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Variational Problems With Linear Growth — Regularity up to the Boundary in Non-Convex Domains

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Abstract. The classical example of a variational problem with linear growth is the minimal surface problem. It is well known that for smooth data such problem possesses a regular (up to the boundary) solution if the domain is convex (or has positive mean curvature). On the other hand, for non-convex domains we know that there always exist data for which the solution does exist only in the space of functions of bounded variations (the desired Dirichlet boundary datum is not attained). Recently, in continuum mechanics there were identified problems (limiting strain) that can be under certain circumstances rewritten as variational problems with linear growth but possibly having different structure than the minimal surface problem. We identify the class of functionals for which we always have regular (up to the boundary) solution in any dimension $d \geq 2$ for arbitrary $C^{1,1}$ domain. Furthermore, we show that this class is sharp in the following sense: whenever the functional does not belong to the class then we can find data for which the $W^{1,1}$ solution does not exist.

Introduction

In the paper we analyse the existence of a solution $U \in W^{1,1}(\Omega)$ to problem

$$-\operatorname{div}\left(\frac{\nabla U}{(1+|\nabla U|^a)^{\frac{1}{a}}}\right) = 0 \ \text{ in } \Omega, \quad U = U_0 \ \text{ on } \partial\Omega, \tag{1}$$

for $\Omega \subset \mathbb{R}^d$ of class $\mathcal{C}^{1,1}$ and the parameter $a > 0$. The special case for $a = 2$ is the well-known minimal surface problem. The problem (1) also has a variational formulation. Define a function $F : \mathbb{R}^d \to \mathbb{R}$,

$$F(\boldsymbol{x}) := \frac{1}{2}\int_0^{|\boldsymbol{x}|^2} \frac{1}{(1+|s|^{\frac{a}{2}})^{\frac{1}{a}}}\,\mathrm{d}s.$$

To find a weak solution to (1) is equivalent to find $U \in W^{1,1}(\Omega)$ such that $U = U_0$ on $\partial\Omega$ and

$$\int_\Omega F(\nabla U)\,\mathrm{d}\boldsymbol{x} \leq \int_\Omega F(\nabla V)\,\mathrm{d}\boldsymbol{x} \tag{2}$$

for all $V \in W^{1,1}(\Omega)$, $V = U_0$ on $\partial\Omega$.

Motivation for studying such problem arises from continuum mechanics. Consider deformation of the body $\Omega \subset \mathbb{R}^3$ with $\Gamma_D \cap \Gamma_N = \emptyset$ and $\overline{\Gamma_D \cup \Gamma_N} = \partial\Omega$ described by the balance of linear momentum and Dirichlet and Neumann boundary conditions,

$$-\operatorname{div}\mathbf{T} = \boldsymbol{f} \ \text{ in } \Omega, \quad \boldsymbol{u} = \boldsymbol{u}_0 \ \text{ on } \Gamma_D, \quad \mathbf{T}\boldsymbol{n} = \boldsymbol{g} \ \text{ on } \Gamma_N, \tag{3}$$

where $\boldsymbol{u}$ is the displacement, $\mathbf{T}$ the Cauchy stress tensor, $\boldsymbol{f}$ the external body forces, $\boldsymbol{g}$ the external surface forces. Let $\boldsymbol{\varepsilon}$ be the linearised strain tensor, i.e.,

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T).$$

To complete the system (3), it remains to prescribe the relation of the Cauchy stress and the displacement gradient or more specifically to the strain tensor defined as $\mathbf{E} := \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$. Note that if $\chi : \Omega \to \mathbb{R}^3$ is a function that describes the deformation then $\boldsymbol{u}(\boldsymbol{x}) := \chi(\boldsymbol{x}) - \boldsymbol{x}$ describes displacement. Now, $\nabla\boldsymbol{u} = \frac{\partial\chi}{\partial\boldsymbol{x}} - \frac{\partial\boldsymbol{x}}{\partial\boldsymbol{x}} = \mathbf{F} - \mathbf{I}$, where $\mathbf{F}$ denotes the deformation gradient and $\mathbf{I}$ is the identity tensor. Therefore, $\mathbf{F} = \nabla\boldsymbol{u} + \mathbf{I}$. Consequently, the strain tensor can be expressed as $\mathbf{E} = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T + (\nabla\boldsymbol{u})^T\nabla\boldsymbol{u})$, which after linearisation with respect to $\nabla\boldsymbol{u}$ ($\nabla\boldsymbol{u}$ is dimensionless and therefore the linearisation is valid provided that $|\nabla u| \ll 1$), can be replaced by $\boldsymbol{\varepsilon}$. It is easy to observe that the condition for linearisation, $|\nabla \boldsymbol{u}| \ll 1$, implies $|\boldsymbol{\varepsilon}| \ll 1$. However, the usual linear models, i.e., models where the Cauchy stress is a linear function of the small strain cannot guarantee that the small strain remains small under a large loading. On the other hand, it is of a real interest to consider models, which admit large stresses but simultaneously the small strain tensor remains small. A class of such models where suggested by *Rajagopal and Walton* [2011] (see also *Kulvait et al.* [2013] for numerical simulations), who considered

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}^*(\mathbf{T}) := \frac{\mathbf{T}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} \tag{4}$$

for a parameter $a > 0$. General existence theory (for weak solutions) for such models was established in *Beck et al.* [2016a], where it was pointed out that the Neumann boundary condition might not be attained (in the sense of distributions) and one has to relax the notion of solution. In this paper we study these models in simplified geometries and we characterize under which assumption on the parameter a such a pathological behaviour does not take place.

Special geometry and equivalent reformulation

First, we consider a very special case of the problem (3). Let $\boldsymbol{f} \equiv \mathbf{0}$ and $\boldsymbol{g}(\boldsymbol{x}) = (0, 0, g(x_1, x_2))$ be given and we look for $\boldsymbol{u}$, $\mathbf{T}$ of the form

$$\boldsymbol{u}(\boldsymbol{x}) = (0, 0, u(x_1, x_2)) \quad \text{and} \quad \mathbf{T}(\boldsymbol{x}) = \begin{pmatrix} 0 & 0 & T_{13}(x_1, x_2) \\ 0 & 0 & T_{23}(x_1, x_2) \\ T_{13}(x_1, x_2) & T_{23}(x_1, x_2) & 0 \end{pmatrix}.$$

Thanks to this we can simplify the problem significantly in the following way. Let $f_{x_i} := \frac{\partial f}{\partial x_i}$. Find a scalar function $U : \Omega \to \mathbb{R}$, $U(\boldsymbol{x}) = U(x_1, x_2)$ such that

$$T_{13} = \frac{1}{\sqrt{2}} U_{x_2} \quad \text{and} \quad T_{23} = -\frac{1}{\sqrt{2}} U_{x_1},$$

then $\operatorname{div} \mathbf{T} = \mathbf{0}$ is fulfilled. Also, it is easy to check that $|\mathbf{T}|^2 = |\nabla U|^2$. Relation (4) reads

$$u_{x_1} = \frac{2T_{13}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}} \quad \text{and} \quad u_{x_2} = \frac{2T_{23}}{(1+|\mathbf{T}|^a)^{\frac{1}{a}}},$$

and the compatibility conditions for $u(x_1, x_2)$ hold, that is, $(u_{x_1})_{x_2} = (u_{x_2})_{x_1}$. Combining all these properties together we realize that U must satisfy

$$-\operatorname{div}\left(\frac{\nabla U}{(1+|\nabla U|^a)^{\frac{1}{a}}}\right) = 0 \ \text{ in } \Omega, \quad U_{x_2}\boldsymbol{n}_1 - U_{x_1}\boldsymbol{n}_2 = \sqrt{2}g \ \text{ on } \Gamma_N.$$

Furthermore, if Ω is simply connected, the existence of a solution U to this problem is equivalent to the existence of a solution $\boldsymbol{u}$ to the original three dimensional problem in the simplified geometry.

Finally, note that the Neumann boundary condition includes the tangential derivative of U, $(U_{x_1}, U_{x_2}) \cdot (-\boldsymbol{n}_2, \boldsymbol{n}_1) = \sqrt{2}g$. If we assume that $\partial\Omega$ is parameterized by a curve $\gamma(s)$, then defining U_0 as

$$U(\gamma(\tau)) = U(\gamma(0)) + \sqrt{2}\int_0^\tau g(\gamma(s))\ |\gamma'(s)|\ \mathrm{d}s =: U_0(\boldsymbol{x})$$

for $\boldsymbol{x} = \gamma(\tau)$ makes the problem Dirichlet. After all, we look for U, a weak solution to (1).

Existence of solution

The text above can serve as a motivation for how the equations in (1) can be obtained. However, now the problem can be viewed at as purely mathematical — let the dimension be arbitrary d ($d \geq 2$) and let the domain $\Omega \subset \mathbb{R}^d$ be arbitrary of the class $\mathcal{C}^{1,1}$. In what follows , we are interested in the investigation how the solution depends on the function space in which we look for the weak solution, the shape of the domain and the parameter a.

The relation between the function space and the parameter a has been studied before for Lipschitz domain (i.e., of class $\mathcal{C}^{0,1}$) by *Bildhauer and Fuchs* [2002a,b]. The result says that the solution in the space $BV(\Omega)$ of functions of bounded variation exists for any $a > 0$. Moreover, if $a \in (0, 2]$, then this solution is locally (in Ω) Lipschitz, however, in any case the trace U_0 need not be attained. In particular if $a = 2$, it was shown already by *Finn* [1965] that if Ω is not a pseudoconvex domain then there always exists the Dirichlet data that are not attained by the BV minimizer. On the other hand, for pseudoconvex domains, it is known due to result of *Miranda* [1979] that for sufficiently smooth data there always exists a classical solution (so attaining the Dirichlet data).

In what follows, we want to look for a solution in the space $W^{1,1}(\Omega)$. To be more precise, we shall study what the domain Ω and the parameter a should fulfil to guarantee the existence of such solution (in particular we want to cover the cases of non-convex domains). In fact, there already are some positive results introduced by *Bulíček et al.* [2015], where the existence of a weak solution is proven for $a \in (0, \infty)$ and Ω Lipschitz uniformly convex and for $a \in (0, 2)$ and Ω Lipschitz piece-wise uniformly convex (i.e., some kind of non-convexity is allowed).

Note that the second result for $a \in (0, 2)$ no longer includes the minimal surface problem (where $a = 2$). However, natural question arises whether for some values of a the existence could be shown for *all* $\mathcal{C}^{1,1}$ domains — even those fully non-convex. Simple answer is yes, it could. Yet, precise answer is rather complicated and will be partially provided below. Unfortunately, providing the complete proof is beyond the size of this paper.

Let us consider the simplest non-convex domain which cannot be divided into convex pieces (and therefore is not the case of the result for $a \in (0, 2)$) — the annulus $\Omega := B_R \setminus B_r \subset \mathbb{R}^d$ centered at the origin,

$$-\operatorname{div}\left(\frac{\nabla U}{(1 + |\nabla U|^a)^{\frac{1}{a}}}\right) = 0 \quad \text{in } B_R \setminus B_r, \quad U = 0 \quad \text{on } \partial B_r, \quad U = K \quad \text{on } \partial B_R, \tag{5}$$

for $0 < r < R$. We demand the solution to attain the boundary value for any $K \in \mathbb{R}$.

Lemma 1 (Existence on (arbitrary) annulus)**.** *For $a > 1$, the problem* (5) *has a weak solution in $W^{1,1}(B_R \setminus B_r)$ if and only if*

$$|K| \leq \int_r^R \frac{r^{d-1}}{\left(z^{a(d-1)} - r^{a(d-1)}\right)^{\frac{1}{a}}} \, dz. \tag{6}$$

If $a \in (0, 1]$, then for any $K \in \mathbb{R}$ there exists a weak solution to problem (5)*.*

Proof. First of all, we show that U, a weak solution to (5), if exists, is independent of rotation. That is, there exists a function $\tilde{U} : [r, R] \to \mathbb{R}$ such that $U(\boldsymbol{x}) = \tilde{U}(|\boldsymbol{x}|)$. To prove this, we need to show that for an arbitrary rotation matrix $\mathbf{Q} \in \mathbb{R}^{d \times d}$ (i.e., $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$ and $\det \mathbf{Q} = 1$), function $V : \Omega \to \mathbb{R}$, defined as

$$V(\boldsymbol{x}) := U(\mathbf{Q}\boldsymbol{x}),$$

is also a weak solution to (5). Obviously, $V \in W^{1,1}(\Omega)$. Also, rotation does not change values on the boundary since both B_r and B_R are balls centered at origin and the boundary values are constants. Therefore $V = 0$ on ∂B_r and $V = K$ on ∂B_R.

The change of variables $\boldsymbol{y} = \mathbf{Q}^T\boldsymbol{x}$ does not change the shape of the domain Ω. Consider the test function $\psi \in \mathcal{D}(\Omega)$ (i.e., smooth compactly supported in Ω) such that $\psi(\boldsymbol{x}) := \varphi(\mathbf{Q}^T\boldsymbol{x})$ for some $\varphi \in \mathcal{D}(\Omega)$. Then for the i-th component of $\nabla\psi$ it holds that

$$[\nabla\psi(\boldsymbol{x})]_i = \sum_{j=1}^{d} \left[\nabla\varphi(\mathbf{Q}^T\boldsymbol{x})\right]_j \mathbf{Q}^T_{ji} = \left[\mathbf{Q}\nabla\varphi(\mathbf{Q}^T\boldsymbol{x})\right]_i.$$

Similarly, $\nabla V(\boldsymbol{x}) = \mathbf{Q}^T\nabla U(\mathbf{Q}\boldsymbol{x})$. In the following calculation we use both these facts. Before that, we multiply the first equation in (5) by ψ, integrate over Ω and integrate by parts,

$$\int_\Omega \frac{\nabla U(\boldsymbol{x})}{(1 + |\nabla U(\boldsymbol{x})|^a)^{\frac{1}{a}}} \cdot \nabla\psi(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = 0. \tag{7}$$

After the change of coordinates $\boldsymbol{y} = \mathbf{Q}^T\boldsymbol{x}$ and the use of the definition of ψ we get

$$\int_\Omega \frac{\nabla U(\mathbf{Q}\boldsymbol{y})}{(1 + |\nabla U(\mathbf{Q}\boldsymbol{y})|^a)^{\frac{1}{a}}} \cdot (\mathbf{Q}\nabla\varphi(\boldsymbol{y})) \,\mathrm{d}\boldsymbol{y} = 0. \tag{8}$$

Finally, we express the last integral in term of V. Since

$$\mathbf{Q}\nabla U \cdot \mathbf{Q}\nabla V = \sum_{i,j,k=1}^{d} \mathbf{Q}_{ij} U_{x_j} \cdot \mathbf{Q}_{ik} V_{x_k} = \sum_{j,k=1}^{d} \delta_{jk} U_{x_j} V_{x_k} = \nabla U \cdot \nabla V$$

(δ_{jk} denotes the Kronecker delta). Therefore $\mathbf{Q}$ will vanish in the denominator in the element $|\nabla V(\boldsymbol{y})|^a$, as

$$|\mathbf{Q}\nabla V(\boldsymbol{y})|^a = \left(|\mathbf{Q}\nabla V(\boldsymbol{y})|^2\right)^{\frac{a}{2}} = \left(|\nabla V(\boldsymbol{y})|^2\right)^{\frac{a}{2}} = |\nabla V(\boldsymbol{y})|^a,$$

and the same happens in the product $\mathbf{Q}\nabla V(\boldsymbol{y}) \cdot \mathbf{Q}\nabla\varphi(\boldsymbol{y})$. Consequently, (8) reduces to

$$\int_\Omega \frac{\nabla V(\boldsymbol{y})}{(1+|\nabla V(\boldsymbol{y})|^a)^{\frac{1}{a}}} \cdot \nabla\varphi(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = 0.$$

which identifies V to be a weak solution to (5). Moreover, due to the strict monotonicity of F' we know that the possible weak solution to (5) is unique and therefore $U(\boldsymbol{x}) = V(\boldsymbol{x}) = U(\mathbf{Q}\boldsymbol{x})$. Since the rotation matrix $\mathbf{Q}$ was chosen arbitrarily, this proves that $U(\boldsymbol{x}) = \tilde{U}(|\boldsymbol{x}|)$.

Then, $\nabla U(|\boldsymbol{x}|) = \tilde{U}'(|\boldsymbol{x}|)\frac{\boldsymbol{x}}{|\boldsymbol{x}|}$ and $|\nabla U|^a = |\tilde{U}'|^a$. Similarly, for any function $g \in \mathcal{D}([r,R])$ we set $\psi(\boldsymbol{x}) := g(|\boldsymbol{x}|)$ in (7). Noticing that $\nabla\psi(|\boldsymbol{x}|) = g'(|\boldsymbol{x}|)\frac{\boldsymbol{x}}{|\boldsymbol{x}|}$ and using it in (7) we are led to

$$\int_\Omega \frac{\tilde{U}'(|\boldsymbol{x}|)}{\left(1+|\tilde{U}'(|\boldsymbol{x}|)|^a\right)^{\frac{1}{a}}} g'(|\boldsymbol{x}|)\,\mathrm{d}\boldsymbol{x} = 0 \quad\Leftrightarrow\quad H_d \int_r^R \frac{s^{d-1}\tilde{U}'(s)}{\left(1+|\tilde{U}'(s)|^a\right)^{\frac{1}{a}}} g'(s)\,\mathrm{d}s = 0,$$

where H_d is Hausdorff measure of the unit sphere in $\mathbb{R}^d$. Therefore,

$$\frac{\tilde{U}'(s)}{\left(1+|\tilde{U}'(s)|^a\right)^{\frac{1}{a}}} = \frac{c}{s^{d-1}} \tag{9}$$

for some constant c and all $s \in [r,R]$. From (9) we know that $\tilde{U}'$ does not change the sign, that $\operatorname{sgn}(\tilde{U}') = \operatorname{sgn} c = \operatorname{sgn} \tilde{U} = \operatorname{sgn} K$ and that $|c| < r^{d-1}$. The fact that $\tilde{U}'$ is of the same sign as $\tilde{U}$ is an easy observation from the formulation of the problem. Also, after a simple manipulation one gets from (9) that

$$\tilde{U}'(s) = \frac{c}{\left(s^{a(d-1)} - |c|^a\right)^{\frac{1}{a}}}. \tag{10}$$

Therefore,

$$\tilde{U}(s) = \int_r^s \frac{c}{\left(z^{a(d-1)} - |c|^a\right)^{\frac{1}{a}}}\,\mathrm{d}z \tag{11}$$

and

$$|K| = |\tilde{U}(R)| = \int_r^R \frac{|c|}{\left(z^{a(d-1)} - |c|^a\right)^{\frac{1}{a}}}\,\mathrm{d}z \le \int_r^R \frac{r^{d-1}}{\left(z^{a(d-1)} - r^{a(d-1)}\right)^{\frac{1}{a}}}\,\mathrm{d}z.$$

The last inequality is legitimate thanks to the fact that the function $|c|/(z^{a(d-1)} - |c|^a)^{\frac{1}{a}}$ is increasing in $|c| \in [0, r^{d-1})$. Moreover, it gives the equivalent condition (see (6)) for the solvability of the problem (5), as the last term is also the lowest upper bound for $|K|$, since we can get arbitrarily close to it by a corresponding choice of the constant c. The question is, for which values of parameter a this term is bounded and for which it is not. Hence to discuss the finiteness of the integral, we use the sequence of the following approximations

$$|K| \le \int_r^R \frac{1}{\left(\left(\frac{z}{r}\right)^{a(d-1)} - 1\right)^{\frac{1}{a}}}\,\mathrm{d}z = \int_r^R \frac{1}{\left(e^{a\ln\left(\frac{z}{r}\right)^{d-1}} - 1\right)^{\frac{1}{a}}}\,\mathrm{d}z \sim \int_r^R \frac{1}{\left(\ln\left(\frac{z}{r}\right)\right)^{\frac{1}{a}}}\,\mathrm{d}z$$

$$\sim \int_1^{\frac{R}{r}} \frac{1}{(\ln w)^{\frac{1}{a}}}\,\mathrm{d}w \sim \int_1^{\frac{R}{r}} \frac{1}{(w-1)^{\frac{1}{a}}}\,\mathrm{d}w \sim \int_0^{\frac{R}{r}-1} \frac{1}{t^{\frac{1}{a}}}\,\mathrm{d}t.$$

Evidently, the last integral is finite if and only if $a > 1$. This completes the proof. $\square$

It is worth noticing that there were no further restrictions on r nor R but the natural one $0 < r < R$. Therefore, this lemma proves existence of solution on any annulus in $\mathbb{R}^d$. Also, not only we know that the solution exists, but (11) gives us the precise formula for it. Moreover, from (10) one can see that the closer $|c|$ is to r, the larger is $|\tilde{U}'(r)|$ (and $|K|$). This information is crucial in the proof of the following result.

Theorem 1 (Existence on general domain). *For any domain $\Omega \subset \mathbb{R}^d$ of class $\mathcal{C}^{1,1}$, boundary condition $U_0 \in \mathcal{C}^{1,1}(\partial\Omega)$ and $a \in (0,1]$, there exists $U \in \mathcal{C}^{0,1}(\overline{\Omega})$ a solution to problem* (1).

The proof is not presented here, however, is included in the work *Beck et al.* [2016b], where the minimization problem (2) is studied in detail for a broad class of convex functions F and not only those presented in this paper. The result is formulated in the following theorem.

Theorem 2. *Let $F \in \mathcal{C}^2(\mathbb{R}^+)$ be a strictly convex function with $\lim_{s\to 0} F'(s) = 0$ which satisfies, for some constants $C_1, C_2 > 0$,*

$$\begin{aligned} C_1 s - C_2 \le F(s) &\le C_2(1+s) && \text{for all } s \in \mathbb{R}^+, \\ \frac{F''(s)}{F''(t)} &\le C_2 && \text{for all } s \ge 1 \text{ and } t \in [s/2, 2s]. \end{aligned}$$

Then the following statements are equivalent:

1. *For arbitrary domains Ω of class $\mathcal{C}^1$ satisfying the exterior ball condition and arbitrary prescribed boundary values $u_0 \in \mathcal{C}^{1,1}(\overline{\Omega})$ there exists a unique function $u \in \mathcal{C}^{0,1}(\overline{\Omega})$ solving* (2).
2. *The function F satisfies*
$$\int_1^\infty tF''(t)\,\mathrm{dt} = \infty.$$

Remark 1. *We define the exterior ball condition as: there exists a number $r_0 > 0$ such that for every point $\boldsymbol{x}_0 \in \partial\Omega$ there is a ball $B_{r_0}(\boldsymbol{x}_1)$ with $\overline{B_{r_0}(\boldsymbol{x}_1)} \cap \overline{\Omega} = \{\boldsymbol{x}_0\}$.*

Convexity or $\mathcal{C}^{1,1}$ regularity of the domain are sufficient for the exterior ball condition, thus, the theorem holds for all convex domains of class $\mathcal{C}^1$ or for arbitrary domains of class $\mathcal{C}^{1,1}$.

Remark 2. *Similar proof would work with $\mathcal{C}^{0,1}$ domains which are piece-wise $\mathcal{C}^{1,1}$ as well; except from the corner points of the boundary, to which one can not attach the ball — hence we control the trace up to the corner points, which is, however, the set of zero $(d-1)$ measure. Although, this solution is the weak one and belongs to the space $W^{1,1}(\Omega)$.*

This result is achieved via the so-called method of barrier functions denoted U^b and U_b, which are super- and sub-solution to (1), respectively.

Lemma 2 (Existence of barrier functions). *Let $\Omega \subset \mathbb{R}^d$ be of the class $C^{1,1}$ with $a_i \in \mathcal{C}^{1,1}(\mathbb{R}^{d-1})$, $i = 1, \ldots, N$, corresponding parameterizations of the boundary. Let $U \in L^\infty(\Omega)$, $\|U\|_\infty =: M$ and $U_0 \in \mathcal{C}^{1,1}(\partial\Omega)$ such that $U = U_0$ on $\partial\Omega$.*

Then there exist $\varepsilon > 0$ and $r_{max} > 0$ such that for every $\boldsymbol{x}_0 \in \partial\Omega$ and for every $r < r_{max}$ there exist functions $U^b, U_b : \mathbb{R}^d \to \mathbb{R}$ such that for $\mathcal{H}(\boldsymbol{x}_0) := \mathcal{U}_\varepsilon(\boldsymbol{x}_0) \cap \partial\Omega$ for $\mathcal{U}_\varepsilon(\boldsymbol{x}_0)$ an ε-neighbourhood of $\boldsymbol{x}_0$,

$$\begin{aligned} -\operatorname{div}\left(\frac{\nabla U^b}{(1+|\nabla U^b|^a)^{\frac{1}{a}}}\right) &> 0 \quad \text{in } \Omega, & -\operatorname{div}\left(\frac{\nabla U_b}{(1+|\nabla U_b|^a)^{\frac{1}{a}}}\right) &< 0 \quad \text{in } \Omega, \\ U^b &> U_0 \quad \text{on } \mathcal{H}(\boldsymbol{x}_0) \setminus \{\boldsymbol{x}_0\}, & U_b &< U_0 \quad \text{on } \mathcal{H}(\boldsymbol{x}_0) \setminus \{\boldsymbol{x}_0\}, \\ U^b &\ge M \quad \text{on } \partial\Omega \setminus \mathcal{H}(\boldsymbol{x}_0), & U_b &\le M \quad \text{on } \partial\Omega \setminus \mathcal{H}(\boldsymbol{x}_0). \end{aligned}$$

Moreover, $U_b(\boldsymbol{x}_0) = U_0(\boldsymbol{x}_0) = U^b(\boldsymbol{x}_0)$.

In order for U (a solution to (1)) to be Lipschitz, we need the boundedness of the gradient. In the tangential direction it is bounded thanks to attaining the Dirichlet boundary condition, in the normal direction thanks to existence of barriers.

Conclusion

In the paper, we briefly introduced the limiting strain problems from continuum mechanics. In very special case we presented how the problem can be simplified to scalar one. We cited the known existence results and explained how this one differs from them. According to function space, the existence in the weaker sense has been known for spaces $BV(\Omega)$ and $W^{1,1}_{loc}(\Omega)$ (for respective values of parameter a), however, with non-attainment of the trace. On the other hand, we often need the solution to have better properties. Therefore, we studied what properties of the domain and the parameter would guarantee the existence of a solution $U \in W^{1,1}(\Omega)$, $U = U_0$ on $\partial\Omega$. In this direction, we were inspired by the previous work of *Bulíček et al.* [2015] and improved the result in the natural way, i.e., for smaller interval of admissible values of a we showed existence of solution on more general domains. That is, domains with no restriction on convexity. Despite not being able to provide here the proof of this result in its full completeness, we have given the reference on the submitted publication that shall accomplish this mission even in a more general way.

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References

Beck, L., Bulíček, M., and Málek, J. Süli, E., On the existence of integrable solutions to nonlinear elliptic systems and variational problems with linear growth, preprint MORE/2016/01, 2016a.

Beck, L., Bulíček, M., and Maringová, E., Globally lipschitz minimizers for variational problems with linear growth, preprint MORE/2016/17, arXiv:1609.07601, 2016b.

Bildhauer, M. and Fuchs, M., Relaxation of convex variational problems with linear growth defined on classes of vector-valued functions, *Algebra i Analiz*, *14*, 26–45, 2002a.

Bildhauer, M. and Fuchs, M., On a class of variational integrals with linear growth satisfying the condition of μ-ellipticity, *Rend. Mat. Appl. (7)*, *22*, 249–274 (2003), 2002b.

Bulíček, M., Málek, J., Rajagopal, K. R., and Walton, J., Existence of solutions for the anti-plane stress for a new class of "strain-limiting" elastic bodies, *Calc. Var. Partial Differential Equations*, *54*, 2115–2147, 2015.

Finn, R., Remarks relevant to minimal surfaces, and to surfaces of prescribed mean curvature, *J. Analyse Math.*, *14*, 139–160, 1965.

Kulvait, V., Málek, J., and Rajagopal, K., Anti-plane stress state of a plate with a V-notch for a new class of elastic solids, *Int. J. Fract.*, *179*, 59–73, 2013.

Miranda, M., Un principio di massimo forte per le frontiere minimali e una sua applicazione alla risoluzione del problema al contorno per l'equazione delle superfici di area minima, *Rend. Sem. Mat. Univ. Padova*, *45*, 355–366, 1979.

Rajagopal, K. R. and Walton, J., Modeling fracture in the context of strain-limiting theory of elasticity, *Int. J. Fract.*, *169*, 39–48, 2011.