Criteria for the exam and the tutorial

Partial differential equations 2

Summer semester 2023/2024

Homework: You will get five homework. To pass the tutorial, you are supposed to solve all of them - not necessarily perfectly but at least in a reasonable way. The deadline for homework will be specified during the semester. In addition, the score from the last homework will play a role during the exam, see below.

Exam: The exam will be only written. In some exceptional cases there can be also an oral exam.
The exam consists of two parts. The first part **A** will be theoretical - typically the proof of a lemma or a theorem presented during the semester. The second part **B** will be "practical", i.e., you should be able to use your knowledge to solve some problem. In part **B** you **must** provide at least the knowledge of the notion of weak solution otherwise you do **not** pass.

The evaluation will be the following: you can obtain $0-100\%$ from each part **A**, **B** and additionally you will obtain a score **C** from your last homework and it can have values from -10% up to $+10\%$. The final evaluation is the following:

$$E := \begin{cases} 0 & \text{if } B < 10 \text{ or } C = -10, \\ \dfrac{A+B}{2} + C & \text{if } B \geq 10 \text{ and } C \geq -10. \end{cases}$$

The corresponding mark M is then obtained from the following:

$$M := \begin{cases} 1 & \text{if } E \in (85, 100], \\ 2 & \text{if } E \in (67, 85], \\ 3 & \text{if } E \in (50, 67], \\ 4 & \text{if } E \leq 50. \end{cases}$$

What can be in the exam:

Part A: Proofs of Trace theorem, embedding theorem, Poincaré inequality, density of smooth functions, extension theorem, Aubin–Lions lemma, Minty method, Weak-lower semicontinuity of convex functionals, Nemytskii operator, Euler–Lagrange equations, primary formulation, dual formulation and application of the above mentioned tools to theoretical aspects of PDEs.

Part B: An example:

$$-\Delta_p u + v \cdot \nabla u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

Define the notion of a weak solution. Under which assumptions on f and v can you say something about the existence and the uniqueness of a weak solution? (if you use some theorem, formulate it precisely and check all assumptions) Consider also the case when you control div v from below/above/in a suitable norm. Can you provide a "sharp" bound on $\|v\|_\infty$ for which you can get the existence of a solution?

Another example: Let $\Omega := (-1,1)^2$. Define $\Omega_1 := (-1,1) \times (0,1)$ and $\Omega_2 := (-1,1) \times (-1,0)$. Define $a(x) = i$ in Ω_i. Take $u_0 \in \mathcal{C}^\infty(\overline{\Omega})$ and consider the following problem

$$-\text{div } \left(a(x)(1 + |\nabla u(x)|^2)^{\frac{p-2}{2}} \nabla u(x)\right) = 0 \text{ in } \Omega, \qquad u = u_0 \text{ on } \partial\Omega.$$

Define the notion of weak solution, prove its existence and uniqueness. Is u a minimizer to some variational problem? Can you find a dual formulation? Is $u \in W^{2,2}_{loc}(\Omega_i)$? Is $u \in W^{2,2}_{loc}(\Omega)$? Can you say something about $\nabla u(x_1, 0)$?

Another example: Let $\Omega \subset \mathbb{R}^d$ be a $\mathcal{C}^{0,1}$ domain and $T > 0$. Consider the following problem

$$\partial_t u - \Delta_p u + e^u = f\} \text{ in } (0,T) \times \Omega, \qquad u(t,x) = x + t \text{ on } (0,T) \times \partial\Omega, \qquad u(0) = u_0.$$

Define the notion of weak solution (also proper function spaces for u_0 and f), prove its existence and uniqueness. Find the optimal (=minimal) assumptions such that the weak solution satisfies $\partial_t u \in L^2_{loc}(0,T; L^2(\Omega))$ and/or $\partial_t u \in L^2(0,T; L^2(\Omega))$.