HOMEWORK: PART I

Rules: This is the first set of problems you must solve in order to pass the tutorial. Below you find five Problems that should be solved. Solution to each Problem is supposed to be uploaded into SIS. Deadline for upload is emphasized after each Problem formulation and the deadline **is sharp**.

Problem 1: a) Show that for any Lipschitz domain $\Omega \subset \mathbb{R}^d$ with $d \geq 2$ the embedding $W^{1,d}(\Omega) \hookrightarrow L^\infty(\Omega)$ does not hold.
b) Show that if $u \in W^{1,d}(\Omega)$ then it has bounded mean oscillations, i.e., for any $q \in [1, \infty)$, there exists a constant C such that for all balls $B_R(x_0) \subset \Omega$, we have

$$\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} \left| u(x) - \frac{1}{|B_R(x_0)|} \left(\int_{B_R(x_0)} u(y)\, dy \right) \right|^q dx \leq C(q,d) \|\nabla u\|^q_{L^d(\Omega)}.$$

It is important to emphasize that C is independent of x_0 and R! The space of functions having bounded mean oscillation is denoted by BMO, or sometimes John–Nirenberg spaces.

Hint: a) Let $x_0 \in \Omega$. Consider a function $u(x) := f(|x - x_0|)$. Find proper function f (it should satisfy $f(s) \to \infty$ as $s \to 0_+$), for which one can prove the counterexample, i.e., for which $u \in W^{1,d}$.
b) Fix $q \in [1, \infty)$. Mimic the proof of the Poincaré inequality and show the statement for a fixed ball $B_1(0)$ - trace the constant C! Re-scale the result to a general ball $B_R(x_0)$.

DEADLINE: November 6

Solution: a) Let us assume that $x_0 \in \Omega$. We consider the function $u(x) := \ln\ln(1 + |x - x_0|^{-1})$. This function is smooth except the point x_0 and in addition, we see that $u \notin L^\infty(\Omega)$. On the other hand we show that $u \in W^{1,d}(\Omega)$. For simplicity, we consider $x_0 = 0$. For $x \neq 0$, we also have

$$\nabla u(x) = -\frac{x}{|x|^2(|x|+1)\ln(1 + |x|^{-1})}.$$

Next, we need to show that ∇u is really a weak derivative (gradient), that $|\nabla u| \in L^d$ and that also $u \in L^d$. For the integrability claim, we may compute - note that since Ω is Lipschitz, we surely know that $\Omega \subset B_R(0)$, where B_R is a ball in $\mathbb{R}^d$

$$\begin{aligned}\int_\Omega |u|^d + |\nabla u|^d \,\mathrm{d}x &\leq \int_{B_R} |\ln\ln(1 + |x|^{-1})|^d + \frac{1}{|x|^d |\ln(1 + |x|^{-1})|^d} \,\mathrm{d}x \\ &\leq C(d) \int_0^R r^{d-1} |\ln\ln(1 + |r|^{-1})|^d + \frac{1}{|r| \ln(1 + r^{-1})|^d} \,\mathrm{d}r < \infty.\end{aligned}$$

Thus, it remains to show that ∇u is the weak derivative. To show that let us consider $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(0) \subset \Omega$. Then for all $\varepsilon \in (0, \varepsilon_0)$, we have $B_\varepsilon(0) \subset \Omega$ as well. Then for any $\varphi \in \mathcal{C}_0^\infty(\Omega)$, we may compute (considering only sufficiently small ε and by using the Lebesgue dominated convergence theprem and the fact that u and $|\nabla u|$ are integrable and the classical integration by parts formula for smooth functions on $\Omega \setminus B_{|varepsilon}$)

$$\int_\Omega \frac{\partial \varphi}{\partial x_i} u = \lim_{\varepsilon \to 0_+} \int_{\Omega \setminus B_\varepsilon} \frac{\partial \varphi}{\partial x_i} u = - \lim_{\varepsilon \to 0_+} \int_{\Omega \setminus B_\varepsilon} \frac{\partial u}{\partial x_i} \varphi - \int_{\partial B_\varepsilon} \frac{x_i}{|x|} u\varphi = - \int_\Omega \frac{\partial u}{\partial x_i} \varphi,$$

where the last equality follows from the fact that

$$\left| \int_{\partial B_\varepsilon} \frac{x_i}{|x|} u\varphi \right| \le C(\varphi, d) \varepsilon^{d-1} |\ln \varepsilon|^d \to 0$$

as $\varepsilon \to 0_+$. Hence, ∇u is really the weak derivative.

b) Let us assume that the following inequality holds true for all $v \in W^{1,d}(B_1(0))$

$$\begin{aligned} \frac{1}{|B_1(0)|} \int_{B_1(0)} \left| v(x) - \frac{1}{|B_1(0)|} \left(\int_{B_1(0)} v(y)\, dy \right) \right|^q dx \\ \le C(q,d) \left(\int_{B_1(0)} |\nabla v(x)|^d \mathrm{d}x \right)^{\frac{q}{d}}. \end{aligned} \tag{1.1}$$

Next, assume that $u \in W^{1,d}(\Omega)$ and that $B_R(x_0) \subset \Omega$. We define

$$v(x) := u(Rx + x_0).$$

Then $v \in W^{1,d}(B_1(0))$ and we may use (1.1). In details, we use the substitution theorem to observe

$$\begin{aligned} &\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} \left| u(x) - \frac{1}{|B_R(x_0)|} \left(\int_{B_R(x_0)} u(y)\, \mathrm{d}y \right) \right|^q \mathrm{d}x \\ &= \frac{1}{|B_1(0)|} \int_{B_1(0)} \left| v(x) - \frac{1}{|B_1(0)|} \left(\int_{B_1(0)} v(y)\, dy \right) \right|^q \mathrm{d}x \overset{(1.1)}{\le} C(q,d) \left(\int_{B_1(0)} |\nabla v(x)|^d \mathrm{d}x \right)^{\frac{q}{d}} \\ &= C(q,d) \left(\int_{B_1(0)} R^d |\nabla u(Rx + x_0)|^d \mathrm{d}x \right)^{\frac{q}{d}} = C(q,d) \left(\int_{B_R(x_0)} |\nabla u(x)|^d \mathrm{d}x \right)^{\frac{q}{d}}. \end{aligned}$$

Thus, it remains to prove (1.1). Let us define

$$w(x) := v(x) - \frac{1}{|B_1(0)|} \int_{B_1(0)} v(y)\, dy$$

then (1.1) is equivalent to

$$\int_{B_1(0)} |w(x)|^q \, \mathrm{d}x \le \tilde{C}(q,d) \left(\int_{B_1(0)} |\nabla w(x)|^d \mathrm{d}x \right)^{\frac{q}{d}}, \tag{1.2}$$

where $w \in W^{1,d}(\Omega)$ and $\int_{B_1(0)} w(x)\, \mathrm{d}x = 0$.

We show two possibilities. In the first case, we may use the standard Poincaré inequality to get (recall that w has zero mean value)

$$\|w\|_{W^{1,d}(B_1(0))} \le C(d) \|\nabla w\|_{L^d(B_1(0))}.$$

Next, we can use the embedding $W^{1,d}(B_1(0)) \hookrightarrow L^q(B_1(0))$ to get

$$\|w\|_{L^q(B_1(0))} \le C(d,q) \|w\|_{W^{1,d}(B_1(0))} \le \tilde{C}(d,q) \|\nabla w\|_{L^d(B_1(0))},$$

where the second inequality is just the Poincaré inequality. Hence, we see that (1.22) holds true.

Another option is to prove (1.22) by contradiction. Hence, let us assume that for all $n \in \mathbb{N}$ there exists $w^n \in W^{1,d}(B_1(0))$ fulfilling $\int_{B_1(0)} w^n = 0$ such that

$$\|w^n\|_{L^q(B_1(0))} > n\|\nabla w^n\|_{L^d(B_1(0))}.$$

Surely, $w^n \neq= 0$, hence we may define

$$\omega^n := \frac{w^n}{\|w^n\|_{W^{1,d}(B_1(0))}}.$$

Thus, we have $\|\omega^n\|_{1,d} = 1$ and

$$\|\omega^n\|_{L^q(B_1(0))} > n\|\nabla \omega^n\|_{L^d(B_1(0))}.$$

Using the embedding $W^{1,d} \hookrightarrow L^q$ we also have $\|\omega^n\|_q \le C\|\omega^n\|_{1,d} \le C$, and thus

$$n\|\nabla \omega^n\|_{L^d(B_1(0))} \le C.$$

Using the compact embedding $W^{1,d} \hookrightarrow\hookrightarrow L^d$, we deduce that for a subsequence

$$\omega^n \to \omega \text{ strongly in } L^d(B_1(0)).$$

Moreover, from the above inequality, we have

$$\nabla\omega^n \to 0 \text{ strongly in } L^d(B_1(0);\mathbb{R}^d).$$

Furthermore,

$$\|\omega\|_d^d = \lim_{n\to\infty} \|\omega^n\|_{1,d}^d - \|\nabla\omega^n\|_d^d = \lim_{n\to\infty} 1 - \|\nabla\omega^n\|_d^d = 1.$$

Since, $\nabla\omega = 0$ we know $\omega \neq 0$ is constant. On the other hand $\int \omega = 0$ which is a contradiction.

Problem 2: Consider the following problem:

$$\begin{aligned} -(1+|x|^2)\frac{\partial^2 u}{\partial x_1^2} - (4-|x|^2)\frac{\partial^2 u}{\partial x_2^2} &= \frac{\partial\sqrt{1-|x|}}{\partial x_1} && \text{in } \Omega \\ (2x_1 - x_2)\frac{\partial u}{\partial x_1} + (x_1 + 3x_2)\frac{\partial u}{\partial x_2} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^2$ is a unit ball centered at zero. Write down the weak formulation of the above problem. Show that if the weak solution is smooth then it satisfies the above problem.

Hint: Follow the example given during the lecture.

Solution: First, we want to rewrite the problem into the form

$$-\operatorname{div}(\mathbb{A}\nabla u) + bu + \vec{c}\cdot\nabla u + \operatorname{div}(\vec{d}u) = f.$$

Applying the derivatives, we get the identity

$$-\sum_{i,j} \mathbb{A}_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i,j}\frac{\partial \mathbb{A}_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} + (b + \operatorname{div}\vec{d})u + (\vec{c}+\vec{d})\cdot\nabla u = f$$

Comparing it with the setting of the Problem 2, we see that the parameters must be set in the following way

$$
\begin{aligned}
&\mathbb{A}_{11}(x) = 1+|x|^2, \quad \mathbb{A}_{22}(x) = (4-|x|^2), \qquad \mathbb{A}_{12}(x)+\mathbb{A}_{21}(x) = 0,\\
&b+\operatorname{div}\vec{d} = 0, \qquad \vec{c}_j(x)+\vec{d}_j(x)-\sum_i \frac{\mathbb{A}_{ij}(x)}{\partial x_i} = 0.
\end{aligned} \tag{1.3}
$$

Next, we evaluate the Neumann (or Newton) condition on the boundary. We use the explicit for of the normal vector $\vec{\nu} = (x_1, x_2)$, and we have for all $x \in \partial\Omega$ that (note that $|x| = 1$ on the boundary)

$$
\begin{aligned}
&(-\mathbb{A}(x)\nabla u(x)+\vec{d}(x)u(x))\cdot x\\
&= -x_1\mathbb{A}_{11}\frac{\partial u}{\partial x_1} - x_1\mathbb{A}_{12}\frac{\partial u}{\partial x_2} - x_2\mathbb{A}_{21}\frac{\partial u}{\partial x_1} - x_2\mathbb{A}_{22}\frac{\partial u}{\partial x_2} + u(x)(\vec{d}_1(x)x_1+\vec{d}_2(x)x_2)\\
&= -(2x_1+x_2\mathbb{A}_{21})\frac{\partial u}{\partial x_1} - (3x_2+x_1\mathbb{A}_{12})\frac{\partial u}{\partial x_2} + u(x)(\vec{d}_1(x)x_1+\vec{d}_2(x)x_2)
\end{aligned}
$$

Hence, to make it compatible with the boundary condition, we see that we have to set

$$
\mathbb{A}_{21}(x) := -1, \quad \mathbb{A}_{12}(x) = 1
$$

and then going back to (1.3), we deduce that

$$
\vec{d} = 0, \qquad b = 0, \qquad \vec{c}(x) = (2x_1, -2x_2), \qquad f(x) := \frac{\partial\sqrt{1-|x|}}{\partial x_1}.
$$

Thus, we can define the notion of weak solution: we look for $u \in W^{1,2}(\Omega)$ fulfilling for all $\varphi \in W^{1,2}(\Omega)$

$$
\int_\Omega \mathbb{A}\nabla u\cdot\nabla\varphi + \varphi\vec{c}\cdot\nabla u \,\mathrm{d}x = \int_\Omega f\varphi \tag{1.4}
$$

For sure, all terms on the right hand side are well defined, but we need to justify the term on the right hand side. Since

$$
|f| = \left| -\frac{x_1}{2|x|\sqrt{1-|x|}} \right| \in L^p(\Omega)
$$

for all $p \in [1,2)$. Consequently, using the Hölder inequality, we have for some $p \in (1,2)$

$$
\int_\Omega f\varphi \le \|f\|_p\|\varphi\|_{p'}.
$$

Therefore, we need to consider $\varphi \in L^{p'}$ for some $p < 2$. Because Ω is a two dimensional domain, we have $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [1,\infty)$ and therefore we see that $\int_\Omega f\varphi$ is well defined for all $\varphi \in W^{1,2}(\Omega)$.

Alternatively, denoting $F := \sqrt{1-|x|}$, we see that $F \equiv 0$ on $\partial\Omega$ and we may write

$$
\int_\Omega f\varphi = -\int_\Omega F\frac{\partial\varphi}{\partial x_1}\mathrm{d}x,
$$

which is also well defined for all $\varphi \in W^{1,2}(\Omega)$. The second part, i.e., the part showing that if $u \in \mathcal{C}^2(\Omega)$ then it is a classical solution, follows line by line the proof given at the lecture.

DEADLINE: November 6

Problem 3: The goal is to show that the maximal regularity[1] cannot hold in Lipschitz domains or when changing the type of boundary conditions. Let $\varphi_0 \in (0, 2\pi)$ be arbitrary and consider $\Omega \subset \mathbb{R}^2$ given by[2]

$$\Omega := \{(r, \varphi) : \ r \in (0,1), \ \varphi \in (0, \varphi_0)\}.$$

Denote $\Gamma_i \subset \partial\Omega$ in the following way (in polar coordinates (r, φ): $\Gamma_1 := \{(r,0); r \in (0,1)\}$, $\Gamma_2 := \{(r, \varphi_0); r \in (0,1)\}$, $\Gamma_3 := \{(1, \varphi); \varphi \in (0, \varphi_0)\}$.

Consider two functions

$$u_1(r,\varphi) := r^{\alpha_1} \sin\left(\frac{\varphi\pi}{\varphi_0}\right),$$
$$u_2(r,\varphi) := r^{\alpha_2} \sin\left(\frac{\varphi\pi}{2\varphi_0}\right)$$

- Find the condition on α_i so that $u_i \in W^{1,2}(\Omega)$ - find an explicit formula for ∇u_i - and prove that it is really the weak derivative!
- Find the proper condition on α_i so that u_i solves the problem

$$-\Delta u_1 = 0 \text{ in } \Omega, \quad u_1 = 0 \text{ on } \Gamma_1 \cup \Gamma_2, \quad u_1 = \sin\left(\frac{\varphi\pi}{\varphi_0}\right) \text{ on } \Gamma_3$$
$$-\Delta u_2 = 0 \text{ in } \Omega, \quad u_2 = 0 \text{ on } \Gamma_1, \quad u_2 = \sin\left(\frac{\varphi\pi}{2\varphi_0}\right) \text{ on } \Gamma_3,$$
$$\nabla u_2 \cdot n = 0 \text{ on } \Gamma_2$$

  Check in details that for such α_i's the weak formulation of the above elliptic equations hold!
- Find all p's for which $u_i \in W^{2,p}(\Omega)$. What is the criterium on α_i so that $u_i \in W^{2,2}(\Omega)$?
- With the help of the above computation, find $f_i \in L^2(\Omega)$ such that the problems with homogeneous boundary conditions, i.e.,

$$-\Delta v_1 = f_1 \text{ in } \Omega, \quad v_1 = 0 \text{ on } \partial\Omega,$$
$$-\Delta v_2 = f_2 \text{ in } \Omega, \quad v_2 = 0 \text{ on } \Gamma_1 \cup \Gamma_3, \nabla v_2 \cdot n = 0 \text{ on } \Gamma_2$$

  posses unique weak solutions $v_i \in W^{1,2}(\Omega)$ but $v_1 \notin W^{2,2}(\Omega)$ if $\varphi_0 > \pi$ and $v_2 \notin W^{2,2}(\Omega)$ for $\varphi_0 > \frac{\pi}{2}$.
- **REMEMBER:** On domains with corner - the $W^{2,2}$ regularity statement does not hold for Dirichlet problem for angels greater than π and does not hold when changing Dirrichlet to Neumann problems on corners with angle greater than $\pi/2$. In general $W^{2,2}$ regularity for Dirichlet problems holds in any dimension either for convex domains or for domains with $\mathcal{C}^{1,1}$ boundary.

DEADLINE: November 27

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[1]Maximal regularity statement means that if

$$\Delta u = f$$

then $f \in L^p(\Omega) \implies u \in W^{2,p}(\Omega)$. The goal of the homework is to show that this is not true on domains with corners.

[2]We use polar coordinates, i.e. $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$

Solution: First of all, one should be able to derive the formula for derivatives in polar coordinates. Since our change of coordinates is given by

$$x_1 = r\cos\varphi, \qquad x_2 = r\sin\varphi,$$

Then for any $\mathcal{C}^1$ function $f(r,\varphi)$, we have (for $r \in (0,1)$ and $\varphi \in (0,2\pi)$)

$$\begin{aligned}\frac{\partial f(r,\varphi)}{\partial x_1} &= \cos\varphi\frac{\partial f(r,\varphi)}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial f(r,\varphi)}{\partial \varphi}\\ \frac{\partial f(r,\varphi)}{\partial x_2} &= \sin\varphi\frac{\partial f(r,\varphi)}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial f(r,\varphi)}{\partial \varphi}\end{aligned}$$

Consequently, we can also deduce that

$$\begin{aligned}\Delta f(r,\varphi) &= \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_1^2}\\ &= \frac{\partial}{\partial x_1}\left(\cos\varphi\frac{\partial f(r,\varphi)}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial f(r,\varphi)}{\partial \varphi}\right) + \frac{\partial}{\partial x_2}\left(\sin\varphi\frac{\partial f(r,\varphi)}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial f(r,\varphi)}{\partial \varphi}\right)\\ &= \cos\varphi\frac{\partial}{\partial r}\left(\cos\varphi\frac{\partial f(r,\varphi)}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial f(r,\varphi)}{\partial \varphi}\right) - \frac{\sin\varphi}{r}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial f(r,\varphi)}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial f(r,\varphi)}{\partial \varphi}\right)\\ &+ \sin\varphi\frac{\partial}{\partial r}\left(\sin\varphi\frac{\partial f(r,\varphi)}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial f(r,\varphi)}{\partial \varphi}\right) + \frac{\cos\varphi}{r}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial f(r,\varphi)}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial f(r,\varphi)}{\partial \varphi}\right)\end{aligned}$$

Hence, evaluating the right hand side, we get that

$$\begin{aligned}\Delta f(r,\varphi) &= \cos\varphi\left(\cos\varphi\frac{\partial^2 f(r,\varphi)}{\partial r^2} + \frac{\sin\varphi}{r^2}\frac{\partial f(r,\varphi)}{\partial\varphi} - \frac{\sin\varphi}{r}\frac{\partial^2 f(r,\varphi)}{\partial\varphi\partial r}\right)\\ &- \frac{\sin\varphi}{r}\left(-\sin\varphi\frac{\partial f(r,\varphi)}{\partial r} + \cos\varphi\frac{\partial^2 f(r,\varphi)}{\partial r\partial\varphi} - \frac{\cos\varphi}{r}\frac{\partial f(r,\varphi)}{\partial\varphi} - \frac{\sin\varphi}{r}\frac{\partial^2 f(r,\varphi)}{\partial\varphi^2}\right)\\ &+ \sin\varphi\left(\sin\varphi\frac{\partial^2 f(r,\varphi)}{\partial r^2} - \frac{\cos\varphi}{r^2}\frac{\partial f(r,\varphi)}{\partial\varphi} + \frac{\cos\varphi}{r}\frac{\partial^2 f(r,\varphi)}{\partial\varphi\partial r}\right)\\ &+ \frac{\cos\varphi}{r}\left(\cos\varphi\frac{\partial f(r,\varphi)}{\partial r} + \sin\varphi\frac{\partial^2 f(r,\varphi)}{\partial r\partial\varphi} - \frac{\sin\varphi}{r}\frac{\partial f(r,\varphi)}{\partial\varphi} + \frac{\cos\varphi}{r}\frac{\partial^2 f(r,\varphi)}{\partial\varphi^2}\right)\\ &= \cos^2\varphi\frac{\partial^2 f(r,\varphi)}{\partial r^2} + \frac{\sin^2\varphi}{r}\frac{\partial f(r,\varphi)}{\partial r} + \frac{\sin^2\varphi}{r^2}\frac{\partial^2 f(r,\varphi)}{\partial\varphi^2}\\ &+ \sin^2\varphi\frac{\partial^2 f(r,\varphi)}{\partial r^2} + \frac{\cos^2\varphi}{r}\frac{\partial f(r,\varphi)}{\partial r} + \frac{\cos^2\varphi}{r^2}\frac{\partial^2 f(r,\varphi)}{\partial\varphi^2}\\ &= \frac{\partial^2 f(r,\varphi)}{\partial r^2} + \frac{1}{r}\frac{\partial f(r,\varphi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f(r,\varphi)}{\partial\varphi^2}\end{aligned}$$

Next, we use the hint an check for which $A, B \in \mathbb{R}$, we have that

$$\Delta(r^A\sin(B\varphi)) = 0 \qquad \text{in } \Omega.$$

Note that such function is smooth outside of the origin. Consequently, using the above computation, we see that for $u_{AB} := r^A\sin(B\varphi)$

$$\begin{aligned}\Delta u_{AB} &= A(A-1)r^{A-2}\sin(B\varphi) + Ar^{A-2}\sin(B\varphi) - r^{A-2}B^2\sin(B\varphi)\\ &= \left(A^2 - B^2\right)r^{A-2}\sin(B\varphi).\end{aligned}$$

Hence, we require $A^2 = B^2$ in what follows. Next, we also check for which A, B we have $u_{AB} \in W^{1,2}(\Omega)$. Since, the classical derivatives exist in Ω, we know that the weak derivative also exists and we just need to specify the conditions on A, B so that

$$\int_\Omega |\nabla u_{AB}|^2 < \infty.$$

Using the transformation into the polar coordinates and the substitution theorem, we have

$$\begin{aligned}
\int_\Omega |\nabla u_{AB}|^2 \,\mathrm{dx} &= \int_\Omega \left(\frac{\partial u_{AB}}{\partial x_1}\right)^2 + \left(\frac{\partial u_{AB}}{\partial x_2}\right)^2 \,\mathrm{dx} \\
&= \int_0^1 \int_0^{\varphi_0} \left(\cos\varphi \frac{\partial u_{AB}}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial u_{AB}}{\partial \varphi}\right)^2 r + \left(\sin\varphi \frac{\partial u_{AB}}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial u_{AB}}{\partial \varphi}\right)^2 r \, d\varphi \, dr \\
&= \int_0^1 \int_0^{\varphi_0} \left(\left(\frac{\partial u_{AB}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u_{AB}}{\partial \varphi}\right)^2\right) r \, d\varphi \, dr \\
&= \int_0^1 \int_0^{\varphi_0} \left(\left(Ar^{A-1}\sin(B\varphi)\right)^2 + \frac{1}{r^2}\left(Br^A\cos(B\varphi)\right)^2\right) r \, d\varphi \, dr \\
&= \int_0^1 \int_0^{\varphi_0} r^{2A-1}\left(A^2\sin^2(B\varphi) + B^2\cos^2(B\varphi)\right) \, d\varphi \, dr \\
&= \int_0^1 \int_0^{\varphi_0} A^2 r^{2A-1} \, d\varphi \, dr = A^2\varphi_0 \int_0^1 r^{2A-1} \, dr,
\end{aligned}$$

where we used the fact that $A^2 = B^2$. Consequently, if we want to have the above integral finite, we must impose the condition $A > 0$. Since the regularity of the solution does not depend on the sign of B, we assume in what follows only the case $A = B > 0$.

Next, we show for which $p's$ the function $u_{AB} \in W^{2,p}(\Omega)$. For that reasons, we compute the second derivatives (in term of variables (r, φ))

$$\begin{aligned}
\frac{\partial^2 u_{AB}}{\partial x_1^2} &= \cos^2\varphi \frac{\partial^2 u_{AB}}{\partial r^2} + \frac{2\cos\varphi\sin\varphi}{r^2}\frac{\partial u_{AB}}{\partial \varphi} - \frac{2\cos\varphi\sin\varphi}{r}\frac{\partial^2 u_{AB}}{\partial\varphi\partial r} \\
&\quad + \frac{\sin^2\varphi}{r}\frac{\partial u_{AB}}{\partial r} + \frac{\sin^2\varphi}{r^2}\frac{\partial^2 u_{AB}}{\partial \varphi^2} \\
&= A(A-1)r^{A-2}\cos^2\varphi\sin(A\varphi) + 4Ar^{A-2}\cos\varphi\sin\varphi\cos(A\varphi) \\
&\quad + Ar^{A-2}\sin^2\varphi\sin(A\varphi) - A^2r^{A-2}\sin^2\varphi\sin(A\varphi)
\end{aligned}$$

In the same manner we shall estimate other second derivatives to finally conclude that

$$\int_\Omega |\nabla^2 u_{AB}|^p \,\mathrm{dx} \sim \int_0^1 r^{(A-2)p} r \, dr$$

and wee that the integral is finite if and only if (for $0 \le A < 2$, since for $A \ge 2$ it is always finite)

$$p < \frac{2}{2-A}.$$

Finally, we apply everything to the functions u_1 and u_2, which are thus given as

$$u_1 = r^{\frac{\pi}{\varphi_0}} \sin\left(\frac{\varphi\pi}{\varphi_0}\right), \qquad u_2 = r^{\frac{\pi}{2\varphi_0}} \sin\left(\frac{\varphi\pi}{2\varphi_0}\right)$$

and due to the above computations we get that $u_1, u_2 \in W^{1,2}(\Omega)$ and

$$u_1 \in \left\{ \begin{matrix} W^{2,p}(\Omega) & \text{for } p < \dfrac{2}{2-\frac{\pi}{\varphi_0}} \text{ and } \varphi_0 > \dfrac{\pi}{2} \\ W^{2,\infty}(\Omega) & \text{for } \varphi_0 \leq \dfrac{\pi}{2} \end{matrix} \right\} \implies u_1 \in W^{2,2}(\Omega) \text{ if } \varphi_0 < \pi,$$

$$u_2 \in \left\{ \begin{matrix} W^{2,p}(\Omega) & \text{for } p < \dfrac{2}{2-\frac{\pi}{2\varphi_0}} \text{ and } \varphi_0 > \dfrac{\pi}{4} \\ W^{2,\infty}(\Omega) & \text{for } \varphi_0 \leq \dfrac{\pi}{4} \end{matrix} \right\} \implies u_2 \in W^{2,2}(\Omega) \text{ if } \varphi_0 < \frac{\pi}{2}.$$

Finally, we check that u_1 and u_2 are the solutions of the corresponding problem. Since, u_1 is continuous in $\overline{\Omega}$ and also $u_1 \in W^{1,2}(\Omega)$, then evidently[3] the trace of u_1 is zero on Γ_1 and Γ_2. Next, for any smooth compactly supported function v we get by integration by parts (we ca apply that because u_1 is smooth in the interior of Ω)

$$\int_\Omega \nabla u_1 \cdot \nabla v = -\int_\Omega \Delta u_1 v = 0,$$

where we used the fact that u_1 is harmonic. Finally since the space $C_0^\infty(\Omega)$ is dense in $W_0^{1,2}(\Omega)$ we can generalize the above relation also for any $v \in W_0^{1,2}(\Omega)$. Indeed, for any $v \in W_0^{1,2}(\Omega)$, we can find a sequence (by density) $\{v^n\} \subset \mathcal{C}_0^\infty(\Omega)$ such that $v^n \to v$ in $W^{1,2}(\Omega)$ and then

$$\int_\Omega \nabla u_1 \cdot \nabla v = \lim_{n\to\infty} \int_\Omega \nabla u_1 \cdot \nabla v^n = 0. \tag{1.5}$$

Finally, let w be a smooth function on $\mathbb{R}^2$ fulfilling $w = 1$ in $B_{\frac{1}{2}}(0)$ and $w = 0$ on $\mathbb{R}^2 \setminus B_{\frac{3}{4}}(0)$ and define $v_1 := u_1 w$. Then evidently $v_1 \in W^{1,2}(\Omega) \cap W^{2,p}(\Omega)$ with p specified above, $v_1 = 0$ on $\partial\Omega$ and we have for any $z \in W_0^{1,2}(\Omega)$

$$\begin{aligned} \int_\Omega \nabla v_1 \cdot \nabla z &= \int_\Omega w \nabla u_1 \cdot \nabla z + u_1 \nabla w \cdot \nabla z \\ &= \underbrace{\int_\Omega \nabla u_1 \cdot \nabla(wz)}_{(1.5)} - \int_\Omega z \nabla u_1 \cdot \nabla w - u_1 \nabla w \cdot \nabla z \\ &= -\int_\Omega \Big(\nabla u_1 \cdot \nabla w + \operatorname{div}(u_1 \nabla w)\Big) z, \end{aligned}$$

which is the weak formulation of

$$-\Delta v_1 = -\nabla u_1 \cdot \nabla w - \operatorname{div}(u_1 \nabla w) =: f_1$$

Note that thanks to the presence of w, the function f_1 is smooth since u_1 is regular outside of 0 but w is constant near zero.

For v_2 we could use exactly the same arguments to get the result, but we did not formulate any result concerning the density of functions vanishing only near Γ_1. Thus, we proceed differently. Let $Q : [0,\infty) \to [0,\infty)$ be smooth function fulfilling $Q = 1$ on $[0, 1/4]$ and $Q = 0$ on $[1,\infty)$ and let $R : [0,\infty) \to [0,\infty)$ be smooth

[3] The trace operator just assign the values of the function on the boundary whenever the function is continuous.

nondecreasing function fulfilling $R = 0$ on $[0, 1/2]$ and $R = 1$ on $[1, \infty)$. Next, we define

$$v_2(r, \varphi) := u_2(r, \varphi)Q(r^2)$$

Then it is easy to check that $v_2 = 0$ on $\Gamma_1 \cup \Gamma_3$. Next we check what kind of problem v_2 satisfies. We use also the function R to cut everything near zero in order to be able to use integration by parts. In addition, it is also a direct consequence of the definition that $\nabla v_2 \cdot n = 0$ on Γ_2. Hence, let $z \in W^{1,2}(\Omega)$be arbitrary fulfilling (in sense of traces) $z = 0$ on $\Gamma_1 \cup \Gamma_3$

$$\begin{aligned}
\int_\Omega \nabla v_2 \cdot \nabla z &= \lim_{\varepsilon\to 0_+} \int_\Omega \nabla v_2 \cdot \nabla z R(r^2/\varepsilon^2) \\
&= -\lim_{\varepsilon\to 0_+} \int_\Omega \operatorname{div}(R(r^2/\varepsilon^2)\nabla v_2) z - \lim_{\varepsilon\to 0_+} \int_{\partial\Omega} \underbrace{\nabla v_1 \cdot n}_{=0 \text{ on } \Gamma_2} \quad \underbrace{z R(r^2/\varepsilon^2)}_{=0 \text{ on } \Gamma_1\cap\Gamma_3} \\
&= -\lim_{\varepsilon\to 0_+} \int_\Omega (\nabla R(r^2/\varepsilon^2) \cdot \nabla v_2 + R(r^2/\varepsilon^2)\underbrace{\operatorname{div}(\nabla v_2)}_{\Delta(u_2 Q)}) z \\
&= -\int_\Omega (u_2 \Delta Q + \nabla u_2 \cdot \nabla Q) z - \lim_{\varepsilon\to 0_+} 2\varepsilon^{-2} \int_{B_\varepsilon(0)\cap\Omega} R'(r^2/\varepsilon^2) x \cdot \nabla v_2 z
\end{aligned}$$

Thus, if we show that the last limit is equal to zero, we see that v_2 is a weak solution to the desired problem with $f_2 := -(u_2\Delta Q + \nabla u_2 \cdot \nabla Q)$ which is a smooth function.

To estimate the limit, we recall the Poincaré inequality and for all $\tilde{z} \in W^{1,2}(\Omega)$ being equal to zero on Γ_1 there holds $\|\tilde{z}\|_2 \le C\|\nabla\tilde{z}\|_2$. Hence, if we define the particular $\tilde{z}$ as

$$\tilde{z}(x) := z(\varepsilon x)$$

and use the substitution and Poincaré inequality on Ω we have

(1.6)

$$\begin{aligned}
\int_{\Omega\cap B_\varepsilon(0)} |z(x)|^2 \,\mathrm{dx} &= \int_{\Omega\cap B_\varepsilon(0)} |\tilde{z}(x/\varepsilon)|^2 \,\mathrm{dx} = \varepsilon^2 \int_\Omega |\tilde{z}(x)|^2 \,\mathrm{dx} \le C\varepsilon^2 \int_\Omega |\nabla\tilde{z}(x)|^2 \,\mathrm{dx} \\
&= C\varepsilon^4 \int_\Omega |\nabla z(\varepsilon x)|^2 \,\mathrm{dx} = C\varepsilon^2 \int_{\Omega\cap B_\varepsilon(0)} |\nabla z(x)|^2 \,\mathrm{dx}
\end{aligned}$$

Hence, by the Hölder inequality and the above proven re-scaled Poincaréinequality, we have

$$\begin{aligned}
&\left|\varepsilon^{-2} \int_{B_\varepsilon(0)\cap\Omega} R'(r^2/\varepsilon) x \cdot \nabla v_2 z\right| \le \|R'\|_\infty \left(\frac{\int_{\Omega\cap B_\varepsilon(0)} |z(x)|^2 \,\mathrm{dx}}{\varepsilon^2}\right)^{\frac{1}{2}} \left(\int_{\Omega\cap B_\varepsilon(0)} |\nabla v_2|^2\right)^{\frac{1}{2}} \\
&\le C\|R'\|_\infty \|\nabla z\|_2 \left(\int_{\Omega\cap B_\varepsilon(0)} |\nabla v_2|^2\right)^{\frac{1}{2}} \stackrel{\varepsilon\to 0_+}{\to} 0
\end{aligned}$$

where the last limit holds since $v_2 \in W^{1,2}(\Omega)$.

Problem 4: Fredholm alternative vs Lax–Milgram lemma vs minimum principle. Consider $\Omega \subset \mathbb{R}^d$ a Lipschitz domain. Let $\mathbb{A} : \Omega \to \mathbb{R}^d$ be an elliptic matrix. Assume that $\vec{c} \in L^\infty(\Omega; \mathbb{R}^d)$ and $b \ge 0$. Consider the problem

$$(\mathcal{M}) \qquad \begin{aligned}
-\operatorname{div}(\mathbb{A}\nabla u) + bu + \vec{c} \cdot \nabla u &= f \quad \text{in } \Omega, \\
u &= u_0 \quad \text{on } \partial\Omega.
\end{aligned}$$

a) Consider the case $b = 0$, $\vec{c} = \vec{0}$ and $f \in L^2(\Omega)$ fulfilling $f \geq 0$. Let $u_0 \in W^{1,2}(\Omega)$ and denote $m := \text{ess inf}_{x\in\partial\Omega} u_0(x)$. Show that the unique weak solution u to $(\mathcal{M})$ satisfies $u(x) \geq m$ almost everywhere in Ω.

b) Consider $b > 0$ and $\vec{c}$ arbitrary. Prove that for any $u_0 \in W^{1,2}(\Omega)$ and any $f \in L^2(\Omega)$ there exists a weak solution to $(\mathcal{M})$.

Hint: a) Define $\varphi(x) := (u(x)-m)_- = \min\{0, (u(x)-m)\}$. Show that $\varphi \in W_0^{1,2}(\Omega)$ and that

$$\nabla\varphi = \nabla u \chi_{\{u(x)<m\}}.$$

Use φ as a test function in the weak formulation of $(\mathcal{M})$ and show that $\nabla\varphi \equiv 0$, which implies the rest.

b1) Justify that to conclude it is enough to show that if $u \in W_0^{1,2}(\Omega)$ solves

$$(\mathcal{M}_0) \qquad -\operatorname{div}(\mathbb{A}\nabla u) + bu + \vec{c}\cdot\nabla u = 0 \quad \text{in } \Omega$$

then $u \equiv 0$.

b2) Consider the test function $\varphi := (u - M)_+ = \max\{0, u - M\}$. By using the Hölder inequality and the embedding theorem, show that there exists M such that $u \leq M$ a.e. in Ω. Similarly also show $u \geq -M$ and consequently $u \in L^\infty(\Omega)$

b3) Based on b2) show that $\varphi := u^k \in W_0^{1,2}(\Omega)$ is a correct test function. By using the Young inequality and the embedding theorem, one can prove that by choosing k sufficiently large one gets $\varphi \equiv= 0$.

Remark: Here it is the case, when the Lax-Milgram theorem cannot be used if $\|\vec{c}\|_\infty \gg 1$.

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Solution: a) the fact that $\varphi \in W^{1,2}(\Omega)$ and that $\nabla\varphi = \nabla u\chi_{\{u<k\}}$ was proven at the lecture. We repeat it also here. For sake of simplicity we set $m = 0$. Due to the density argument, we can find a sequence $\{u^n\}_{n=1}^\infty \subset \mathcal{C}^\infty$ such that $u^n \to u$ in $W^{1,1}_{loc}$. Finally, we set

$$\varphi^{n,\varepsilon} := \frac{(\min\{0,u^n\})^3}{\varepsilon + (\min\{0,u^n\})^2}, \qquad \varphi^\varepsilon := \frac{(\min\{0,u\})^3}{\varepsilon + (\min\{0,u\})^2}$$

Next, we set arbitrary $\psi \in \mathcal{C}_0^\infty(\Omega)$. Then using the Lebesgue dominated convergence theorem and the fact that $u^n \to u$ strongly in $L^1_{loc}(\Omega)$, we observe

$$\int_\Omega -\frac{\partial\psi}{\partial x_i}\varphi = \lim_{\varepsilon\to 0_+}\int_\Omega -\frac{\partial\psi}{\partial x_i}\varphi^\varepsilon = \lim_{\varepsilon\to 0_+}\left(\lim_{n\to\infty}\int_\Omega -\frac{\partial\psi}{\partial x_i}\varphi^{n,\varepsilon}\right). \tag{1.7}$$

On the other hand, we can use the fact that $\varphi^{n,\varepsilon} \in \mathcal{C}^1(\Omega)$ and that

$$\frac{\partial\varphi^{n,\varepsilon}}{\partial x_i} = \frac{3\varepsilon\,(\min\{0,u^n\})^2 + (\min\{0,u^n\})^4}{(\varepsilon + (\min\{0,u^n\})^2)^2}\frac{\partial u^n}{\partial x_i}.$$

Then we can use the integration by parts for smooth function to get

$$\int_\Omega -\frac{\partial\psi}{\partial x_i}\varphi^{n,\varepsilon} = \int_\Omega \psi\frac{3\varepsilon\,(\min\{0,u^n\})^2 + (\min\{0,u^n\})^4}{(\varepsilon + (\min\{0,u^n\})^2)^2}\frac{\partial u^n}{\partial x_i}.$$

Therefore, using the strong convergence $\nabla u^n \to \nabla u$ in L^1_{loc}, the strong convergence $u^n \to u$ in L^1 and also the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_\Omega \psi \frac{3\varepsilon\left(\min\{0,u^n\}\right)^2 + (\min\{0,u^n\})^4}{(\varepsilon + (\min\{0,u^n\})^2)^2}\frac{\partial u^n}{\partial x_i} = \int_\Omega \psi \frac{3\varepsilon\left(\min\{0,u\}\right)^2 + (\min\{0,u\})^4}{(\varepsilon + (\min\{0,u\})^2)^2}\frac{\partial u}{\partial x_i}$$

$$= \int_{\Omega\cap\{u<0\}} \psi \frac{3\varepsilon u^2 + u^4}{(\varepsilon+u^2)^2}\frac{\partial u}{\partial x_i} = \int_{\Omega\cap\{u<0\}} \psi \left(1 + \frac{\varepsilon u^2 - \varepsilon^2}{(\varepsilon+u^2)^2}\right)\frac{\partial u}{\partial x_i}$$

Since $|\frac{\varepsilon u^2-\varepsilon^2}{(\varepsilon+u^2)^2}| \le 1$ and since

$$\frac{\varepsilon u^2 - \varepsilon^2}{(\varepsilon+u^2)^2} \overset{\varepsilon\to 0_+}{\to} 0 \text{ almost everywhere on } \{x\in\Omega;\ u(x)<0\},$$

we can use the Lebesgue dominated convergence theorem to observe

$$\lim_{\varepsilon\to 0_+} \int_{\Omega\cap\{u<0\}} \psi \left(1 + \frac{\varepsilon u^2 - \varepsilon^2}{(\varepsilon+u^2)^2}\right)\frac{\partial u}{\partial x_i} = int_{\Omega\cap\{u<0\}}\psi\frac{\partial u}{\partial x_i}.$$

Consequently, comparing the result with (1.7), we have

$$\int_\Omega -\frac{\partial\psi}{\partial x_i}\varphi = \int_\Omega \psi\frac{\partial u}{\partial x_i}\chi_{\{u<m\}},$$

which is the desired claim.

Next, we continue. Since $u \ge m$ on $\partial\Omega$ then we even get that $\varphi \in W^{1,2}_0(\Omega)$. Using φ in the weak formulation of the problem $(\mathcal{M})$, we have

$$\int_\Omega \mathbb{A}\nabla u\cdot\nabla\varphi = \int_\Omega f\varphi \le 0,$$

where the last inequality follows from the fact that $f \ge 0$ $\varphi \le 0$ a.e. in Ω. Using also the identification of the gradient of φ, it follows from the ellipticity of $\mathbb{A}$ and the above inequality that

$$C_1\int_\Omega |\nabla\varphi|^2 \overset{ellipticity}{\le} \int_\Omega \mathbb{A}\nabla\varphi\cdot\nabla\varphi = \int_\Omega \mathbb{A}\nabla u\cdot\nabla\varphi\chi_{\{u<k\}} = \int_\Omega \mathbb{A}\nabla u\cdot\nabla\varphi \le 0.$$

Then, since φ has zero trace, we can use the Poincaré inequality and conclude that $\varphi \equiv 0$ in Ω. Therefore $u - m \ge 0$ and consequently $u \ge m$ a.e. in Ω.

b1) Let us define $\tilde u := u - u_0$. Then $u \in W^{1,2}(\Omega)$ is a weak solution to $(\mathcal{M})$ if and only if $\tilde u \in W^{1,2}_0(\Omega)$ solves for all $\varphi \in W^{1,2}_0(\Omega)$

(1.8)
$$\int_\Omega \mathbb{A}\nabla\tilde u\cdot\nabla\varphi + b\tilde u\varphi + \vec c\cdot\nabla\tilde u\varphi = \int_\Omega f\varphi - \mathbb{A}\nabla u_0\cdot\nabla\varphi - bu_0\varphi - \vec c\cdot\nabla u_0\varphi =: \langle F,\varphi\rangle_{W^{1,2}_0(\Omega)}.$$

The important change to the proof in the lecture is that $F \in (W^{1,2}_0(\Omega))^*$ but we do not know that whether $F \in L^2(\Omega)$. In case the second claim is true, we could use the Fredholm alternative from the lecture. Here, we need to improve it, i.e., we need to show that if the only weak solution to $(\mathcal{M}_0)$ is $u \equiv 0$ then for all $F \in (W^{1,2}_0(\Omega))^*$ there exists a unique solution to (1.8). We prove this claim in two steps. First, start with proving that there exists a constant $C > 0$ such that for any $f \in L^2(\Omega)$ and $u_0 \equiv 0$ there exists unique weak solution $u \in W^{1,2}_0(\Omega)$ to the problem $(\mathcal{M})$ fulfilling

$$\|u\|_{1,2} \le C\|F\|_{(W^{1,2}_0(\Omega))^*} := \sup_{\varphi\in W^{1,2}_0(\Omega);\ \|\varphi\|_{1,2}\le 1} C\int_\Omega F\varphi. \tag{1.9}$$

Clearly, since we assume that the only solution to $(\mathcal{M}_0)$ is zero, we have that for any $f \in L^2(\Omega)$ there is unique weak solution. We only need to show the inequality (1.9). Let us assume a contradiction, i.e., we have $F^n \in L^2(\Omega)$ and a weak solution $u^n \in W_0^{1,2}(\Omega)$ to $(\mathcal{M})$ such that (we can rescale to problem and assume that the left hand side is equal to one thanks to the linearity of the equation)

$$1 = \|u^n\|_{1,2} > n\|F^n\|_{(W_0^{1,2}(\Omega))^*}. \tag{1.10}$$

Due to the reflexivity, and thanks to the compact embedding, we can find subsequence, that we do not relabel, such that

$$\begin{aligned} u^n &\rightharpoonup u && \text{weakly in } W_0^{1,2}(\Omega),\\ \nabla u^n &\rightharpoonup \nabla u && \text{weakly in } L^2(\Omega;\mathbb{R}^d),\\ F^n &\to 0 && \text{strongly in } (W_0^{1,2}(\Omega))^*,\\ u^n &\to u && \text{strongly in } L^2(\Omega). \end{aligned}$$

Recalling the definition of a weak solution: for all $\varphi \in W_0^{1,2}(\Omega)$

$$\int_\Omega \mathbb{A}\nabla u^n \cdot \nabla\varphi + bu^n\varphi + \vec{c}\cdot\nabla u^n\varphi = \int_\Omega F^n\varphi =: \langle F^n, \varphi\rangle, \tag{1.11}$$

we see that we can use the above convergence results and let $n \to \infty$ in the above identity to deduce

$$\int_\Omega \mathbb{A}\nabla u \cdot \nabla\varphi + bu\varphi + \vec{c}\cdot\nabla u\varphi = 0 \tag{1.12}$$

and consequently, using our assumption, we see that $u \equiv 0$. On the other hand, we shall show that $\|u\|_{1,2} = 1$, which is the desired contradiction. To do so, we use the ellipticity of the matrix $\mathbb{A}$ and set $\varphi := u^n - u$ in (1.11) to deduce (again with the help of the weak convergence results)

$$\begin{aligned} &\lim_{n\to\infty} C_1\|\nabla u^n - \nabla u\|_2^2 \le \lim_{n\to\infty}\int_\Omega \mathbb{A}\nabla(u^n-u)\cdot\nabla(u^n-u)\\ &= \lim_{n\to\infty}\int_\Omega \mathbb{A}\nabla u^n\cdot\nabla(u^n-u) - \int_\Omega \underbrace{\mathbb{A}\nabla u}_{\in L^2}\cdot\underbrace{\nabla(u^n-u)}_{\rightharpoonup 0 \text{ in } L^2}\\ &= \lim_{n\to\infty}\int_\Omega \mathbb{A}\nabla u^n\cdot\nabla(u^n-u) \overset{(1.11)}{=} \lim_{n\to\infty}\int_\Omega -bu^n(u^n-u) + \vec{c}\cdot\nabla u^n(u^n-u) + \langle F^n, u^n-u\rangle\\ &\le (\|b\|_\infty + \|\vec{c}\|_\infty)\lim_{n\to\infty}\|u^n\|_{1,2}(\|u^n-u\|_2 + \|F^n\|_{(W_0^{1,2}(\Omega))^*}) = 0. \end{aligned}$$

Therefore, we got that $u^n \to u$ strongly in $W_0^{1,2}(\Omega)$ and consequently $\|u\|_{1,2} = \lim_{n\to\infty}\|u^n\|_{1,2} = 1$ and therefore u^n cannot be equal to zero. Hence, (1.9) holds true whenever $F \in L^2(\Omega)$.

Next, we show that for any $F \in (W_0^{1,2}(\Omega))^*$ there exists a unique solution to (1.8) fulfilling the estimate (1.9), provided that the only solution to $(\mathcal{M}_0)$ is $u \equiv 0$. To do so, we use theLax-Milgram theorem and find $v \in W_0^{1,2}$ such that

$$\int_\Omega \nabla v\cdot\nabla\varphi + v\varphi = \langle F, \varphi\rangle \quad \text{for all } \varphi \in W_0^{1,2}(\Omega). \tag{1.13}$$

Due to the density, we can find a sequence $\{v^n\}_{n=1}^{\infty} \subset \mathcal{C}_0^{\infty}$ such that $v^n \to v$ in $W_0^{1,2}(\Omega)$. For such v^n we can define

$$\langle F^n, \varphi\rangle := \int_\Omega \nabla v^n \cdot \nabla \varphi + v^n \varphi = \int_\Omega (-\Delta v^n + v)\,\varphi,$$

where we used integration by parts since v^n is smooth compactly supported function. Consequently we see that $F^n = -\Delta v^n + v^n \in L^2(\Omega)$ and therefore there exists unique u^n solving

$$\int_\Omega \mathbb{A}\nabla u^n \cdot \nabla\varphi + bu^n\varphi + \vec{c}\cdot\nabla u^n\varphi = \int_\Omega F^n\varphi = \langle F^n, \varphi\rangle = \int_\Omega \nabla v^n\cdot\nabla\varphi + v^n\varphi. \tag{1.14}$$

Thus using (1.9), we see that

$$\|u^n\|_{1,2} \le C\|F^n\|_{(W_0^{1,2})^*} \le C\|v^n\|_{1,2} \le \tilde{C} < \infty.$$

Therefore, we have for a subsequence that

$$u^n \rightharpoonup u \text{ weakly in } W_0^{1,2}(\Omega).$$

Hence, we can let $n \to \infty$ in (1.14) to get we see that we can use the above convergence results and let $n \to \infty$ in the above identity to deduce

$$\int_\Omega \mathbb{A}\nabla u \cdot \nabla\varphi + bu\varphi + \vec{c}\cdot\nabla u\varphi = \int_\Omega \nabla v\cdot\nabla\varphi + v\varphi = \langle F, \varphi\rangle \tag{1.15}$$

In addition, using the weak lower semicontinuity of norms, we have that (1.9) is valid also for the limiting u and F.

b2) First we show that $u \in L^\infty(\Omega)$. Let $m > 0$ be fixed and consider the test function $\varphi := (u-m)_+$. Then we have from $(\mathcal{M}_0)$ that

$$\int_\Omega \mathbb{A}\nabla u\cdot\nabla(u-m)_+ + bu(u-m)_+ = -\int_\Omega \vec{c}\cdot\nabla u(u-m)_+$$

We estimate all terms. Trivially, we have $bu(u-m)_+ \ge 0$ a.e. in Ω. Also

$$\|(u-m)_+\|_{1,2}^2 \le C\int_\Omega |\nabla(u-m)_+|^2 \le \tilde{C}\int_\Omega \mathbb{A}\nabla u\cdot\nabla(u-m)_+$$

Finally, using the Hölder inequality and the embedding theorem (we define $p := d/(d-2)$ if $d > 2$ and $p = 4$ if $d = 2$) we have

$$\begin{aligned}
-\int_\Omega \vec{c}\cdot\nabla u(u-m)_+ &\le \|\vec{c}\|_\infty \int_{\{u>m\}} |\nabla(u-m)_+||(u-m)_+| \\
&\le C\|\nabla(u-m)_+\|_2\|(u-m)_+\|_{2p}|\{u>m\}|^{\frac{1}{2p'}} \\
&\le \hat{C}\|(u-m)_+\|_{1,2}^2\left(\int_\Omega \frac{|u|}{m}\right)^{\frac{1}{2p'}} \\
&\le \frac{C(\vec{c},\Omega,u)}{m^{\frac{1}{2p'}}}\|(u-m)_+\|_{1,2}^2
\end{aligned}$$

Thus, comparing all inequalities, we have

$$\|(u-m)_+\|_{1,2}^2 \le \frac{K}{m^{\frac{1}{2p'}}}\|(u-m)_+\|_{1,2}^2,$$

where K is a constant depending on the data but independent of m. Thus, setting m sufficiently large, we see that $(u-m)_+ \equiv 0$ and consequently $u \le m$ a.e. in Ω.

Similarly, we proceed also with the negative part of u and we have $u \in L^\infty$.
b3) Due to the above estimate, we have for arbitrary $m \in \mathbb{N}$

$$\|\nabla u^m\|_2 \le \|m|\nabla u||u^{m-1}|\|_2 \le m\|u\|_\infty^{m-1}|\nabla u\|_2 < \infty.$$

Hence $u^{2k+1} \in W_0^{1,2}(\Omega)$ is a possible test function and we have the identity

$$I_1 + I_2 := \int_\Omega \mathbb{A}\nabla u \cdot \nabla u^{2k+1} + buu^{2k+1} = -\int_\Omega \vec{c} \cdot \nabla uu^{2k+1} =: I_3.$$

Again, we estimate all terms. First, using the elipticity, we have

$$I_1 = (2k+1)\int_\Omega u^{2k}\mathbb{A}\nabla u \cdot \nabla u \ge C_1(2k+1)\int_\Omega u^{2k}|\nabla u|^2.$$

Similarly, defining the positive $m > 0$ by the formula $m := \text{ess inf}_{x\in\Omega} b(x)$, we have

$$I_2 \ge m\int_\Omega u^{2k+2}.$$

Finally, using the Young inequality, and denoting $M := \|\vec{c}\|_\infty$, we have

$$\begin{aligned} I_3 &\le \int_\Omega (\sqrt{2C_1(2k+1)}|\nabla u||u|^k)\left(\frac{M|u|^{k+1}}{\sqrt{2C_1(2k+1)}}\right) \\ &\le C_1(2k+1)\int_\Omega u^{2k}|\nabla u|^2 + \frac{M^2}{4C_1(2k+1)}\int_\Omega u^{2k+2}. \end{aligned}$$

Putting everything together, we deduce

$$m\int_\Omega u^{2k+2} \le \frac{M^2}{4C_1(2k+1)}\int_\Omega u^{2k+2}.$$

Setting

$$k = \frac{M^2}{4mC_1} - \frac{1}{2}$$

it follows

$$\frac{m}{2}\int_\Omega u^{2k+2} \le 0 \implies u \equiv 0$$

and the proof is complete.

b2) + b3) Alternative approach: In previous two steps, we needed to justify to use of u^{2k+1} as a test function and we used the fact that u is bounded. However, we can overcome the proof of boundedness by the following approach. Let $n \in \mathbb{N}$ and $s \ge 0$ be arbitrary. We denote the truncation function T_n as

$$T_n(s) := \min\{n, |s|\}\text{sign } s.$$

Then the function $\varphi := u|T_n(u)|^p$ belongs to $W_0^{1,2}(\Omega)$, which can be proven similarly as the case in Step a). Thus, using such φ in (1.12), we obtain

$$\int_\Omega \mathbb{A}\nabla u \cdot \nabla(u|T_n(u)|^p) + buu|T_n(u)|^p + \vec{c} \cdot \nabla uu|T_n(u)|^p = 0. \tag{1.16}$$

Our goal is to let $n \to \infty$ in the above identity and then to use the procedure described above. Thus, it only remains to justify rigorously the limit passage $n \to \infty$.

Using the ellipticity of the matrix $\mathbb{A}$ and the Hölder inequality, we deduce (similarly as above)

(1.17)
$$
\begin{aligned}
&\frac{4C_1(p+1)}{(p+2)^2}\int_\Omega \left|\nabla |T_n(u)|^{\frac{p+2}{2}}\right|^2 \le \frac{4(p+1)}{(p+2)^2}\int_\Omega \mathbb{A}\nabla |T_n(u)|^{\frac{p+2}{2}}\cdot \nabla |T_n(u)|^{\frac{p+2}{2}}\\
&= (p+1)\int_\Omega \mathbb{A}\nabla u\cdot\nabla u|T_n(u)|^p\chi_{\{|u|<n\}} \le \int_\Omega \mathbb{A}\nabla u\cdot\nabla u\left(|T_n(u)|^p + p|T_n(u)|^p\chi_{\{|u|<n\}}\right)\\
&\le \int_\Omega \mathbb{A}\nabla u\cdot\nabla(u|T_n(u)|^p) + buu|T_n(u)|^p \overset{(1.16)}{=} -\int_\Omega \vec{c}\cdot\nabla uu|T_n(u)|^p\\
&\le C\int_\Omega |\nabla u||u|^{1+p},
\end{aligned}
$$

whenever the last term on the right hand side is well defined and finite. Finally, we show, that the last term is well defined for any $p\ge 0$. To do so, we continue inductively. We know that in any dimension we have $W^{1,2}\hookrightarrow L^{2+\alpha}$ with some $\alpha>0$. Next, we define $p_0:=0$ and $p_{k+1}:=p_k+a/2$. Evidently,

$$
\int_\Omega |\nabla u||u|^{1+p_0}<\infty.
$$

Next we show that

$$
\int_\Omega |\nabla u||u|^{1+p_k}<\infty \implies \int_\Omega |\nabla u||u|^{1+p_{k+1}}<\infty. \tag{1.18}
$$

Consequently, we can choose p arbitrarily large in (1.17) and the proof is complete.

To show (1.18), we use (1.17) with $p:=p_k$ and since the right hand side is finite, we can let $n\to\infty$ to deduce that

$$
\int_\Omega \left|\nabla |u|^{\frac{p_k+2}{2}}\right|^2<\infty \implies \int_\Omega |u|^{p_k}|\nabla u|^2<\infty \tag{1.19}
$$

but we can also use the embedding theorem and the Pincaré inequality to get

$$
\int_\Omega \left|\nabla |u|^{\frac{p_k+2}{2}}\right|^2<\infty \implies \||u|^{\frac{p_k+2}{2}}\|_{1,2}<\infty \implies \int_\Omega |u|^{\frac{(p_k+2)(2+a)}{2}}<\infty. \tag{1.20}
$$

To show (1.18) we use the Young inequality and (1.19)–(1.20) as follows

$$
\begin{aligned}
&\int_\Omega |\nabla u||u|^{1+p_{k+1}} \le \int_\Omega |\nabla u|^2|u|^{p_k} + |u|^{2+2p_{k+1}-p_k} = \int_\Omega |\nabla u|^2|u|^{p_k}+|u|^{2+p_k+a}\\
&\le \int_\Omega |\nabla u|^2|u|^{p_k}+|u|^{\frac{(p_k+2)(2+a)}{2}}+1 \overset{(1.19),(1.20)}{<} \infty.
\end{aligned}
$$

Therefore (1.18) holds true.

Problem 5: Lax–Milgram lemma vs Fredholm alternative II. Consider $\Omega\subset\mathbb{R}^d$ a Lipschitz domain. Let $a,b\in\mathbb{R}$ Consider the problem: For given $\vec{f}=(f_1,f_2)\in L^2(\Omega)\times L^2(\Omega)$, find $\vec{u}=(u_1,u_2)\in W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)$ solving

$$
\begin{aligned}
-\Delta u_1 - a\Delta u_2 + u_1 &= f_1 &&\text{in }\Omega,\\
-\Delta u_2 - b\Delta u_1 + u_2 &= f_2 &&\text{in }\Omega,\\
u_1=u_2&=0 &&\text{on }\partial\Omega.
\end{aligned} \tag{$\mathcal{S}$}
$$

Under which conditions on a, b the system $(\mathcal{S})$ has for any $\vec{f}$ a weak solution? Under which condition on $\vec{f}$, the system $(\mathcal{S})$ has a solution?
Hint: a) First, try to use the Lax-Milgram lemma. Consider the space $V := W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ and proper bilinear form (follow the lecture).
b) Use the spectral theory or the Fredholm alternative and/or be creative:)

DEADLINE: November 27

Solution: Method a): Let us denote $V := W_0^{1,2}(\Omega)$. The weak solution is defined as $u = (u_1, u_2) \in V \times V$ solving for all $\zeta, \psi \in V$

$$\int_\Omega \nabla u_1 \cdot \nabla \zeta + a \nabla u_2 \cdot \nabla \zeta + u_1 \zeta = \langle f_1, \zeta \rangle,$$
$$\int_\Omega \nabla u_2 \cdot \nabla \psi + b \nabla u_1 \cdot \nabla \psi + u_2 \psi = \langle f_2, \psi \rangle.$$

Next we try to use the Lax–Milgram theorem. The first attempt is as follows. Let $A, B > 0$ be arbitrary. We define (here $u := (u_1, u_2)$ and $\varphi := (\varphi_1, \varphi_2)$)

$$B(u, \varphi) := A \int_\Omega \nabla u_1 \cdot \nabla \varphi_1 + a \nabla u_2 \cdot \nabla \varphi_1 + u_1 \varphi_1 + B \int_\Omega \nabla u_2 \cdot \nabla \varphi_2 + b \nabla u_1 \cdot \nabla \varphi_2 + u_2 \varphi_2$$

and

$$\langle F, \varphi \rangle := A \langle f_1, \varphi_1 \rangle + B \langle f_2, \varphi_2 \rangle.$$

Then to find weak solutions to $(\mathcal{S})$ is equivalent to find $u = (u_1, u_2) \in V \times V$ such that for all $\varphi = (\varphi_1, \varphi_2) \in V \times V$ there holds

$$(\mathcal{S}_1) \qquad B(u, \varphi) = \langle F, \varphi \rangle_{V \times V}.$$

Next, we try to solve $(\mathcal{S}_1)$ by using the Lax–Milgram theorem and proper choice of A, B. We define the Hilbert space

$$H := V \times V.$$

We check whether the assumptions of the Lax–Milgram lemma are satisfied. First, H is evidently a Hilbert space that can be endowed (by using also the Poincaré inequality) by the scalar product

$$(u, \varphi) := \int_\Omega \sum_{i=1}^d \nabla u_i \cdot \nabla \varphi_i.$$

The Hölder inequality and the linearity of the integral also implies that $F \in H^*$. Similarly, we can directly deduce that $\mathcal{B}$ is bilinear and H-bounded. It remains to check whether it is also H-elliptic. Using the definition of $\mathcal{B}$ we see that

$$\begin{aligned}
\mathcal{B}(u, u) &= \int_\Omega A|\nabla u_1|^2 + A u_1^2 + B|\nabla u_2|^2 + B u_2^2 + (Aa + Bb) \nabla u_1 \cdot \nabla u_2 \\
&\geq A\|u_1\|_V^2 + B\|u_2\|_V^2 - |Aa + Bb| \|u_1\|_V \|u_2\|_V \\
&\geq A\|u_1\|_V^2 + B\|u_2\|_V^2 - A(1-\varepsilon)\|u_1\|_V^2 - \frac{|Aa + Bb|^2}{4A(1-\varepsilon)} \|u_2\|_V^2 \\
&= A\varepsilon \|u_1\|_V^2 + \left(B - \frac{|Aa + Bb|^2}{4A(1-\varepsilon)} \right) \|u_2\|_V^2 \geq C_1 \|u\|_H^2,
\end{aligned}$$

where the last inequality is true if we can find positive ε, A, B such that

$$B - \frac{|Aa + Bb|^2}{4A(1-\varepsilon)} > 0$$

and since ε can be arbitrarily small, it is equivalent to

$$4AB > (Aa + Bb)^2 \iff (2\sqrt{AB} + Aa + Bb)(2\sqrt{AB} - Aa - Bb) > 0.$$

First, in case $ab < 0$, we can simply set $A := |b|$ and $B := |a|$ and we see that the above inequality holds true. Next we assume $ab \geq 0$. Without loss of generality we may assume that $a \geq 0$ and $b \geq 0$. The case $a = 0$ or $b = 0$ is solved trivially, so assume that $ab > 0$. Then the above inequality reduces to

$$0 > Aa + Bb - 2\sqrt{AB} \geq Bb - B/a = \frac{B}{a}(ab - 1).$$

Hence, we see that we need

$$\boxed{ab < 1}$$

Next, we try to use the Lax–Milgram theorem again but we modify the bilinear form such that the case when $ab \gg 1$ will be handled. Without loss of generality we assume that $a > 0$ and $b > 0$ (the opposite case is treated similarly). We define new bilinear form as

$$B(u, \varphi) := A \int_\Omega \nabla u_1 \cdot \nabla \varphi_2 + a \nabla u_2 \cdot \nabla \varphi_2 + u_1 \varphi_2 + B \int_\Omega \nabla u_2 \cdot \nabla \varphi_1 + b \nabla u_1 \cdot \nabla \varphi_1 + u_2 \varphi_1$$

and

$$\langle F, \varphi \rangle := A \langle f_1, \varphi_2 \rangle + B \langle f_2, \varphi_1 \rangle.$$

Hence we just switched the role of u_1 and u_2. Clearly, F belongs again to H^*, B is bilinear and bounded. We just need to check the coercivity. Using the Hölder inequality, we have

$$\begin{aligned} B(u, u) &:= A \int_\Omega \nabla u_1 \cdot \nabla u_2 + a \nabla u_2 \cdot \nabla u_2 + u_1 u_2 + B \int_\Omega \nabla u_2 \cdot \nabla u_1 + b \nabla u_1 \cdot \nabla u_1 + u_2 u_1 \\ &\geq Aa \|\nabla u_2\|_2^2 + Bb \|\nabla u_1\|_2^2 + (A + B) \|\nabla u_1\|_2 \|\nabla u_2\|_2 + (A + B) \|u_1\|_2 + \|u_2\|_2 \end{aligned}$$

Next, we recall Poincaré inequality. Since we shall use it also later, we recall that for all $v \in W_0^{1,2}(\Omega)$, we have

$$\|\nabla v\|_2^2 \geq \lambda_1 \|v\|_2^2,$$

where $\lambda > 0$ is the smallest eigenvalue of the Laplace operator. Consequently, we deduce with the help of the Young inequality

$$\begin{aligned} B(u, u) &\geq Aa \|\nabla u_2\|_2^2 + Bb \|\nabla u_1\|_2^2 + (A + B)(1 + 1/\lambda_1) \|\nabla u_1\|_2 \|\nabla u_2\|_2 \\ &\geq Aa\varepsilon \|\nabla u_2\|_2^2 + \left(Bb - \frac{(A + B)^2 (1 + \lambda_1)^2}{\lambda_1^2 4Aa(1 - \varepsilon)} \right) \|\nabla u_1\|_2^2 \geq \alpha \|u\|_H^2, \end{aligned}$$

provided we have

$$Bb - \frac{(A + B)^2 (1 + \lambda_1)^2}{\lambda_1^2 4Aa} > 0 \iff ab > \frac{(A + B)^2 (1 + \lambda_1)^2}{\lambda_1^2 4AB}.$$

Doing the optimal choice $A = B$, we deduce that the sufficient condition is

$$ab > \left(1 + \frac{1}{\lambda_1} \right)^2.$$

Thus, we see that for any $f_1, f_2 \in V^*$ we have the existence of a unique solution $(u_1, u_2) \in H$, provided

$$ab \notin \left[1, (1 + 1/\lambda_1)^2\right].$$

We discuss also the case $ab = 1$. We multiply the first equation by b to have

$$\int_\Omega b\nabla u_1 \cdot \nabla\zeta + \nabla u_2 \cdot \nabla\zeta + bu_1\zeta = b\langle f_1, \zeta\rangle,$$

$$\int_\Omega \nabla u_2 \cdot \nabla\psi + b\nabla u_1 \cdot \nabla\psi + u_2\psi = \langle f_2, \psi\rangle.$$

Denoting $w_1 := bu_1 + u_2$ and $w_2 := u_2 - bu_1$, we see that we may write

$$\int_\Omega 2\nabla w_1 \cdot \nabla\varphi + w_1\varphi = \langle bf_1 + f_2, \varphi\rangle, \tag{1.21}$$

$$\int_\Omega w_2\psi = \langle f_2 - bf_1, \psi\rangle. \tag{1.22}$$

Thus we see that the first equation is uniquely solvable whenever $f_1, f_2 \in V^*$ but to obtain $w_2 \in W_0^{1,2}$ we require more smoothness for the right hand sides, i.e., we must have

$$\boxed{ab = 1 \qquad \text{and} \qquad f_2 - bf_1 \in W_0^{1,2}(\Omega).}$$

Method b): Here, we simply set $A = B = 1$ and consider the basis of $W_0^{1,2}(\Omega)$ of the form $\{w^k\}_{k=1}^\infty$, which consists of eigen functions of the Laplace operator, i.e., they fulfill

$$-\Delta w^k = \lambda_k w^k \text{ in } \Omega \text{ and } w_k = 0 \text{ on } \partial\Omega.$$

Here $\{\lambda_k\}_{k=1}^\infty$ are eigen-values, which are positive,nondecreasing and tend to infinity. We know from the lecture, that such basis exists, can be made orthogonal in $W_0^{1,2}(\Omega)$ and orthonormal in $L^2(\Omega)$.

We look for solution in the form of Fourier series in $W_0^{1,2}$ of the form

$$u_i = \sum_{j=1}^\infty u_i^j w_j.$$

Due to orthonormality, we can also write f_i in terms of Fourier series as follows

$$f_i = \sum_{j=1}^\infty f_i^j w_j$$

where f_i^j are given as $f_i^j := \int_\Omega f_i w^j = \langle f_i, w^j\rangle$. In addition, setting $\zeta, \psi := w_j$ with $j = 1, \ldots, \infty$, we see that the weak solution satisfy for all $j = 1, \ldots, \infty$

$$\int_\Omega \nabla u_1 \cdot \nabla w_j + a\nabla u_2 \cdot \nabla w_j + u_1 w_j = \langle f_1, w_j\rangle,$$

$$\int_\Omega \nabla u_2 \cdot \nabla w_j + b\nabla u_1 \cdot \nabla w_j + u_2 w_j = \langle f_2, w_j\rangle.$$

which is reduced (by using the properties of the basis) to

$$(\lambda_j + 1)u_1^j + a\lambda_j u_2^j = f_1^j,$$

$$b\lambda_j u_1^j + (\lambda_1 + 1)u_2^j = f_2^j.$$

Hence, defining for each $j \in \mathbb{N}$

$$\mathbb{A}_j := \begin{pmatrix} \lambda_j + 1 & a\lambda_j \\ b\lambda_j & (\lambda_j + 1) \end{pmatrix}, \qquad \vec{F}_j := \begin{pmatrix} f_1^j \\ f_2^j \end{pmatrix}$$

we see that to solve the original problem, it is equivalent to solve for each $j \in \mathbb{N}$

$$\mathbb{A} \begin{pmatrix} u_1^j \\ u_2^j \end{pmatrix} = \vec{F}_j$$

Since $\det \mathbb{A} = (\lambda_j + 1)^2 - ab\lambda_j^2$ we can deduce that if for each j there holds

$$ab \neq (1 + 1/\lambda_j)^2$$

then we can find uniquely defined u_i^j. In case $ab = (1 + 1/\lambda_j)^2$ then we require

$$b\lambda_j f_1^j = (\lambda_1 + 1) f_2^j \Longleftrightarrow b f_1^j = \sqrt{ab} f_2^j.$$

It remains to check that (we use the properties of the eigen vectors)

$$\|\nabla u_i\|_2^2 = \sum_{j=1}^{\infty} \lambda_j (u_i^j)^2 < \infty.$$

Since

$$\mathbb{A}_j^{-1} := \frac{1}{(\lambda_j + 1)^2 - \lambda_j^2 ab} \begin{pmatrix} \lambda_j + 1 & -a\lambda_j \\ -b\lambda_j & \lambda_j + 1 \end{pmatrix}$$

and then we can also show that if $ab \neq 1$ then there exists a constant such that

$$|u_i^j| \leq C(|f_1^j| + |f_2^j|)\lambda_j.$$

Consequently,

$$\sum \lambda_j (u_i^j)^2 \leq C \sum (|f_1^j|^2 + |f_2^j|^2)/\lambda_j = C(\|f_1\|_{V^*}^2 + \|f_2\|_{V^*}^2).$$

In case $ab = 1$, we however see that

$$\begin{pmatrix} u_1^j \\ u_2^j \end{pmatrix} = \mathbb{A}^{-1} \begin{pmatrix} f_1^j \\ f_2^j \end{pmatrix} = \frac{1}{1 + 2\lambda_j} \begin{pmatrix} \lambda_j + 1 & -a\lambda_j \\ -b\lambda_j & \lambda_j + 1 \end{pmatrix} \begin{pmatrix} f_1^j \\ f_2^j \end{pmatrix}$$

and we see that $|u_i^j| \lesssim (|f_1^j| + |f_2^j)$. Hence we require that f_1, f_2 belong to the better space. This was already discussed in the first part.