HOMEWORK II

Problem 6: Consider general boundary condition for the parabolic equation

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, $T > 0$ be given and denote $Q := (0,T) \times \Omega$. Assume that $\mathbb{A} \in L^\infty(Q; \mathbb{R}^{d\times d}_{sym})$ be elliptic matrix and $f \in L^2(0,T; L^2(\Omega))$, $b \in L^2(0,T; L^\infty(\partial\Omega))$ and $g \in L^2(0,T; L^2(\partial\Omega))$ be given. Consider the problem

$$\begin{aligned} \partial_t u - \operatorname{div}(\mathbb{A}\nabla u) &= f && \text{in } Q, \\ \mathbb{A}\nabla u \vec{\nu} + bu &= g && \text{on } \Gamma := (0,T) \times \partial\Omega, \\ u(0,x) &= u_0(x) && \text{in } \Omega, \end{aligned}$$

where $u_0 \in L^2(\Omega)$.

GOAL 1: Define a notion of a weak solution for general setting. Assume that $b \geq 0$ and prove the existence and the uniqueness of the weak solution.

GOAL 2: Assume that $b \geq \varepsilon > 0$ a.e. on Γ and $f \in L^2_{loc}(0,\infty; L^2(\Omega))$, $b \in L^2_{loc}(0,\infty; L^\infty(\partial\Omega))$, $g \in L^2_{loc}(0,\infty; L^2(\partial\Omega))$ and satisfies for some $\tau > 0$ that $f(t,x) = f(t+\tau,x)$ for almost all $(t,x) \in (0,\infty) \times \Omega$ and $g(t,x) = g(t+\tau,x)$ and $b(t,x) = b(t+\tau,x)$ for almost all $(t,x) \in (0,\infty) \times \partial\Omega$, i.e. f, b and g are time periodic with the period τ. Show that there exists unique $u_0 \in L^2(\Omega)$, for which the weak solution u satisfies $u(t,x) = u(t+\tau,x)$, i.e. there is a unique initial data for which there is unique periodic solution.

GOAL 3: Improve the result of the GOAL 1 and prove the existence and the uniqueness **without** the assumption $b \geq 0$ and for arbitrary $f \in L^1(0,T; L^2(\Omega))$ and $g \in L^{\frac{4}{3}}(0,T; L^2(\partial\Omega))$. Then consider $f = 0$, $g = 1$, $b = 1$ and look for the behaviour of $u(t)$ as $t \to \infty$.

DEADLINE: January 8

Hint: Please, do everything rigorously! For Goal 1, follow the lecture, and be inspired by the very similar case we had in the elliptic setting. Do not forget to formulate proper function spaces, proper Gelfand triple, etc. Please, do not repeat all steps from the lecture - repeat/redo them only in case there are essential differences. (Maybe there is different Poincré inequality, etc.)

For Goal 2, consider the mapping $F : L^2(\Omega) \to L^2(\Omega)$ prescribed as $F(u_0) := u(\tau)$, where $u(\tau)$ is a weak solution at time τ corresponding the initial data u_0. Show that F is a contraction and then be little bit creative.

For Goal 3 and the existence part, use the inequality

$$\|u\|^2_{L^2(\partial\Omega)} \leq C\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}.$$

to estimate the boundary term. The above inequality prove at least for the domains with flat boundaries.

In the second part, you should derive the estimate

$$\|u(t) - 1\|^2_2 \leq \|u_0 - 1\|^2_2 e^{-\alpha t} \quad \text{for some } \alpha.$$

Problem 2: Prove rigorously the finite speed of propagation of weak solution to linear hyperbolic equations of the second order.

Let $\Omega \subset \mathbb{R}^d$ be an open set fulfilling $B_1(0) \subset \Omega$. Assume that $\mathbb{A} \in L^\infty(\Omega; \mathbb{R}^{d\times d}_{sym})$ be elliptic and that u is a weak solution to

$$\partial_{tt} u - \operatorname{div}(\mathbb{A}\nabla u) = 0 \qquad \text{in } Q := (0,T)\times\Omega,$$

i.e.,

$$u \in L^2(0,T; W^{1,2}(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega)) \cap W^{2,2}(0,T; (W^{1,2}_0(\Omega))^*) \tag{1.1}$$

satisfies for almost all $t \in (0,T)$ and all $w \in W^{1,2}_0(\Omega)$

$$\langle \partial_{tt} u, w\rangle + \int_\Omega \mathbb{A}\nabla u \cdot \nabla w = 0 \tag{1.2}$$

GOAL: Find proper/optimal relation[1] between $\Omega_0 \subset B_1(0)$ and $Q_0 \subset Q$ such that the following implication holds true

$$\boxed{u(0) = \partial_t u(0) = 0 \text{ in } \Omega_0 \quad \Longrightarrow \quad u = 0 \text{ in } Q_0.}$$

Subgoal1 (obligatory): Show the result for constant matrix $\mathbb{A}$.
Subgoal2 (not obligatory but recommended and can improve your final mark): Show it for general $\mathbb{A}$.

DEADLINE: three days before exam

Hint: Please, do everything rigorously, i.e., assume just (1.1)–(1.2). For the first subgoal, I would suggest two options. Either follow the lecture and on the places where we used $|x|$ use a different norm in $\mathbb{R}^d$, that will depend on $\mathbb{A}$, or try to find a linear transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ such that the function $\tilde{u}(t,x) := u(t,Tx)$ is a weak solution to the classical wave equation and then use the result of the lecture.

For the second subgoal: Try to prove everything formally and then try to justify it rigorously. The formal procedure can be: Multiply (1.1) by $\partial_t u$ to get

$$\frac{1}{2}\partial_t(|\partial_t u|^2 + \mathbb{A}\nabla u \cdot \nabla u) - \operatorname{div}(\mathbb{A}\nabla u \partial_t u) = 0$$

Integrate the result over Q_0 and then use the integration by parts formula (in $\mathbb{R}^{d+1}$) to create an integral over ∂Q_0. Try to find a proper shape of the boundary ∂Q_0 such that this procedure formally leads to $\partial_t u$ and ∇u are zero on ∂Q_0. This will then help you to show the desired result. But you should do everything rigorously!

[1] This relation depends essentially on the structure of $\mathbb{A}$