with a unit change of X_j covariate, however, with all other predictors being fixed. In practice, the predictor variables typically change simultaneously. variables

Example: Body fat vs. weight and height

- **Body fat vs. person's height**

```
lm(formula = fat ~ height, data = Policie)

Coefficients:
Estimate Std. Error t value Pr(>|t|)
(Intercept) -47.6791    23.9707  -1.989   0.0524 .
height        0.3405     0.1343   2.535   0.0146 *
```

- **Body fat vs. person's weight**

```
lm(formula = fat ~ weight, data = Policie)

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) -20.75217    3.42327  -6.062 2.02e-07 ***
weight        0.42674    0.04266  10.003 2.51e-13 ***
```

What about a multiple model?

- **Body fat vs. person's height and weight**

```
lm(formula = fat ~ height + weight, data = Policie)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 16.55309   15.24621   1.086   0.2831
height      -0.24362    0.09728  -2.504   0.0158 *
weight       0.50418    0.05095   9.896 4.49e-13 ***
```

What about a multiple model?

- **Body fat vs. person's height and weight**

```
lm(formula = fat ~ height + weight, data = Policie)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 16.55309   15.24621   1.086   0.2831
height      -0.24362    0.09728  -2.504   0.0158 *
weight       0.50418    0.05095   9.896 4.49e-13 ***
```

- **What is the estimated effect of the height on the overall body fat?**
- **What is the estimated effect of the weight on the overall body fat?**
- **How well the conclusions correspond among different models?**

What about a multiple model?

- **Body fat vs. person's height and weight**

```
lm(formula = fat ~ height + weight, data = Policie)

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 16.55309   15.24621   1.086   0.2831
height      -0.24362    0.09728  -2.504   0.0158 *
weight       0.50418    0.05095   9.896 4.49e-13 ***
```

- **What is the estimated effect of the height on the overall body fat?**
- **What is the estimated effect of the weight on the overall body fat?**
- **How well the conclusions correspond among different models?**
- The estimated correlation between the weight and height is 0.6068

How to overcome the problems? Interactions!

- **Body fat vs. person's height and weight with the interaction**

```
lm(formula = fat ~ height + weight + height:weight)

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)   -48.604790  87.698149  -0.554    0.582
height          0.123659   0.496447   0.249    0.804
weight          1.324727   1.088637   1.217    0.230
height:weight  -0.004608   0.006106  -0.755    0.454
```

How to overcome the problems? Interactions!

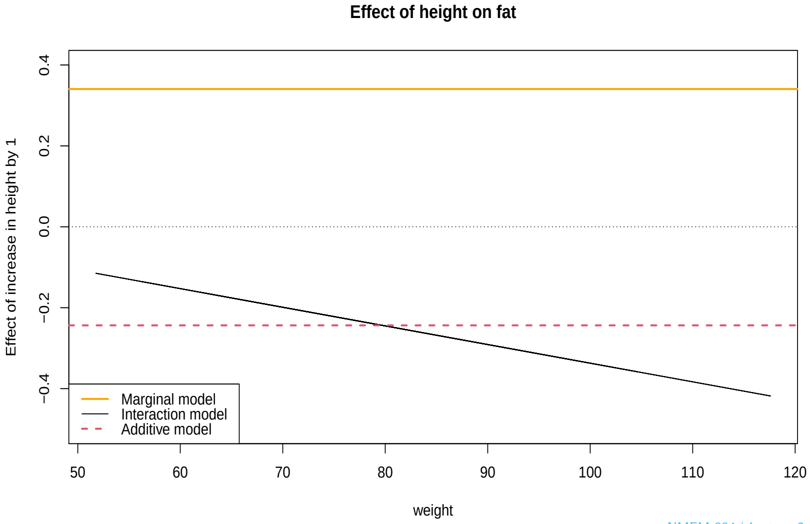
- **Body fat vs. person's height and weight with the interaction**

```
lm(formula = fat ~ height + weight + height:weight)

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)   -48.604790  87.698149  -0.554    0.582
height          0.123659   0.496447   0.249    0.804
weight          1.324727   1.088637   1.217    0.230
height:weight  -0.004608   0.006106  -0.755    0.454
```

- **What is the interaction term? How to explain it?**
- **Is the model good one?**
- **What are the main advantages and disadvantages of the model with interactions?**

Illustration of the models



Regression model with interactions: Formally

- **Implementation in the R software**
  - using the expression `height:weight`
  - using the expression `height * weight`
  - defining new covariate as a product of `height` and `weight`

- **Formulation within a linear regression model**
  - using a regression model expression: $Y \approx \beta_0 + \beta_1 X_h + \beta_2 X_w + \beta_3 X_h X_w$
  - using a new covariate $Y \approx \beta_0 + \beta_1 X_h + \beta_2 X_w + \beta_3 Z$ where $Z = X_h \times X_w$

- **More general formulations and models**
  - effect of height: $Y \approx \beta_0 + (\beta_1 + \beta_3 X_w) X_h + \beta_2 X_w$
  - effect of weight: $Y \approx \beta_0 + (\beta_2 + \beta_3 X_h) X_w + \beta_3 X_h$

Regression model with interactions: Formally

- **Implementation in the R software**
  - using the expression `height:weight`
  - using the expression `height * weight`
  - defining new covariate as a product of `height` and `weight`

- **Formulation within a linear regression model**
  - using a regression model expression: $Y \approx \beta_0 + \beta_1 X_h + \beta_2 X_w + \beta_3 X_h X_w$
  - using a new covariate $Y \approx \beta_0 + \beta_1 X_h + \beta_2 X_w + \beta_3 Z$ where $Z = X_h \times X_w$

- **More general formulations and models**
  - effect of height: $Y \approx \beta_0 + (\beta_1 + \beta_3 X_w) X_h + \beta_2 X_w$
  - effect of weight: $Y \approx \beta_0 + (\beta_2 + \beta_3 X_h) X_w + \beta_3 X_h$

  - parameter β_3 can be seen as a linear function of X_w (or X_h respectively)
  - more generaly, β_3 is a function of X_w (or X_h respectively)
  - thus, we can write $\beta_3(X_w)$ (or $\beta_3(X_h)$ respectively), where $\beta_3 x = cx$

Regression model with interactions: Formally

- **Implementation in the R software**
  - using the expression `height:weight`
  - using the expression `height * weight`
  - defining new covariate as a product of `height` and `weight`

- **Formulation within a linear regression model**
  - using a regression model expression: $Y \approx \beta_0 + \beta_1 X_h + \beta_2 X_w + \beta_3 X_h X_w$
  - using a new covariate $Y \approx \beta_0 + \beta_1 X_h + \beta_2 X_w + \beta_3 Z$ where $Z = X_h \times X_w$

- **More general formulations and models**
  - effect of height: $Y \approx \beta_0 + (\beta_1 + \beta_3 X_w) X_h + \beta_2 X_w$
  - effect of weight: $Y \approx \beta_0 + (\beta_2 + \beta_3 X_h) X_w + \beta_3 X_h$

  - parameter β_3 can be seen as a linear function of X_w (or X_h respectively)
  - more generaly, β_3 is a function of X_w (or X_h respectively)
  - thus, we can write $\beta_3(X_w)$ (or $\beta_3(X_h)$ respectively), where $\beta_3 x = cx$
  - so, is it necessary to stay with the linearity restrictions? What if $\beta(x) = g(x)$ for some general function g?

Regression model with interactions: Formally

- **Implementation in the R software**
  - using the expression `height:weight`
  - using the expression `height * weight`
  - defining new covariate as a product of `height` and `weight`

- **Formulation within a linear regression model**
  - using a regression model expression: $Y \approx \beta_0 + \beta_1 X_h + \beta_2 X_w + \beta_3 X_h X_w$
  - using a new covariate $Y \approx \beta_0 + \beta_1 X_h + \beta_2 X_w + \beta_3 Z$ where $Z = X_h \times X_w$

- **More general formulations and models**
  - effect of height: $Y \approx \beta_0 + (\beta_1 + \beta_3 X_w) X_h + \beta_2 X_w$
  - effect of weight: $Y \approx \beta_0 + (\beta_2 + \beta_3 X_h) X_w + \beta_3 X_h$

  - parameter β_3 can be seen as a linear function of X_w (or X_h respectively)
  - more generaly, β_3 is a function of X_w (or X_h respectively)
  - thus, we can write $\beta_3(X_w)$ (or $\beta_3(X_h)$ respectively), where $\beta_3 x = cx$
  - so, is it necessary to stay with the linearity restrictions? What if $\beta(x) = g(x)$ for some general function g?

$\hookrightarrow$ Thus, when being interested in the effect of height on the overall fat, the other covariate (weight) acts as a **effect modifier** in the model (and vise versa)

When to use a model with interactions?

- **Effect modifier**
  When there is an expectation that the effect of one specific covariate X_j will be different in different sub-populations that we control for in the model by using the remaining covariates
- **Colinearity issues** If the model design is not optimal and there is a belief that some covariates may be correlated (linearly dependent – multicolinearity) then the interaction(s) may help to improve the model
- **Model interpretability** Interactions can be also used just for the purpose of some better model interpretability (despite the fact that mostly interactions make the model interpretability more complex)

When to use a model with interactions?

- **Effect modifier**
  When there is an expectation that the effect of one specific covariate X_j will be different in different sub-populations that we control for in the model by using the remaining covariates
- **Colinearity issues** If the model design is not optimal and there is a belief that some covariates may be correlated (linearly dependent – multicolinearity) then the interaction(s) may help to improve the model
- **Model interpretability** Interactions can be also used just for the purpose of some better model interpretability (despite the fact that mostly interactions make the model interpretability more complex)

Interactions are not necessarily just between to explanatory covariates (so-called double interactions, or first-order interactions). In practice, we can technically use even higher-order intractions between three and more covariates – but they subtantially complicates the interpretability

Simple interpretation of the interaction term

- Consider a simple regression model with one interaction

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 (X_1 \times X_2) + \varepsilon$$

- We are primariliy interested in the effect of X_1 on $E[Y|X_1, X_2]$ thus, we can rewrite the model in the equivalent form

$$Y = \beta_0 + (\beta_1 + \beta_3 X_2) X_1 + \beta_2 X_2 + \varepsilon$$

- To describe the effect of X_1 on $E[Y|X_1, X_2]$ we need to quantify/estimate $(\beta_1 + \beta_3 X_2)$ which, however, depends on the value of X_2 – taking (hypotetically) infinitelly many values Which ones to use?
- For $X_2 = 2$ the effect of X_1 on $E[Y|X_1, X_2]$ only reduces to the quantification/estimation of β_1 Can we somehow achieve this?

Transformations of the covariates

- **Nonlinear transformations**
  many different transformation functions $g \in \mathcal{G}$ can be considered within the regression model

$$Y = \beta_0 + \beta_1 g_1(X_1) + \beta_2 g_2(X_2) + \varepsilon$$

  but different transformations (different choice of $g_1, g_2 \in \mathcal{G}$) change the overall model (its properties, interpretation, etc.) and the models are not directly comparable among each other

- **Linear transformations**
  a very specific class of transformations that preserve most of the model qualitites are of the form $g(x) = a + bx$, i.e.,

$$Y = \beta_0 + \beta_1(a_1 + b_1 X_1) + \beta_2(a_2 + b_2 X_2) + \varepsilon$$

  for $a_1, a_2, b_1, b_2 \in \mathbb{R}$ – models under such transformations are equivalent (if $b_1 \neq 0 \neq b_2$) and can be directly compared among each other...

Linear transformations of the covariates

Typically they are used to

- to improve the stability of the estimated parameters
  (e.g., measuring the distance between Prague and Brno in millimeters/kilometers)
- for better representation of the model outputs
  (mostly using different units, scales, proportions for better visualization)
- to improve the interpretation of the final model
  (typically, we want to have a reasonable interpretation of the intercept and interactions)

Linear transformations of the covariates

Typically they are used to

- to improve the stability of the estimated parameters
  (e.g., measuring the distance between Prague and Brno in millimeters/kilometers)
- for better representation of the model outputs
  (mostly using different units, scales, proportions for better visualization)
- to improve the interpretation of the final model
  (typically, we want to have a reasonable interpretation of the intercept and interactions)

However, it only works with a hierarchically well structured model.

- What is a hierarchically well structured model?
- What are the consequences of a non-hierarchical model?

Model hierarchy

- **Advantages**
  - linear transformations of the covariates does not effect the model
  - different models are better comparable within their hierarchical structure
  - systematic model building procedures are well defined and work well

- **Disadvantages**
  - some models can not be fitted under the restriction of hierarchy
  - models with various irregularities (discontinuous, non-smooth
  - sometimes it is necessary to use a model without the intercept

Model hierarchy

- **Advantages**
  - linear transformations of the covariates does not effect the model
  - different models are better comparable within their hierarchical structure
  - systematic model building procedures are well defined and work well
- **Disadvantages**
  - some models can not be fitted under the restriction of hierarchy
  - models with various irregularities (discontinuous, non-smooth
  - sometimes it is necessary to use a model without the intercept

$\hookrightarrow$ when fitting a linear regression model, we always need to be aware of its structure – whether we are building a model that is hierarchically well formulated or not... and depending on the model we have different tools available for the fitting process and the consecutive inference as well

Summary

- Models with interactions
  - the yhelp to overcome some issues with the covariates
  - the improve the overall flexibility of the model
  - interpretation of the model becomes more challenging
- Linear transformations of the covariates
  - they help with the model stability
  - when used wisely, they improve the interpretability of the model
  - they require a hierarchically well formulated model to work properly
- Hierarchically well formulated model
  - it has its specific advantages and disadvantages
  - inference in a hierarchical model is more straightfoward
  - some practical applications require a non-hierarchical model