Linear parabolic equations with general data

$$\boxed{\begin{aligned} &\partial_t u + Lu = f && \text{in } Q := (0,T)\times\Omega \\ &u = u_D && \text{on } (0,T)\times\Gamma_1 \\ &(A\nabla u - du)n + \alpha u = g && \text{on } (0,T)\times\Gamma_2 \\ &(A\nabla u - du)n = g && \text{on } (0,T)\times\Gamma_3 \\ &u(0) = u_0 && \text{in } \Omega \end{aligned}}$$

$Lu = -\operatorname{div} A\nabla u + bu + c\cdot\nabla u + \operatorname{div}(\vec{d}u)$

Γ_i: mutually disjoint open in $\partial\Omega$ and $|\Gamma_1\cup\Gamma_2\cup\Gamma_3| = |\partial\Omega|$

A-elliptic, c, d, b, μ - bounded

- we did everything for $\Gamma_1 = \partial\Omega$ and $u_D \equiv 0$
- we need to include also Neumann or Newton data
- For u_D and g independent of "t" - almost no change in the proof
- For u_D and g depending on "t" - more assumptions w.r.t. time needed

Weak formulation

We set $V := \{v \in W^{1,2}(\Omega);\ v = 0 \text{ on } \Gamma_1\}$

Let $u_D \in L^2(0,T; W^{1,2}(\Omega)) \cap W^{1,2}(0,T; V^*) \cap C([0,T]; L^2(\Omega))$ and $f \in L^2(0,T; V^*)$, $g \in L^2(0,T; (W^{\frac{1}{2},2}(\partial\Omega))^*)$ and $u_0 \in L^2(\Omega)$. We say that u is a weak solution to $\boxed{}$ iff

Gelfand triple

$V \overset{\text{dense}}{\hookrightarrow} H = L^2(\Omega) = (L^2(\Omega))^* \hookrightarrow V^*$

- $\boxed{u - u_D \in L^2(0,T; V) \cap W^{1,2}(0,T; V^*)}$

- for all $v \in V$ and almost all $t \in (0,T)$

$$\langle \partial_t u, v\rangle_V + \int_\Omega A\nabla u\cdot\nabla v + buv + \vec{c}\cdot\nabla u\, v - \vec{d}u\cdot\nabla v + \int_{\Gamma_2} \mu u v = \langle f, v\rangle + \langle g, v\rangle_{W^{\frac{1}{2},2}(\Gamma_2\cup\Gamma_3)}$$

- $\boxed{u(0) = u_0 \text{ in } \Omega}$

$\Big((u - u_D) \in L^2(0,T; V) \cap W^{1,2}(0,T; V^*)$ and $V \overset{\text{dense}}{\hookrightarrow} L^2(\Omega) \hookrightarrow V^*$

$\Rightarrow (u - u_D) \in C([0,T]; L^2(\Omega))$ and since $u_D \in C([0,T]; L^2(\Omega))$

we have $u \in C([0,T]; L^2(\Omega))$ so it is meaningful

To talk about $u(0)$? $\Big)$

THEOREM: There exists unique weak solution to $\boxed{}$

NOTES TO PROOF:

① Uniqueness: u_1, u_2 – weak solutions, set $v := u_1 - u_2$. From $\boxed{}$ it follows

$$\langle \partial_t v, w \rangle + \int_\Omega A \nabla v \cdot \nabla w + b v w + \vec{c} \cdot \nabla v\, w + \vec{d} v \cdot \nabla w + \int_{\Gamma_2} \lambda v \cdot w = 0$$

Since $v = (u_1 - u_D) - (u_2 - u_D) \Rightarrow v \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$ and also $v(0) = (u_1(0) - u_D(0)) - (u_2(0) - u_D(0)) = 0$

Set $w := v$ in ... Long computation based on YOUNG, HÖLDER, TRACE THM

$$\frac{d}{dt}\|v\|_2^2 + C_1 \|\nabla v\|_2^2 \le \hat{C}\left(\|v\|_2^2 + \int_{\partial\Omega} |v|^2\right) \quad \text{for a.a. } t \in (0,T)$$

$$\overset{\text{Trace + interpolation}}{\le} \tilde{\tilde{C}}\left(\|v\|_2^2 + \|v\|_2 \|v\|_{1,2}\right)$$

$$\overset{\text{Young}}{\le} \hat{c}\, \|v\|_2^2 + C_1 \|\nabla v\|_2^2$$

$$\Rightarrow \frac{d}{dt}\|v\|_2^2 \le \hat{C}\|v\|_2^2 \overset{\text{Gronwall}}{\Rightarrow} \|v(t)\|_2^2 \le e^{\hat{c}t}\|v(0)\|_2^2 = 0$$

$\Rightarrow v = 0$ a.e. in $Q \Rightarrow u_1 = u_2$ a.e. in Q.

② Existence: V – separable; $\{w_j\}_1^\infty$ – basis of V – orthonormal in $L^2(\Omega)$ and orthogonal in V ($\int_\Omega \nabla w_i \cdot \nabla w_j = 0$ for $i \ne j$)

P^n: projection to the span of $\{w_j\}_{j=1}^n$ such that IMPORTANT!

$\exists C > 0\ \forall n \in \mathbb{N}\ \forall u \in V: \|P^n u\|_V \le C\|u\|_V$ (for free $\|P^n u\|_L \le \|u\|_2$)

Galerkin approximation:

$u^n := u_D + \sum_{i=1}^n a_i^n(t) w_i(x)$ with initial condition $(u^n - u_D)(0) := P^n(u_0 - u_D(0))$

fulfilling for a.a. $t \in (0,T)$ and all $j = 1, \dots, n$

$$\boxed{\int_\Omega \partial_t u^n w_j + A\nabla u^n \cdot \nabla w_j + b u^n w_j + \vec{c} \cdot \nabla u^n w_j - u^n \vec{d} \cdot \nabla w_j + \int_{\Gamma_2} \lambda u^n w_j = \langle f, w_j \rangle + \langle g, w_j \rangle_{\Gamma_2}}$$

STEP 2a Existence of Galerkin approximation

STEP 2b Convergence $n \to \infty$

2a Existence: (abbreviation $\Sigma := \sum_{i=1}^{n} a_i^n w_i$)

We can rewrite $\boxed{}$ as

$$\int_\Omega \partial_t(\Sigma) w_j + \int_\Omega A\nabla\Sigma\cdot\nabla w_j \dots = \dots - \int_\Omega A\nabla u_D\cdot\nabla w_j \dots - \langle \partial_t u_D, w_j\rangle$$

$\|$ ← orthonormality

$$\sum_{i=1}^{n} (a_i^n)'(t) \int_\Omega w_i w_j = (a_j^n)'(t)$$

Hence $\boxed{}$ is a system of ODE's: $\vec{a} := (a_1^n, \dots, a_n^n)$

$$\frac{d}{dt}\vec{a}(t) = \vec{F}(a(t), t) \quad \text{with } F \text{ - measurable w.r.t "t"}$$
$$\text{- continuous w.r.t "}\vec{a}\text{"}$$

Carathéodory theory ⇒ existence of a solution on a short time interval

Estimates from next step ⇒ existence on the whole time interval $(0,T)$

2b Convergence $n \to \infty$

(i) Uniform estimates: From $\boxed{}$ we have the identity

$$\langle \partial_t u^n, u^n - u_D\rangle + \int_\Omega A\nabla u^n\cdot\nabla(u^n - u_D) + b u^n(u^n - u_D) + \vec{c}\cdot\nabla u^n(u^n - u_D) + \vec{d}\cdot\nabla(u^n - u_D)u^n$$
$$+ \int_\Omega g u^n(u^n - u_D) = \langle f, u^n - u_D\rangle$$

$$\geq c_1\|\nabla u^n\|_2^2 - \underbrace{\tilde{c}\int_\Omega |\nabla u^n||\nabla u_D|}_{\text{TO RHS}}$$

$$= \langle \partial_t(u^n - u_D), u^n - u_D\rangle + \underbrace{\langle \partial_t u_D, u^n - u_D\rangle}_{\text{To the RIGHT HAND SIDE}}$$

Similar procedure as those used for homogeneous Dirichlet (Hölder, Young, trace)

⇒

$$\frac{d}{dt}\|u^n - u_D\|_2^2 + \frac{c_1}{4}\|\nabla u^n\|_2^2 \leq C\Big(\|u^n\|_2^2 + \|f\|_{V^*}^2 + \|g\|^2_{W^{1/2,2}(\partial\Omega)^*} + \|u_D\|_V^2 + \|\partial_t u_D\|_{V^*}\Big)$$

depends on $A, b, \vec{c}, \vec{d}, \lambda$

$$\leq 2\|u^n - u_D\|_2^2 + 2\|u_D\|_V^2$$

Gronwall ⇒

$$\sup_t \|u^n - u_D\|_2^2 + \int_0^T \|\nabla u^n\|_2^2 \leq \tilde{C}\left(1 + \|(u^n - u_D)(0)\|_2^2\right) \leq \tilde{C}\left(1 + \|u_0 - u_D(0)\|_2^2\right) \leq C$$

Properties of P^n

INDEPENDENT OF n !!

(ii) $\{u^n\}$ - bounded sequence in nice reflexive spaces → weak convergence

→ the same procedure as for homogeneous Dirichlet → existence

MAXIMUM AND MINIMUM PRINCIPLES (EXCURSION)

$$\left.\begin{aligned} \partial_t u + Lu &= f \quad \text{in } Q \\ u(0) &= u_0 \quad \text{in } \Omega \end{aligned}\right\} \quad \text{and} \quad \begin{aligned} &u \in L^2(0,T; H^{1,2}(\Omega)) \\ &\partial_t u \in L^2(0,T; (H_0^{1,2}(\Omega))^*) \end{aligned} \quad \text{is } \underline{\text{weak solution}}$$

Q1: Set $m := \operatorname{essinf} u(t,x)$ on $\{(0,T)\times\partial\Omega\} \cup \{\{0\}\times\Omega\}$.
(= essential infimum on parabolic boundary)
Under which assumption we know that $u \geq m$ a.e. in $(0,T)\times\Omega$

Q2: If the answer on Q1 is negative, is there any quantity $F(u,t)$, which satisfies the minimum principle?

PARTIAL ANSWER 1) CASE: $Lu := -\Delta u$ and $f \geq 0$

Weak formulation: $\langle \partial_t u, w\rangle + \int_\Omega \nabla u \cdot \nabla w = \int_\Omega f w \quad \forall w \in H_0^{1,2}(\Omega)$ and a.e. $t \in (0,T)$

SET $w := (u-m)_- = \min\{0, u-m\}$

!! IMPORTANT $\underline{w = 0 \text{ on } (0,T)\times\partial\Omega}$ $\Rightarrow$ $w \in W_0^{1,2}(\Omega)$ for a.a. t

$$\underbrace{\langle \partial_t u, (u-m)_-\rangle}_{= \langle \partial_t (u-m), (u-m)_-\rangle = \frac{1}{2}\frac{d}{dt}\|(u-m)_-\|_2^2} + \underbrace{\int_\Omega \nabla u \cdot \nabla (u-m)_-}_{= \|\nabla (u-m)_-\|_2^2 \geq 0} = \int_\Omega \underbrace{f}_{\geq 0} \underbrace{(u-m)_-}_{\leq 0} \leq 0$$

$$\Rightarrow \frac{d}{dt}\|(u-m)_-\|_2^2 \leq 0 \quad \Rightarrow \quad \|(u-m)_-(t)\|_2^2 \leq \underbrace{\|(u-m)_-(0)\|_{L^1}}_{\text{DEFINITION OF } m} = 0$$

$$\Rightarrow (u-m)_- = 0 \quad \text{a.e. in } (0,T)\times\Omega$$

$$\Rightarrow u \geq m \quad \text{a.e. in } (0,T)\times\Omega$$

PARTIAL ANSWER 2: CASE $Lu = -\Delta u + bu$

We try to mimic CASE 1: Let $m \in \mathbb{R}$ be a number satisfying $m \leq u(t,x)$ for all $(t,x) \in \partial Q$ (parabolic boundary) and test by $(u-m)_-$

$$\frac{1}{2}\frac{d}{dt}\|(u-m)_-\|_2^2 + \|\nabla (u-m)_-\|_2^2 + \int_\Omega b\, u\, (u-m)_- = \int_\Omega f (u-m)_-$$

$$\Rightarrow \frac{d}{dt}\|(u-m)_-\|_2^2 \leq \int_\Omega (f - bu)(u-m)_- \overset{?}{\leq} \underset{\in L^1(0,T)}{G(t)} \|(u-m)_-\|_2^2$$

IF ? holds, we can use the Gronwall lemma

$$\Rightarrow \|(u-m)_-(t)\|_2 \leq 0 \quad \Rightarrow \quad u \geq m \quad \text{a.e.}$$

HOW TO CHOOSE m ??

$$\int_\Omega (f-bm)(u-m)_- = -\int_\Omega b(u-m)(u-m)_- + \int_\Omega (f-bm)(u-m)_-$$

$$\leq \underbrace{c\|(u-m)_-\|_2^2}_{\partial_t} + \int_\Omega (f-bm)(u-m)_- \leq c\|(u-m)_-\|_L^2$$

IF m is such that $f-bm \geq 0$ a.e in Q then

PARTIAL ANSWER 3: CASE $\partial_t u - \Delta u + bu = f$

Formal proof: multiply equation by $e^{\int_0^t g(\tau)d\tau}$ with some $g \in L^1(0,T)$.

Define $v := u e^{\int_0^t g(\tau)d\tau}$, then

$$\boxed{\partial_t v + (b-g)v - \Delta v = f e^{\int_0^t g(\tau)d\tau}}$$

Repeat CASE 2: so if $\boxed{v \geq m}$ on ∂Q and $\boxed{f e^{\int_0^t g(\tau)d\tau} - (b-g)m \geq 0}$

Then $v \geq m$ a.e. in Q

We rewrite it in terms of u:

So if ① $u \geq m e^{-\int_0^t g(\tau)d\tau}$ on ∂Q
② $f \geq e^{-\int_0^t g(\tau)d\tau}(b-g)m$ } then $u \geq m e^{-\int_0^t g(\tau)d\tau}$ a.e. in Q

EXAMPLE 1: $f \geq 0$, set $g(t) := \sup_x b(t,x)$. Then ② is (and $u \geq 0$ on ∂Q) satisfied for any $m \geq 0$.

Set $m := \max\{0, \inf_{(x,t)\partial Q} u(t,x)\} \Rightarrow u \geq m e^{-\int_0^t \|b(t)\|_\infty}$

EXAMPLE 2: $f \geq 0$ and $u \leq 0$ on ∂Q set $g(t) := \inf_x b(t,x)$. Then ② is true for any $m \leq 0$

set $m := \min\{0, \inf u(t,x)\}$

EXAMPLE 3: $f \geq -A$; $b \leq 0$, SET $g := Aa$

if $m := \frac{e^{A_a \tau}}{a} \leq u$ on $(0,T)\times\partial\Omega \cup \{0\}\times\Omega$

Then $u \geq m e^{-\int_0^t g} = \frac{1}{a}$