PARTIAL DIFFERENTIAL EQUATIONS 2

20.2.2019

WINTER (babies)

- $W^{k,p}(\Omega)$
- linear elliptic equations

  Lax-Milgram
- linear parabolic + hyperbolic eq.

  Galerkin
- Fredholm

$$\Updownarrow$$

Minimization of quadratic functionals

$$\underbrace{\begin{array}{c} -\Delta u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{array}}$$

$$\Updownarrow$$

$$\overbrace{\min_{u\in W_0^{1,2}} \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fu}$$

SUMMER (teenagers)

- prove it
- nonlinearity:
- 12-14: nonlinearity in lower order term

  $-\Delta u + \sin u = f$
- 14-18: nonlinearity in leading term

  $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f$

  Minty, monotone operator

$$\Updownarrow$$

Minimization of convex functionals

$$\underbrace{\begin{array}{c} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{array}}$$

$$\Updownarrow$$

$$\overbrace{\min_{u\in W_0^{1,p}} \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega f u}$$

- Regularity theory (introduction to)

0. LEBESGUE SPACES, FIXED POINT THEOREMS, FUNCTIONAL ANALYSIS

Luzin theorem: Let Ω be measurable and $f \in L^1_{loc}(\Omega)$. Then

$\forall \varepsilon > 0 \quad \exists U \subseteq \Omega, \ |U| < \varepsilon$ and $f \in C(\Omega \setminus U)$.

Egorov (Jegorov) theorem: Let Ω be measurable and $f, f^n \in L^1_{loc}(\Omega)$,

$f^n \to f$ in $L^1_{loc}(\Omega)$ $\quad (\Leftrightarrow \forall$ compact $K \subset\subset \Omega : \int_K |f^n - f| \to 0)$.

Then $\forall \varepsilon > 0 \quad \exists U \subseteq \Omega, \ |U| < \varepsilon$ and $f^n \to f$ in $C(\Omega \setminus U)$.

Lebesgue dominated convergence theorem

Vitali convergence theorem: Let Ω be measurable f^n be a sequence of measurable functions, $f^n(x) \to f(x)$ for a.a. $x \in \Omega$.

Then $\lim\limits_{n\to\infty} \int_\Omega f^n = \int_\Omega f$, provided that the sequence f^n is uniformly equiintegrable $(\Leftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall U \subseteq \Omega \ |U| < \delta \ \forall n \ \int_U |f^n| < \varepsilon)$

Fatou lemma: -"- and $f^n \geq 0$. Then $\int_\Omega f \leq \liminf\limits_{n\to\infty} \int_\Omega f^n$.

Literature: Lukeš, Malý: Measure and integral

Kufner, John, Fučík: Function spaces

Regularization.

Def: Regularization kernel: $\eta \in C_0^\infty(B_1(0))$ nonnegative, radially symmetric & $\int_{B_1(0)} \eta(x)dx = 1$.

Reg. of f: Let $f \in L^p(\Omega)$ with $p \in [1,\infty)$. We extend f by zero outside Ω and define

$f_\varepsilon := \eta_\varepsilon * f$, where $\eta_\varepsilon(x) = \frac{1}{\varepsilon^d} \eta\left(\frac{x}{\varepsilon}\right)$. $\quad (\Leftrightarrow f_\varepsilon(x) = \int_{\mathbb{R}^d} \eta_\varepsilon(x-y) f(y) dy)$.

$\forall p \in [1,\infty)$: if $f \in L^p(\Omega)$ then $f_\varepsilon \to f$ in $L^p(\Omega)$

$p = \infty$: $f_\varepsilon \to f$ a.e. and $f_\varepsilon \rightharpoonup^* f$ in $L^\infty(\Omega)$ $\quad (\Leftrightarrow \forall g \in L^1(\Omega) : \int_\Omega f_\varepsilon g \to \int_\Omega fg)$.

Reflexivity, separability, weak and weak* convergences

Theorem: $L^p(\Omega)$ is Banach, separable for $p \in [1,\infty)$ and reflexive for $p \in (1,\infty)$.

If $\{f^n\}_{n=1}^\infty$ is a bounded sequence in $L^p(\Omega)$ then there exists a subsequence such that

$f^{n_k} \rightharpoonup f$ in $L^p(\Omega)$, $p \in (1,\infty)$ $\quad (\Leftrightarrow \forall g \in L^{p'} \ \int_\Omega f^{n_k} g \to \int_\Omega fg)$

$f^{n_k} \rightharpoonup^* f$ in $L^\infty(\Omega)$

$f^{n_k} \rightharpoonup^* f$ in $M(\Omega)$ (Radon measures) $p = 1$.

Fixed point theorems.

1. Let F be continuous from $\mathbb{R}^d \to \mathbb{R}^d$. Assume that $\exists$ convex closed set Ω such that $F(\Omega) \subseteq \Omega$. Then $\exists x \in \Omega$: $F(x) = x$.
2. Let $F: X \to X$ (X - Banach space), F is continuous and compact, $\exists$ convex closed $\Omega \subseteq X$, $F(\Omega) \subseteq \Omega$. Then $\exists x \in \Omega$: $F(x) = x$.

Remark: Note that in infinite dimension (2.) we need compactness.

Nemytskii operator.

Def. & theorem: Let $\Omega \subseteq \mathbb{R}^d$ be open and $f: \Omega \times \mathbb{R}^N \to \mathbb{R}$. We say that f is Carathéodory

if 1. $\forall y \in \mathbb{R}^N$ $f(\cdot, y)$ is measurable wrt x

and 2. for a.a. $x \in \Omega$ $f(x, \cdot)$ is continuous wrt y.

Assume that $|f(x,y)| \leq g(x) + c \sum_{i=1}^{N} |y_i|^{\frac{p_i}{p}}$ for some $p_i \in [1,\infty)$, $p \in [1,\infty)$ with $g \in L^p(\Omega)$.

Then $\forall u_i \in L^{p_i}(\Omega)$ the function $f(x, u_1(x), u_2(x), \dots, u_N(x))$ is measurable and the mapping $(u_1, u_2, \dots, u_N) \mapsto f(\cdot, u_1, \dots, u_N)$ is continuous from $L^{p_1} \times L^{p_2} \times \dots \times L^{p_N} \to L^p$ ($\Leftrightarrow$ $u_i^N \to u_i$ in L^{p_i}, $i = 1, \dots, N$, then $f(\cdot, u_1^n, u_2^n, \dots, u_N^n) \to f(\cdot, u_1, \dots, u_N)$ in $L^p(\Omega)$).
and it is called Nemytskii operator.

Proof: a) measurability is obvious

b) $f(\cdot, u_1, \dots, u_N)$ is in $L^p(\Omega)$:

$$\int_\Omega |f(\cdot, u_1, \dots, u_N)|^p \leq \int_\Omega |g(x) + c \sum_{i=1}^{N} |u_i|^{\frac{p_i}{p}}|^p \leq c(p) \int_\Omega |g|^p + \sum_{i=1}^{N} |u_i|^{p_i} < \infty$$

c) $u_i^n \to u_i$ in L^{p_i} $\Rightarrow$ $f(\cdot, u^n) \to f(\cdot, u)$ in L^p

$$Q: \quad \limsup_{n\to\infty} \int_\Omega |f(\cdot, u_1^n, \dots, u_N^n) - f(\cdot, u_1, \dots, u_N)|^p \overset{?}{=} 0$$

due to $u_i^n \to u_i$ in L^{p_i}, for a subsequence (that we do not relabel),

$u_i^n(x) \to u_i(x)$ for a.a. $x \in \Omega$.

$$\overset{f \text{ Carath.}}{\Longrightarrow} \quad \underbrace{|f(x, u_1^n(x), \dots, u_N^n(x)) - f(x, u_1(x), \dots, u_N(x))|^p}_{} \to 0 \quad \text{a.e.}$$

if we show that the difference is equiintegrable then the use of Vitali's theorem finishes the proof.

Uniform equiintegrability:

$u_i^n \to u_i$ in $L^{p_i}(\Omega)$ $\Rightarrow$ $|u_i^n|^{p_i} + |u_i|^{p_i}$ is uniformly equiintegrable

$$|f(x,u^n) - f(x,u)|^p \leq c(p) \left(|f(x,u^n)|^p + |f(x,u)|^p\right)$$
$$\leq c(p) \left(2|g(x)|^p + \sum_{i=1}^{N} |u_i^n|^{p_i} + |u_i|^{p_i}\right) \leftarrow \text{and this is UEI}$$

1. SOBOLEV SPACES (second reading = with proofs)

You should know $W^{k,p}(\Omega)$

Theorem (local approximation of $W^{k,p}(\Omega)$ by smooth functions):

Let $f \in W^{k,p}(\Omega)$ and extend it by zero outside of Ω. Define $f_\varepsilon := \eta_\varepsilon * f$ and set $\Omega_\varepsilon := \{x \in \Omega : B_\varepsilon(x) \subseteq \Omega\}$. Then $D^\alpha(f_\varepsilon) = (D^\alpha f)_\varepsilon$ in Ω_ε $\forall \alpha, |\alpha| \leq k$ and for all $\Omega' \subseteq \overline{\Omega'} \subseteq \Omega$, $f_\varepsilon \to f$ in $W^{k,p}(\Omega')$.

Proof: $\frac{\partial}{\partial x_i}(f_\varepsilon(x)) = \frac{\partial}{\partial x_i} \int_{\mathbb{R}^d} \eta_\varepsilon(x-y) f(y)\, dy = \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} \eta_\varepsilon(x-y) f(y)\, dy$

$$= -\int_{\mathbb{R}^d} \frac{\partial}{\partial y_i}(\eta_\varepsilon(x-y)) f(y)\, dy = -\int_{B_\varepsilon(x)} \frac{\partial}{\partial y_i}(\eta_\varepsilon(x-y)) f(y)\, dy$$

$$= -\int_\Omega \frac{\partial}{\partial y_i}(\eta_\varepsilon(x-y)) f(y)\, dy \overset{DEF}{=} \int_\Omega \eta_\varepsilon(x-y) \frac{\partial}{\partial y_i} f(y)\, dy = \left(\frac{\partial f}{\partial y_i}\right)_\varepsilon (x)$$

$\Omega' \subseteq \overline{\Omega'} \subseteq \Omega$: Find $\varepsilon_0 > 0$: $\forall \varepsilon < \varepsilon_0$ $\Omega' \subseteq \Omega_\varepsilon$

then $\lim_{\varepsilon \to 0+} \| f_\varepsilon - f \|_{W^{k,p}(\Omega')} \leq \lim_{\varepsilon \to 0+} \sum_\alpha \| D^\alpha(f_\varepsilon) - D^\alpha f \|_{L^p(\Omega')} = \lim \sum_\alpha \| (D^\alpha f)_\varepsilon - D^\alpha f \|_{L^p(\Omega'}$

$= 0$ (Lebesgue spaces and regularization)

Theorem (composition of Lipschitz and Sobolev functions): Let $\Omega \subseteq \mathbb{R}^d$ open and $f : \mathbb{R} \to \mathbb{R}$ Lipschitz. Assume that $u \in W^{1,p}(\Omega)$. Then $(f(u) - f(0)) \in W^{1,p}(\Omega)$ and

(weak der.) $\frac{\partial f(u)}{\partial x_i} = f'(u) \frac{\partial u}{\partial x_i} \chi_{\{x;\, u(x) \notin S_f\}}$, where

$S_f := \{ s \in \mathbb{R};\ f'(s) \text{ does not exist} \}$.

Moreover, define $\Omega_a := \{ x \in \Omega ; u(x) = a\}$, then $\nabla u = 0$ a.e. in Ω_a.

Example: $\frac{\partial |u|}{\partial x_i} = \operatorname{sgn} u \frac{\partial u}{\partial x_i} \chi_{\{x;\, u(x) \neq 0\}}$

Proof: Rademacher said that $|S_f| = 0$.

1. we prove it for $f \in C^1$, $f_{Lip} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{x-y}$

$|f(u) - f(0)| \leq f_{Lip} |u - 0| = f_{Lip} |u|$

if $u \in L^p(\Omega) \Rightarrow (f(u) - f(0)) \in L^p(\Omega)$

Next we show $\frac{\partial f(u)}{\partial x_i} = f'(u) \frac{\partial u}{\partial x_i}$

if u_ε is smooth then $\frac{\partial f(u_\varepsilon)}{\partial x_i} = f'(u_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i}$

$\varphi \in C_0^\infty(\Omega)$ $\int_\Omega f(u) \frac{\partial \varphi}{\partial x_i} = \int_\Omega f(u_\varepsilon) \frac{\partial \varphi}{\partial x_i} + (f(u) - f(u_\varepsilon)) \frac{\partial \varphi}{\partial x_i}$

consider $\varepsilon \ll 1$, $\operatorname{supp} \varphi \subseteq \Omega_\varepsilon$

$$= -\int_\Omega f'(u_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \varphi + \int_\Omega (f(u) - f(u_\varepsilon)) \frac{\partial \varphi}{\partial x_i} =: (*)$$

$$(*) = -\int_\Omega f'(u) \frac{\partial u}{\partial x_i} \varphi + \int_\Omega \left(f'(u) \frac{\partial u}{\partial x_i} - f'(u_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i}\right)\varphi + \int_\Omega (f(u) - f(u_\varepsilon)) \frac{\partial \varphi}{\partial x_i}$$

$$\left| \int_\Omega \left(f'(u) \frac{\partial u}{\partial x_i} - f'(u_\varepsilon) \frac{\partial u_\varepsilon}{\partial x_i} \right) \varphi \right| \leq \|\varphi\|_\infty \int_{\operatorname{supp}\varphi} |f'(u_\varepsilon)| \left| \frac{\partial u_\varepsilon}{\partial x_i} - \frac{\partial u}{\partial x_i} \right| + |f'(u_\varepsilon) - f'(u)| \left| \frac{\partial u}{\partial x_i} \right|$$

$$\leq c \int_{\operatorname{supp}\varphi} \underbrace{|\nabla u^\varepsilon - \nabla u|}_{\to 0 \text{ local approx thm}} + \underbrace{|\nabla u| \, |f'(u_\varepsilon) - f'(u)|}_{\to 0 \text{ Lebesgue dom. conv. thm}}, \xrightarrow{\varepsilon \to 0+} 0 + 0$$

$$\int_\Omega (f(u) - f(u_\varepsilon)) \frac{\partial \varphi}{\partial x_i} \to 0 \qquad \text{trivial}$$

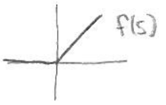
$$\Rightarrow \quad (*) \quad \to \quad -\int_\Omega f'(u) \frac{\partial u}{\partial x_i} \varphi$$

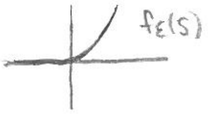
2. extension to $f \in C^{0,1}(\mathbb{R})$, $f_{lip} := \sup\limits_{x \neq y} \dfrac{|f(x) - f(y)|}{x - y}$

we know $(f(u) - f(0)) \in L^p(\Omega)$

if $\dfrac{\partial f(u)}{\partial x_i} = f'(u) \dfrac{\partial u}{\partial x_i} \chi_{\{x;\, u(x) \notin S_f\}}$, then $\dfrac{\partial f(u)}{\partial x_i} \in L^p$

the formula is true for $f(s) := \max(0, s)$





$$f_\varepsilon(s) := \begin{cases} \sqrt{s^2 + \varepsilon^2} - \varepsilon & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases} \qquad f'_\varepsilon(s) = \begin{cases} \frac{s}{\sqrt{s^2 + \varepsilon^2}} \\ 0 \end{cases}$$

$f'_\varepsilon(s) \nearrow \chi_{\{s > 0\}}$, $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega f(u) \frac{\partial \varphi}{\partial x_i} = \lim_{\varepsilon \to 0+} \int_\Omega f_\varepsilon(u) \frac{\partial \varphi}{\partial x_i} = -\lim_{\varepsilon \to 0+} \int_\Omega f'_\varepsilon(u) \frac{\partial u}{\partial x_i} \varphi$$

$$= -\lim_{\varepsilon \to 0+} \int_\Omega \frac{u}{\sqrt{u^2 + \varepsilon^2}} \frac{\partial u}{\partial x_i} \varphi \chi_{\{u(x) > 0\}} \overset{\text{Lebesgue}}{=} \int_\Omega \frac{\partial u}{\partial x_i} \varphi \chi_{\{u > 0\}}$$

$$\Rightarrow \left.\begin{aligned} \frac{\partial \max(0, u)}{\partial x_i} &= \frac{\partial u}{\partial x_i} \chi_{\{u > 0\}} \\ \frac{\partial \min(0, u)}{\partial x_i} &= \frac{\partial u}{\partial x_i} \chi_{\{u < 0\}} \end{aligned}\right\} \quad \begin{aligned} \frac{\partial u}{\partial x_i} &= \frac{\partial}{\partial x_i} \left(\max(0, u) + \min(0, u) \right) \\ &= \frac{\partial u}{\partial x_i} \chi_{\{u > 0\}} + \frac{\partial u}{\partial x_i} \chi_{\{u < 0\}} \end{aligned}$$

$$\Rightarrow \frac{\partial u}{\partial x_i} \chi_{\{u = 0\}} = 0 \qquad \text{a.e. in } \Omega$$

$$\Leftrightarrow \quad \frac{\partial u}{\partial x_i} = 0 \quad \text{a.e. in } \{x;\, u(x) = 0\}$$

$$\Rightarrow \forall c \in \mathbb{R} : \frac{\partial u}{\partial x_i} = 0 \quad \text{a.e. in } \{x;\, u(x) = c\}$$

full generality:

$f_\varepsilon \in C^1$, $f_\varepsilon \to f$ in $C(\mathbb{R})$

$\|f'_\varepsilon\|_{L^\infty(\mathbb{R})} \leq f_{lip}$, $f'_\varepsilon \to f'$ a.e. in $\mathbb{R}$ (except from S_f)

$$\int_\Omega f(u) \frac{\partial \varphi}{\partial x_i} = \lim_{\varepsilon \to 0+} \int_\Omega f_\varepsilon(u) \frac{\partial \varphi}{\partial x_i} = -\lim_{\varepsilon \to 0} \int_\Omega f'_\varepsilon(u) \frac{\partial u}{\partial x_i} \varphi$$

$$f'_\varepsilon(u) \frac{\partial u}{\partial x_i} \to f'(u) \frac{\partial u}{\partial x_i} \chi_{\{u \notin S_f\}}$$

- easy if $u \notin S_f$

- if $u \in S_f$ then $\frac{\partial u}{\partial x_i} = 0$

27.2.2019 Theorem (equivalent characterization of Sobolev functions): Let $\Omega \subseteq \mathbb{R}^d$ be an open set. Denote $\Omega_\delta := \{x \in \Omega, B_\delta(x) \subseteq \Omega\}$ and set $u_i^h(x) := \frac{u(x+he_i) - u(x)}{h}$. Then

1. if $u \in W^{1,p}(\Omega)$ then $\forall \delta > 0 \ \forall h < \frac{\delta}{2} : \|u_i^h\|_{L^p(\Omega_\delta)} \leq \|\frac{\partial u}{\partial x_i}\|_{L^p(\Omega)}$
2. if $p \in (1,\infty]$ and $\sup\limits_{\delta > 0} \sup\limits_{h<\frac{\delta}{2}} \|u_i^h\|_{L^p(\Omega_\delta)} \leq K$ then $\frac{\partial u}{\partial x_i}$ exists and $\|\frac{\partial u}{\partial x_i}\|_{L^p(\Omega)} \leq K$
3. for $p \in [1,\infty)$ if $u \in W^{1,p}(\Omega)$ then $u_i^h \xrightarrow{h \to 0+} \frac{\partial u}{\partial x_i}$ in $L^p_{loc}(\Omega)$.

Proof: 2. Fix $\Omega_1 \subset\subset \Omega$. Find $\delta > 0 : \Omega_1 \subseteq \Omega_\delta$. Then $\|u_i^h\|_{L^p(\Omega_1)} \leq K \quad \forall h \leq \frac{\delta}{2}$

For $p \in (1,\infty)$ we have L^p is reflexive $\Rightarrow$ find a subsequence $u_i^{h_n} \xrightarrow{h_n \to 0+} \bar{u}$ in $L^p(\Omega_1)$

From weak lower semicontinuity $\|\bar{u}\|_{L^p(\Omega_1)} \leq \liminf\limits_{h_n \to 0+} \|u_i^{h_n}\|_{L^p(\Omega_1)} \leq K$

Last: we need $\bar{u} = \frac{\partial u}{\partial x_i}$ a.e. in Ω_1. For $\varphi \in C_0^\infty(\Omega_1)$,

$$\int_{\Omega_1} \bar{u}\varphi = \lim_{h_n \to 0+} \int_{\mathbb{R}^d} u_i^{h_n} \varphi = \lim_{h_n \to 0+} \int_{\mathbb{R}^d} \frac{u(x+h_n e_i) - u(x)}{h_n} \varphi(x)\,dx$$

$$= -\lim_{h_n \to 0+} \int_{\mathbb{R}^d} u(x) \underbrace{\left(\frac{\varphi(x-h_n e_i) - \varphi(x)}{-h_n}\right)}_{\text{uniformly} \to \frac{\partial \varphi}{\partial x_i}} dx = -\int_{\mathbb{R}^d} u(x) \frac{\partial \varphi}{\partial x_i}\,dx = -\int_{\Omega_1} u \frac{\partial \varphi}{\partial x_i}$$

The case $p = \infty$ is the same, just replace weak by weak*

$\Rightarrow$ we know $\frac{\partial u}{\partial x_i}$ exists $\forall \Omega_1 \subset\subset \Omega$ with $\|\frac{\partial u}{\partial x_i}\|_{L^p(\Omega_1)} \leq K \quad \Rightarrow$ let $\Omega_1 \nearrow \Omega$

1. $u \in W^{1,p}(\Omega)$ $(p < \infty)$, extend u by 0

$$u_\varepsilon := u * \eta_\varepsilon \qquad D^\alpha u_\varepsilon = (D^\alpha u)_\varepsilon \quad \text{in } \Omega_\varepsilon$$

$$D^\alpha u_\varepsilon \to D^\alpha u \quad \text{in } L^p_{loc}(\Omega)$$

$$\frac{u_\varepsilon(x+he_i) - u_\varepsilon(x)}{h} = \frac{1}{h} \int_0^1 \frac{d}{dt} u_\varepsilon(x+hte_i)\,dt = \frac{1}{\cancel{h}} \int_0^1 \cancel{h} \frac{\partial u}{\partial x_i}(x+the_i)\,dt$$

$$\left|\frac{u_\varepsilon(x+he_i) - u_\varepsilon(x)}{h}\right|^p \leq \left|\int_0^1 \frac{\partial u}{\partial x_i}(x+the_i)\,dt\right|^p \overset{\text{Jensen}}{\leq} \int_0^1 \left|\frac{\partial u}{\partial x_i}(x+the_i)\right|^p dt$$

$$\int_{\Omega_\delta} \left|\frac{u_\varepsilon(x+he_i) - u_\varepsilon(x)}{h}\right|^p dx \leq \int_0^1 \int_{\Omega_\delta} \left|\frac{\partial u_\varepsilon}{\partial x_i}(x+the_i)\right|^p dx\,dt =: (*)$$

$\delta > 0, \quad h < \frac{\delta}{2}, \quad \varepsilon < \frac{\delta}{2}$:

$$(*) \leq \int_0^1 \int_{\Omega_{\frac{\delta}{2}}} \left|\frac{\partial u_\varepsilon}{\partial x_i}(x)\right|^p dx\,dt = \int_{\Omega_{\frac{\delta}{2}}} \left|\frac{\partial u_\varepsilon}{\partial x_i}(x)\right|^p dx \overset{\varepsilon \to 0+}{\leq} \int_{\Omega_{\delta/2}} \left|\frac{\partial u}{\partial x_i}\right|^p dx \leq \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p$$

$p = \infty$: Define $\Omega_R := \Omega \cap B_R$, where B_R is a ball

$u \in W^{1,\infty}(\Omega) \Rightarrow u \in W^{1,\infty}(\Omega_R) \overset{\Omega_R \text{ bounded}}{\Rightarrow} u \in W^{1,p}(\Omega_R) \quad \forall p \in [1,\infty)$

$\Rightarrow \|u_i^h\|_{L^p(\Omega_R^\delta)} \leq \|\frac{\partial u}{\partial x_i}\|_{L^p(\Omega_R)} \qquad \lim\limits_{p \to \infty} \|f\|_p = \|f\|_\infty$

$\|u_i^h\|_{L^\infty(\Omega_R^\delta)} \leq \|\frac{\partial u}{\partial x_i}\|_{L^\infty(\Omega_R)} \quad , \quad R \to \infty$

3. easy homework. Hint: show that u_i^h is Cauchy w.r.t. h in L^p

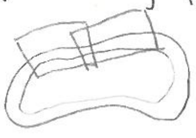
Properties up to the boundary and extensions

Theorem (approximation by smooth functions). Let $\Omega \subseteq \mathbb{R}^d$ be open and $p \in [1,\infty)$. Then

$\forall u \in W^{k,p}(\Omega)$ 1. $\exists \{u^n\}_{n=1}^{\infty} \subseteq C^{\infty}(\Omega)$ such that $\|u^n - u\|_{W^{k,p}(\Omega)} \to 0$

2. if $\Omega \in C^0$ $\exists \{u^n\}_{n=1}^{\infty} \subseteq C^{\infty}(\overline{\Omega})$ such that $\|u^n - u\|_{W^{k,p}(\Omega)} \to 0$

Proof: 1st by picture

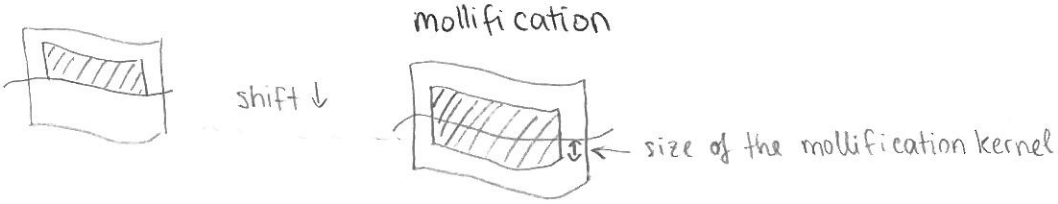


open covering of Ω

introduce partition of unity

$u = \sum u (\text{cut-off})_i$

mollification



Proof: 2nd rigorous.

1. description of $\partial\Omega$. There exists M orthogonal transformations T_r, $r=1,\dots,M$, continuous functions a_r, $r=1,\dots,M$ and α, β such that

$V_r^+ := \{ (x_r', x_{rd}) \in \mathbb{R}^d,\ |x_r'| < \alpha,\ a_r(x_r') < x_{rd} < a_r(x_{rd}') + \beta \}$

$V_r^- := \{ \quad - \text{"} - \quad ,\ a_r(x_r') - \beta < x_{rd} < a_r(x_r') \}$

$\Lambda_r := \{ \quad - \text{"} - \quad ,\ a_r(x_r') = x_{rd} \}$

$T_r(V_r^+) \subset \Omega$, $T_r(V_r^-) \subseteq \mathbb{R}^d \setminus \overline{\Omega}$, $T_r(\Lambda_r) \subseteq \partial\Omega$

$V_r := V_r^+ \cup \Lambda_r \cup V_r^-$, then $\bigcup_{r=1}^{M} T_r(V_r) \supset \partial\Omega$

Define $\Omega_r := T_r(V_r)$, $r=1,\dots,M$. We can find open $\Omega_{M+1} \subset\subset \Omega$, $\bigcup_{r=1}^{M+1} \Omega_r \supseteq \overline{\Omega}$.

Then we have finite open covering of a compact set.

Lemma: $\exists \varphi_r \in C_0^{\infty}(\Omega_r)$, $r=1,\dots,M+1$ such that $\forall x \in \overline{\Omega}$ $\sum_{r=1}^{M+1} \varphi_r(x) = 1$ ... partition of unity

For given $u \in W^{k,p}(\Omega)$ and arbitrary $\varepsilon > 0$ we want to find $u^n \in C^{\infty}(\overline{\Omega})$, $\|u^n - u\|_{k,p} \leq \varepsilon$ (Dream 1)

Define $u_r(x) := u(x)\varphi_r(x)$, show that $\forall u_r\ \exists u_r^n \in C^{\infty}(\overline{\Omega})$ such that $\|u_r^n - u_r\|_{k,p} \leq \frac{\varepsilon}{M+1}$ (Dream 2)

Dream 2 $\Rightarrow$ Dream 1

Define $u^n := \sum_{r=1}^{M+1} u_r^n$ then $\|u^n - u\|_{k,p} = \|\sum_{r=1}^{M+1} (u_r^n - u_r)\|_{k,p} \leq \sum_{r=1}^{M+1} \frac{\varepsilon}{M+1} = \varepsilon$

Mollification - on the Ω_{M+1}

$u_{M+1} = u\varphi_{M+1} \in W^{k,p}(\mathbb{R}^d)$, $\forall \varepsilon > 0\ \exists \delta > 0$ $\|u_{M+1} - u_{M+1} * \eta_\delta\|_{k,p} \leq \frac{\varepsilon}{M+1}$, $u_{M+1}^n := u_{M+1} * \eta_\delta$

Mollification - on the bdary

Ω_1, without loss of generality $T_1 = I$. Then $\Omega_1 = V_1$ and $(x_1', x_{1d}) = (x', x_d)$.

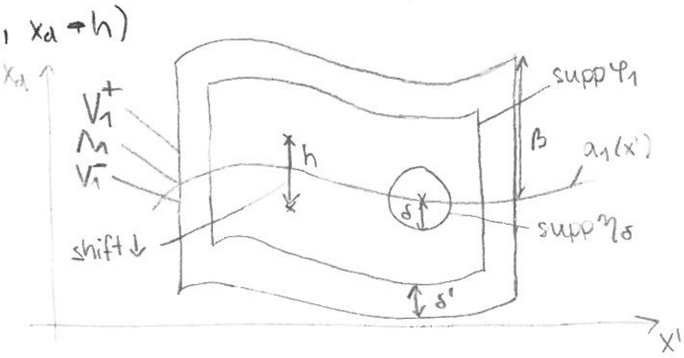
$u_1 = u\varphi_1 \quad \varphi_1 \in C_0^\infty(V_1) \qquad u_1^h(x) := u_1(x_1, \dots, x_{d-1}, x_d + h)$

$\forall \varepsilon > 0 \ \exists h_0 > 0 \ \forall h < h_0 \qquad \| u_1 - u_1^h \|_{k,p} \leq \frac{\varepsilon}{2(M+1)}$

$u_1^n := u_1^h * \eta_\delta$

$V_1^+ := \{ (x', x_d), \ a_1(x') < x_d < a_1(x') + \beta \}$

$V_1^- := \{ (x', x_d), \ a_1(x') - \beta < x_d < a_1(x') \}$



$\varphi_1 \in C_0^\infty(V_1) \quad \exists \delta' > 0 \quad \varphi_1 = 0$ on the set $x_d > a_1(x') + \beta - \delta'$ & $x_d < a_1(x') - \beta + \delta'$

$\Rightarrow$ h_0 must be less than δ'

Take point $(x_1, \dots, x_{d-1}, x_d - h)$ where $(x_1, \dots, x_d) \in \partial\Omega$

we need to check that $\operatorname{dist}((x_1, \dots, x_{d-1}, x_d - h), \partial\Omega) < \delta$

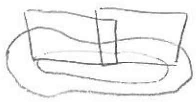
$\delta > 0$ a_1 is continuous $\Rightarrow \exists h_{max} \ \forall h < h_{max}$ $\downarrow$ is true

take $h < \overset{\min}{\cancel{\max}} (h_{max}, h_0) \qquad \| u_1^n - u_1 \| \leq \frac{\varepsilon}{M+1}$ provided $\delta > 0$ is small

Theorem (Extension) : Let $\Omega \in C^{0,1}$ and $p \in [1, \infty]$. Then there exists continuous linear operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ such that $\forall u \in W^{1,p}(\Omega)$

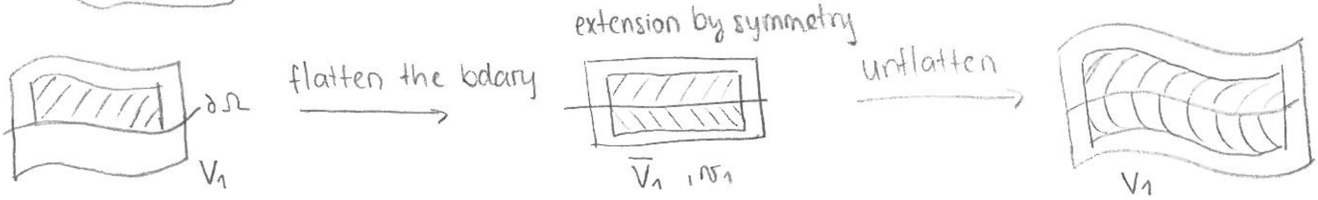
1. $Eu = u$ in Ω
2. $\exists B_R \subseteq \mathbb{R}^d$ such that $Eu = 0$ on $\mathbb{R}^d \setminus B_R$
3. $\| Eu \|_{W^{1,p}(\mathbb{R}^d)} \leq c(d, p, \Omega) \| u \|_{W^{1,p}(\Omega)}$

Proof : picture



partition of unity $u = \sum_{r=1}^{M+1} u_r$

extend u_r



"rigorous"

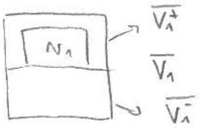
$u = \sum_{r=1}^{M+1} u_r \ , \quad u_1 \in W^{1,p}(\Omega_1)$ to extend u_1 to $W^{1,p}(\mathbb{R}^d)$

$V_1, T_1 = I \qquad F : V_1 \to \bar{V}_1 \ \begin{cases} y' = x' \\ y_d = x_d - a(x') \end{cases} \qquad \begin{aligned} V_1 &: x', x_d \\ \bar{V}_1 &: y', y_d \end{aligned}$

F is Lipschitz! $\det \nabla F = 1$

$F^{-1} : \bar{V}_1 \to V_1 \ \begin{cases} x' = y' \\ x_d = y_d + a(y') \end{cases} \qquad$ F^{-1} is Lipschitz $\quad \det \nabla F^{-1} = 1$

$u_1 \in W^{1,p}(\underbrace{\Omega_1 \cap \Omega}_{V_1^+})$



$v_1(y) := u_1(F^{-1}(y))$ defined where $y_d > 0$

easy homework: check that $v_1 \in W^{1,p}(\overline{V_1^+})$, $\|v_1\|_{W^{1,p}(\overline{V_1^+})} \le \|u_1\|_{W^{1,p}(\Omega)} \, c(\Omega, p)$

$$\left|\frac{\partial v_1}{\partial y_i}\right| \sim |\nabla u(F^{-1}(y))| \, |\nabla F^{-1}|$$ composition Sobolev(Lipschitz) = Sobolev

Define $$Ev_1(y) := \begin{cases} v_1(y) & \text{if } y_d > 0 \\ v_1(y_1, \dots, -y_d) & \text{if } y_d < 0 \end{cases}$$

Ev_1 is compactly supported in $\overline{V_1}$, Sobolev in $\overline{V_1^+}$ and $\overline{V_1^-}$. What happens when $y_d = 0$?

Check $\frac{\partial E v_1}{\partial y_d}$: $\varphi \in C_0^\infty(\overline{V_1})$

$$\int_{\overline{V_1}} Ev_1 \frac{\partial \varphi}{\partial y_d} = \int_{\overline{V_1^+}} v_1(y) \frac{\partial \varphi}{\partial y_d} + \int_{\overline{V_1^-}} v_1(y_1, \dots, -y_d) \frac{\partial \varphi}{\partial y_d}$$

$$= \int_{\overline{V_1^+}} v_1(y) \left(\frac{\partial \varphi}{\partial y_d}(y_1, \dots, y_d) + \frac{\partial \varphi}{\partial y_d}(y_1, \dots, -y_d) \right)$$

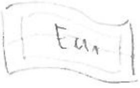
$$= \int_{\overline{V_1^+}} v_1(y) \left(\frac{\partial \varphi}{\partial y_d}(y) - \left[\frac{\partial \varphi}{\partial(-y_d)}(y_1, \dots, -y_d) \right) \right) (\tau + (1-\tau))$$

$$\tau = \begin{cases} 1 & |y_d| > \varepsilon \\ 0 & |y_d| < \frac{\varepsilon}{2} \end{cases} \qquad \left|\frac{\partial \tau}{\partial y_d}\right| \le \frac{c}{\varepsilon}$$

$$= -\int_{\overline{V_1^+}} \frac{\partial v_1(y)}{\partial y_d} (\varphi(y) - \varphi(y_1, \dots, -y_d))\tau + \int_{\overline{V_1^+}} v_1(y)(\varphi(y) - \varphi(y_1, \dots, -y_d)) \frac{\partial \tau}{\partial y_d} + \int_{\overline{V_1^+}} v_1(y) \left(\frac{\partial \varphi}{\partial y_d} - \frac{\partial \varphi}{\partial(-y_d)} \right)(1-\tau)$$

$Eu_1(x) := E v_1(F(x))$

again $\|Eu_1\|_{1,p} \le c \, \|Ev_1\|_{1,p} \le c \|v_1\|_{1,p} \le c \|u_1\|_{1,p}$



Embeddings

We know, $\Omega \in C^{0,1}$: $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ if $\begin{cases} p < d & q < \frac{dp}{d-p} \\ p > d & q = \infty \end{cases}$

$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ if $p > d$ $\quad \alpha = 1 - \frac{d}{p}$

General scheme: take $u \in W^{1,p}(\Omega)$, extend to $Eu \in W^{1,p}(\mathbb{R}^d)$, compactly supported, prove embeddings for Eu in $W^{1,p}(\mathbb{R}^d)$ and then go back to $u \in W^{1,p}(\Omega)$

1. Embeddings of type $W^{1,p} \hookrightarrow C^{0,\alpha}$ (Morrey)

Lemma: Let $u \in W^{1,1}(B_R(0))$ and 0 be a Lebesgue point of u. Then

$$\left| \fint_{B_R} u - u(0) \right| \le R^A c(A, d) \sup_{\rho \le R} \int_{B_\rho} \frac{|\nabla u|}{\rho^{d-1+A}} \qquad (A > 0).$$

Proof: $$\left| \fint_{B_r} u - u(0) \right| = \lim_{r \to 0+} \left| \fint_{B_R} u - \fint_{B_r} u \right| = \lim_{r \to 0+} \left| \int_r^R \frac{d}{d\rho} \fint_{B_\rho} u \, d\rho \right| = \lim_{r \to 0+} \left| \int_r^R \frac{d}{d\rho} \left(\fint_{B_1(0)} u(\rho x) dx \right) d\rho \right|$$

$$= \lim_{r \to 0+} \left| \int_r^R \fint_{B_1(0)} \sum \frac{\partial u}{\partial x_i}(\rho x) \, x_i \, dx \, d\rho \right| \le \lim_{r \to 0+} c \int_r^R \fint_{B_1(0)} |\nabla u(\rho x)| \, dx \, d\rho =$$

$$= \lim_{r \to 0+} c \int_r^R \fint_{B_\rho} |\nabla u| \, dx \, d\rho = \lim_{r \to 0+} c(d) \int_r^R \int_{B_\rho} \frac{|\nabla u|}{\rho^{d-1+A}} \rho^{A-1} \, dx \, d\rho$$

$$\le \left(\sup_{\rho \le R} \int_{B_\rho} \frac{|\nabla u|}{\rho^{d-1+A}} \right) \lim_{r \to 0+} (c(d)) \int_r^R \rho^{A-1} d\rho = \frac{R^A}{A} c(d) \sup_{\rho \le R} \left(\int_{B_\rho} \frac{|\nabla u|}{\rho^{d-1+A}} \right)$$

13.3.2019 We want $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega)$ if $p>d$ $\alpha = 1-\frac{d}{p}$

We know: If x is a Lebesgue point of u, then $\underbrace{|u(x) - \fint_{B_R(x)} u(y)dy| \le R^\alpha c(\alpha,d) \sup_{r\le R} \int_{B_r(x)} \frac{|\nabla u|}{r^{d-1+\alpha}}}_{(*)}, \alpha>0.$

Proof of $W^{1,p} \hookrightarrow C^{0,\alpha}$:

1. we extend u by Eu: $u\in W^{1,p}(\Omega) \Rightarrow Eu \in W^{1,p}(\mathbb{R}^d)$, Eu is compactly supported in $\mathbb{R}^d$,

$Eu = u$ in Ω, $\|Eu\|_{W^{1,p}(\mathbb{R}^d)} \le c(\Omega,p)\|u\|_{1,p}$

2. to show that if x,y are lebesgue points of Eu then

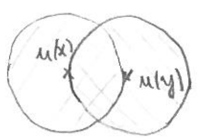
$|u(x)-u(y)| \le |x-y|^\alpha c(\alpha,\Omega,p)\max\{I_x,I_y\}$, $\alpha>0$

$I_x := \sup_{r\le 2|x-y|}\int_{B_r(x)} \frac{|\nabla u|}{r^{d-1+\alpha}}$, $I_y := \sup_{r\le 2|x-y|} \int_{B_r(y)} \frac{|\nabla u|}{r^{d-1+\alpha}}$

Proof of 2: set $R := |x-y|$, use (*):

$$|u(x)-u(y)| \le |u(x) - \fint_{B_R(x)} u| + |u(y) - \fint_{B_R(y)} u| + |\fint_{B_R(x)} u - \fint_{B_R(y)} u|$$

$$\le c(\alpha) R^\alpha \max\{I_x,I_y\} + |\fint_{B_R(x)} u - \fint_{B_R(y)} u|$$



$$|\fint_{B_R(x)} u - \fint_{B_R(y)} u| = |\int_0^1 \frac{d}{dt}\fint_{B_R(tx+(1-t)y)} u| = |\int_0^1 \frac{d}{dt}\fint_{B_R(0)} u(tx+(1-t)y+z)\,dz|$$

$$= |\int_0^1 \fint_{B_R(0)} \nabla u(tx+(1-t)y+z)\cdot(x-y)\,dz| \le c(d)|\int_0^1\int_{B_R(0)} \frac{|\nabla u(tx+(1-t)y+z)|}{R^{d-1}}$$

$$\le \tilde c(d)\,(2R)^\alpha\, 2^{d-1}\int_0^1 \int_{B_{2R}(x)} \frac{|\nabla u|}{(2R)^{d-1}\,(2R)^\alpha} \le \tilde c(d)\, R^\alpha I_x$$

3. Morrey embedding: $\sup_{x\neq y}\frac{|Eu(x)-Eu(y)|}{|x-y|^\alpha} \le \frac{c(\Omega,p)}{\alpha}\sup_{x,R}\frac{|\nabla Eu|}{R^{d-1+\alpha}}$

Proof = Step 2.

Note: true for Lebesgue points, but from above we can redefine Eu to be continuous.

4. end of the proof

$$\sup_{x,R}\int_{B_R(x)}\frac{|\nabla Eu|}{R^{d-1+\alpha}} \overset{Eu \text{ is supp in } B_{R_0}}{\le} \sup_{\substack{x\in B_{R_0}\\ R\le R_0}}\int_{B_R(x)}\frac{|\nabla Eu|}{R^{d-1+\alpha}} \overset{\text{Hölder}}{\le} \sup R^{1-d-\alpha}\left(\int_{B_R(x)}|\nabla Eu|^p\right)^{1/p}\cdot\left(\int_{B_R(x)}1^{p'}\right)^{1/p'}$$

$$\le c\,\|u\|_{1,p}\sup_{x,R} R^{1-d-\alpha}R^{\frac{d}{p'}} = c\,\|u\|_{1,p}\, R^0 \quad \text{if } \alpha = 1-\frac{d}{p}$$

$$\Rightarrow \sup_{x\neq y}\frac{|Eu(x)-Eu(y)|}{|x-y|^\alpha} \le \frac{c}{\alpha}\|u\|_{1,p} \qquad (\alpha = 1-\frac{d}{p})$$

what remains: $\|u\|_{L^\infty(\Omega)} \le c\|u\|_{1,p}$

$$|u(x)| \le |u(x)-u(y)| + |u(y)| \le c\|u\|_{1,p}|x-y|^\alpha + |u(y)| \le c(\Omega,p)\|u\|_{1,p} + |u(y)|$$

$$|u(x)| = \fint_\Omega |u(x)|dy \le \fint_\Omega c(\Omega,p)\|u\|_{1,p} + |u(y)|dy \le c(\Omega,p)(\|u\|_{1,p}+\|u\|_1) \le c(\Omega,p)\|u\|_{1,p}$$

5. you should know $C^{0,\alpha} \hookrightarrow\hookrightarrow C^{0,\beta}$ if $\beta<\alpha$

Note. What if $p=d$? $W^{1,p}(\Omega)\hookrightarrow BMO(\Omega)$. $BMO(\Omega) = \{u\in L^1(\Omega), \fint_{B_R}|u-\fint_{B_R}u| < \infty\}$

PDE people like it, function spaces people don't

Embedding $\underline{W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)}$, $\underline{q = \frac{dp}{d-p}}$

$\underline{W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^{q-\varepsilon}(\Omega)}$

Scheme: 1. extension, 2. mollification , 3. show the estimates for smooth functions

Lemma (Gagliardo-Nirenberg inequality): $\exists\, c(d) \quad \forall u \in C_0^\infty\, \mathbb{R}^d$

1. $\|u\|_{L^{\frac{d}{d-1}}} \leq c(d)\, \|\nabla u\|_1$

2. $\|u\|_{\frac{dp}{d-p}} \leq c(d,p)\, \|\nabla u\|_p \qquad p<d$

Proof: "1. => 2."

Define $v := |u|^q$, apply 1. to v : $\|v\|_{\frac{d}{d-1}} \leq c(d)\, \|\nabla v\|_1$, $q>1$

$$\left(\int_{\mathbb{R}^d} |u|^{\frac{qd}{d-1}}\right)^{\frac{d-1}{d}} = \|v\|_{\frac{d}{d-1}} \leq c(d)\, \|\nabla v\|_1 \leq c(d) \int_{\mathbb{R}^d} |\nabla |u|^q| \leq c(d)\, q \int_{\mathbb{R}^d} |u|^{q-1} |\nabla u|$$

$$\leq c(d)\, q\, \|\nabla u\|_p \left(\int_{\mathbb{R}^d} |u|^{p'(q-1)}\right)^{\frac{1}{p'}}$$

choose q: $\frac{qd}{d-1} = p'(q-1)$, $q := \frac{p(d-1)}{d-p}$ $\qquad q-1 = \frac{dp-d}{d-p} = \frac{d(p-1)}{d-p}$

$$\Rightarrow \left(\int_{\mathbb{R}^d} |u|^{\frac{dp}{d-p}}\right)^{\frac{d-1}{d}} \leq c(d)\, \frac{p(d-1)}{d-p}\, \|\nabla u\|_p \left(\int_{\mathbb{R}^d} |u|^{\frac{dp}{d-p}}\right)^{1/p'} \qquad / \cdot \frac{1}{\left(\int |u|^{\frac{dp}{d-p}}\right)^{1/p'}}$$

$$\Rightarrow \|u\|_{\frac{dp}{d-p}} \leq \frac{c(d)\, p(d-1)}{d-p}\, \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

Proof of 1.

Lemma (Gagliardo): Let $u_i \in C_0^\infty(\mathbb{R}^{d-1})$ $i = 1,\dots,d$. Define $v_i(x_1,\dots,x_d) := u_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d)$.

Then $\int_{\mathbb{R}^d} \prod_{i=1}^d |v_i(x)|\, dx \leq \prod_{i=1}^d \|u_i\|_{L^{d-1}(\mathbb{R}^{d-1})}$

Proof: by induction w.r.t. d

1. if $d=2$. $\int_{\mathbb{R}^d} \prod_{i=1}^d |v_i(x)|\, dx = \int_{\mathbb{R}^2} |u_1(x_1)|\, |u_2(x_2)|\, dx_1 dx_2 \overset{\text{Fubini}}{=} \int_{\mathbb{R}} |u_1| \int_{\mathbb{R}} |u_2| = \|u_1\|_{L^1(\mathbb{R})} \|u_2\|_{L^1(\mathbb{R})}$

2. $d \Rightarrow d+1$ $\int_{\mathbb{R}^{d+1}} \prod_{i=1}^{d+1} |v_i(x)|\, dx = \int_{\mathbb{R}^d} \Big(\underbrace{|v_{d+1}(x)|}_{L^d} \underbrace{\int_{\mathbb{R}} \prod_{i=1}^d |v_i(x)|\, dx_{d+1}}_{L^{d'}}\Big)\, dx_1 \dots dx_d$

$$\leq \|v_{d+1}\|_{L^d(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \Big(\underbrace{\int_{\mathbb{R}} \prod_{i=1}^d |v_i(x)|\, dx_{d+1}}_{I}\Big)^{d'} dx_1 \dots dx_d\right)^{1/d'} =: (*)$$

$$I = \int_{\mathbb{R}} |v_1(x)| \dots |v_d(x)|\, dx_{d+1} \leq \prod_{i=1}^d \left(\int_{\mathbb{R}} |v_i(x)|^d\, dx_{d+1}\right)^{1/d} \qquad \sum_{i=1}^d \frac{1}{d} = 1$$

$$(*) \leq \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \Big(\prod_{i=1}^d \Big(\int_{\mathbb{R}} |v_i(x)|^d\, dx_{d+1}\Big)^{d'/d} dx_1 \dots dx_d\right)^{1/d'}$$

$$\leq \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \prod_{i=1}^d \Big(\int_{\mathbb{R}} |v_i(x)|^d\, dx_{d+1}\Big)^{\frac{1}{d-1}} dx_1 \dots dx_d\right)^{1/d'}$$

$$\left[z_i(x) := \left(\int_{\mathbb{R}} |v_i(x)|^d\, x_{d+1}\right)^{\frac{1}{d-1}}\right]$$

$$\leq \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \prod_{i=1}^d |z_i|\, dx_1 \dots dx_d\right)^{1/d'}$$

induction for z_i

$$\leq \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \prod_{i=1}^d \|z_i\|_{L^{d-1}(\mathbb{R}^{d-1})}^{1/d'} = \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \prod_{i=1}^d \Big(\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} |v_i|^d\, dx_{d+1}\, dx_1 \overset{\dots dx_{i-1} \cdot dx_{i+1} \dots}{\dots} dx_d\Big)$$

$$= \prod_{i=1}^{d+1} \|u_i\|_{L^d(\mathbb{R}^d)} \qquad \left[\frac{1}{d-1} \cdot \frac{1}{d'} = \frac{1}{d}\right]$$

use of G-L : $u \in C_0^\infty(\mathbb{R}^d)$

$$u(x) = \int_{-\infty}^{x_i} \frac{\partial u}{\partial x_i}(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d)\, ds$$

$$|u(x)| \leq \int_{-\infty}^{\infty} |\nabla u(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d)|\, ds$$

$$|u(x)|^{\frac{d}{d-1}} \leq \prod_{i=1}^{d} \left(\int_{-\infty}^{\infty} |\nabla u(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d)|\, ds \right)^{\frac{1}{d-1}}$$

$$\underbrace{\int_{\mathbb{R}^d} |u|^{\frac{d}{d-1}}}_{\| u \|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)}^{\frac{d}{d-1}}} \leq \int_{\mathbb{R}^d} \prod_{i=1}^{d} \underbrace{\left(\int_{-\infty}^{\infty} |\nabla u(\dots)| \right)^{\frac{1}{d-1}}}_{v_i} \overset{G-L}{\leq} \prod_{i=1}^{d} \|v_i\|_{L^{d-1}}$$

$$= \prod_{i=1}^{d} \left(\int_{\mathbb{R}^{d-1}} \left[\left(\int_{-\infty}^{\infty} |\nabla u(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d)|\, ds \right)^{\frac{1}{d-1}} \right]^{d-1} dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_d \right)^{\frac{1}{d-1}}$$

$$= \prod_{i=1}^{d} \left(\|\nabla u\|_{L^1(\mathbb{R}^d)}^{\frac{1}{d-1}} \right) = \|\nabla u\|_{L^1(\mathbb{R}^d)}^{\frac{d}{d-1}}$$

Proof a) $\underline{W^{1,p}(\Omega) \hookrightarrow L^{\frac{dp}{d-p}}(\Omega) \qquad p \in [1,d) \text{ and } \Omega \in C^{0,1}}$

$$\|u\|_{L^{\frac{dp}{d-p}}(\Omega)} \leq \|Eu\|_{L^{\frac{dp}{d-p}}(\mathbb{R}^d)} \overset{2.+\text{ mollification}}{\leq} \frac{c(d)}{d-p} \|\nabla Eu\|_{L^p(\mathbb{R}^d)} \overset{\text{extension}}{\leq} \frac{c(\Omega,p)}{d-p} \|u\|_{W^{1,p}(\Omega)}$$

b) $\underline{W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \qquad q < \frac{dp}{d-p}}$

$$\int_\Omega \underbrace{|u|^q}_{\frac{dp}{(d-p)q}} \cdot \underbrace{1}_{\left(\frac{dp}{(d-p)q}\right)'} \overset{\text{Hölder}}{\leq} \left(\int_\Omega |u|^{\frac{dp}{d-p}} \right)^{\frac{d-p}{dp} q} c(|\Omega|, q, p, d) \overset{a)}{\leq} \|u\|_{1,p}^q \, c(|\Omega|, p, q, d)$$

$\underline{\text{compact embedding} \quad W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \quad \text{if} \quad q < \frac{dp}{d-p}}$

1. step show $W^{1,1}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$

2. step show $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$

"1. ⇒ 2." $\quad \forall u \in W^{1,p}$: $\|u\|_q \leq c \|u\|_{1,p}^{\alpha} \|u\|_1^{1-\alpha}$

Lebesgue interpolation $\quad \|u\|_q \leq \|u\|_p^{\alpha} \|u\|_z^{1-\alpha} \quad \forall \; p \leq q \leq z \;,\; \frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{z}$

$$\|u\|_q \leq \|u\|_1^{\alpha} \|u\|_{\frac{dp}{d-p}}^{1-\alpha} \qquad \frac{1}{q} = \alpha + \frac{(1-\alpha)(d-p)}{dp} \;, \qquad \alpha < 1 \text{ if } q < \frac{dp}{d-p}$$

$$\overset{\text{cont. emb.}}{\leq} c(\Omega, p) \|u\|_1^{\alpha} \|u\|_{1,p}^{1-\alpha}$$

Assume 1. holds, let B be bounded subset of $W^{1,p}(\Omega)$.

1. ⇒ $\forall \varepsilon > 0 \;\; \exists \{u_i\}_{i=1}^{N} \subseteq W^{1,p}(\Omega) \;\; \forall u \in B \quad \min_i \|u - u_i\|_{L^1} \leq \varepsilon$

$$\|u - u_i\|_q \overset{\text{interp.}}{\leq} c \|u - u_i\|_1^{\alpha} \|u - u_i\|_{1,p}^{1-\alpha} \leq c \|u - u_i\|_1^{\alpha}$$

$$\min_i \|u - u_i\|_q \leq C \varepsilon^{\alpha}$$

Proof of 1. B a bdd subset of $W^{1,1}(\Omega)$, EB a bdd subset of $W^{1,1}(\mathbb{R}^d)$ (created by extension)

$u \in EB$, $u_\delta := u * \eta_\delta \qquad \left(u_\delta(x) = \int_{\mathbb{R}^d} u(x+y) \eta_\delta(y)\, dy \right)$

try to estimate $u - u_\delta$ in L^1

$$\int_{\mathbb{R}^d} |u(x) - u_\delta(x)|\, dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x+y) - u(x)|}{|y|} \eta_\delta(y) |y|\, dx\, dy \qquad \left(\int \frac{|u(x+z) - u(x)|}{|z|} dx \leq \|\nabla u\|_{L^1(\mathbb{R}^d)} \right)$$

$$\leq \|\nabla Eu\|_{L^1(\mathbb{R}^d)} \int_{\mathbb{R}^d} |y| \eta_\delta(y)\, dy \leq \delta \|\nabla Eu\|_1 \int_{\mathbb{R}^d} \eta_\delta(y)\, dy \leq \delta \|\nabla u\|_1$$

Give me $\varepsilon > 0$, set $\delta := \frac{\varepsilon}{2}$, $(EB)_\delta = \{u_\delta,\ u \in EB\}$, $\|u_\delta\|_{C^1_0} \le c(\delta, \|u\|_1)$

find finite covering $\{v_i\}_{i=1}^N \subseteq W^{1,1}(\mathbb{R}^d)$, $\min_i \|v_i - u_\delta\|_1 \le \frac{\varepsilon}{2}$

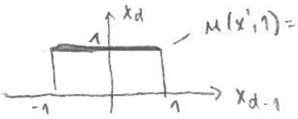
$\Rightarrow \|u - v_i\|_1 \le \|u - u_\delta\|_1 + \|u_\delta - v_i\|_1 \le \varepsilon$

Trace theorems

1. on cube for smooth functions

$\Omega := (-1,1)^{d-1} \times (0,1)$, $u \in C^1$ in Ω and $u(x',1) = 0$

Question: what is the best q such that $\int_{(-1,1)^{d-1}} |u|^q ds \le \|u\|_{W^{1,p}(\Omega)}\, c(p,d)$



$$|u(x',0)|^q = \left|\int_0^1 \frac{d}{dx_d} |u(x',x_d)|^q dx_d\right| \le q \int_0^1 |u(x',x_d)|^{q-1} |\nabla u(x',x_d)| dx_d$$

$$\int_{(-1,1)^{d-1}} |u(x',0)|^q dx_1 \dots dx_{d-1} \le q \int_\Omega |u|^{q-1} |\nabla u|\, dx \le q \|\nabla u\|_p \, \||u|^{q-1}\|_{p'} \qquad | p < d,\ W^{1,p} \hookrightarrow L^{\frac{dp}{d-p}})$$

$(q-1)p' = \frac{dp}{d-p}$, set $q := \frac{d(p-1)}{d-p} + 1 = \frac{p(d-1)}{d-p}$

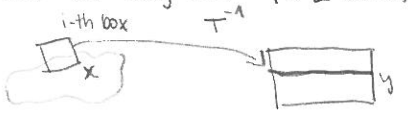
$$\Rightarrow \|u\|^q_{L^q(-1,1)^{d-1}} \le q\, c(\Omega,p) \|\nabla u\|_p \|u\|^{q-1}_{\frac{dp}{d-p}} \le c \|u\|^q_{1,p}$$

Last time: on cube $(-1,1)^{d-1} \times (0,1)$ if u is smooth 27.3.2019

$$\int_{(-1,1)^{d-1}} |u(x',0)|^q dx' \le c \|u\|^q_{1,p} \qquad p < d,\ 1 \le q \le \frac{p(d-1)}{d-p}$$

2. Definition: Let $\Omega \in C^{0,1}$ and $f: \partial\Omega \to \mathbb{R}$, we say that $f \in L^p(\partial\Omega)$, $p \in [1,\infty]$ if $\forall i = 1,\dots,N$

$f \circ T \in L^p(-\alpha,\alpha)^{d-1}$



$y' = x'$
$y_d - a_i(y') = x_d$

We define [illegible] (φ_i = partition of unity)

$$\int_{\partial\Omega} f\, dS := \int_{\partial\Omega} \sum_{i=1}^N (f \varphi_i) = \sum_{i=1}^N \int_{V_i} f\varphi_i = \sum_{i=1}^N \int_{(-\alpha,\alpha)^{d-1}} f(T_i(y)) \sqrt{1+|\nabla a_i|^2}\, \varphi_i(T_i(y))$$

$\int_{\partial\Omega} f\, dS$ is independent of the covering!

Lemma (integration by parts): Let $\Omega \in C^{0,1}$ and $f \in C^1(\bar\Omega)$ $\int_\Omega \frac{\partial f}{\partial x_i} = \int_{\partial\Omega} f n_i\, dS$

n = outer normal, $n \sim \frac{(\nabla a, 1)}{\sqrt{1+|\nabla a|^2}}$

Difficult homework – prove it (include all details)

3. Trace theorem: Let $\Omega \in C^{0,1}$. Then there exists a linear operator $Tr: W^{1,p}(\Omega) \to L^p(\partial\Omega)$, for all $p \in [1,\infty]$ such that for all $u \in C(\bar\Omega)$, $Tr\, u = u|_{\partial\Omega}$.

Proof: a) $p > d$ $W^{1,p}(\Omega) \hookrightarrow C(\bar\Omega)$

b) $p \le d$, we have $C^1(\bar\Omega)$ is dense in $W^{1,p}(\Omega)$, $\forall u \in W^{1,p}(\Omega)\ \exists \{u^n\}_{n=1}^\infty \subseteq C^1(\bar\Omega)\ u^n \to u$ in $W^{1,p}$

$$\int_{\partial\Omega} |u^n - u^m|^q = \sum_{i=1}^N \int_{V_i} |u^n - u^m|^q \varphi_i = \sum_{i=1}^N \int_{(-\alpha,\alpha)^{d-1}} |u^n \circ T_i - u^m \circ T_i|^q\, \varphi_i \circ T_i \sqrt{1+|\nabla a_i|^2} \quad , q \le \frac{(d-1)p}{d-p}$$

$$\le c \sum_{i=1}^N \|u^n \circ T_i - u^m \circ T_i\|^q_{1,p} \le c(\Omega) \|u^n - u^m\|^q_{L^p(\Omega)} + c(\Omega) \|\nabla u^n - \nabla u^m\|^q_{L^p}$$

$\uparrow$
estimate on cube

$\Rightarrow \{u^n\}$ is Cauchy in $L^q(\partial\Omega)$ – Banach space, $Tr\, u := \lim_{n\to\infty} u^n|_{\partial\Omega}$

Theorem (integration by parts for Sobolev functions): Let $\Omega \in C^{0,1}$, $u \in W^{1,p}(\Omega)$, $v \in W^{1,q}(\Omega)$, let $W^{1,p} \hookrightarrow L^{q'}$ and $W^{1,q} \hookrightarrow L^{p'}$. Then

$$\left[\begin{array}{ll} \frac{1}{p}+\frac{1}{q} \le 1+\frac{1}{d} & \text{if } p,q>1 \\ \frac{1}{p}+\frac{1}{q} < 1+\frac{1}{d} & \text{if } p=1 \text{ or } q=1 \end{array}\right]$$

$$\int_\Omega \frac{\partial u}{\partial x_i} v = -\int_\Omega \frac{\partial v}{\partial x_i} u + \int_{\partial\Omega} \operatorname{Tr} u \operatorname{Tr} v \, n_i$$

second part of the diff. hw - prove it

Inverse trace operator (+ functions with non-integer derivative)

What is the target of Tr? Warning! Not $L^{\frac{(d-1)p}{d-p}}(\partial\Omega)$!

Theorem: Let $\Omega \in C^{0,1}$, $p \in [1,\infty]$, $s \in (\frac{1}{p}, 1]$. Then Tr is bounded from $W^{s,p}(\Omega) \to W^{s-\frac{1}{p},p}(\partial\Omega)$.

Moreover, $\exists \operatorname{Tr}^{-1}: W^{s-\frac{1}{p},p}(\partial\Omega) \to W^{s,p}(\Omega)$, linear, bounded and $\operatorname{Tr}^{-1}(\operatorname{Tr} u) = u$.

(For $p=1$, $\exists \operatorname{Tr}^{-1}: W^{0,1}(\partial\Omega)$ $(= L^1(\partial\Omega)) \to W^{0,1}(\Omega)$ $(= L^1(\Omega))$ which is nonlinear.)

Definition (Sobolev-Slobodeckii): We say that $u \in W^{s,p}(\Omega)$, $s \in (0,1)$ if

$$\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{d+ps}} \, dx\, dy < \infty .$$

Remark: Similarly on $\partial\Omega$.

Definition (Nikolskii): Let $u \in L^p(\Omega)$, we say that $u \in \mathcal{N}^{s,p}(\Omega)$ if $\int_\Omega \frac{|u(x+he_i)-u(x)|^p}{h^{ps}} \varphi < \infty$, $\forall h>0$, φ compactly supported in $\Omega + \{he_i\}$.

Theorem: $W^{s,p}(\Omega) \hookrightarrow \mathcal{N}^{s,p}(\Omega) \hookrightarrow W^{s-\varepsilon,p}(\Omega) \quad \forall \varepsilon > 0$

2. NONLINEAR ELLIPTIC EQUATIONS AS COMPACT PERTURBATIONS

Example.

$$\begin{aligned} -\Delta u + g(u) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

$g: \mathbb{R} \to \mathbb{R}$, continuous, $|g(s)| \le c(1+|s|)^\alpha \quad \alpha \in [0,1)$

Nemytskii $g: L^2(\Omega) \to L^2(\Omega)$

"Apriori estimates" for smooth solution:

$$\int_\Omega -\Delta u \cdot u + g(u)\, u = \int_\Omega f u$$

$$c_1 \|u\|_{1,2}^2 \overset{\text{Poincaré}}{\le} \int_\Omega |\nabla u|^2 = \int_\Omega f u - g(u)\, u \le \|f\|_2 \|u\|_2 + c \int_\Omega (1+|u|)^\alpha |u|$$

$$\int_\Omega (1+|u|)^\alpha |u| \le \int_\Omega (1+|u|)^{(1+\alpha)<2} = \int_\Omega \underbrace{\left((1+|u|)^2 \varepsilon\right)^{\frac{1+\alpha}{2}}}_{\frac{2}{\alpha+1}} \cdot \underbrace{\left(\frac{1}{\varepsilon}\right)^{\frac{\alpha+1}{2}}}_{\frac{2}{1-\alpha}} \overset{\text{Young}}{\le} \frac{\alpha+1}{2} \int_\Omega \varepsilon (1+|u|)^2 + \frac{1-\alpha}{2} \int_\Omega \left(\frac{1}{\varepsilon}\right)^{\frac{\alpha+1}{2} \cdot \frac{2}{\alpha-1}}$$

$$\le c\varepsilon \|u\|_2^2 + c(\Omega,\varepsilon)$$

choose $0 < \varepsilon \ll 1$

$$\Rightarrow \|u\|_{1,2}^2 \le c(\Omega,\alpha)\left(1+\|f\|_2^2\right)$$

Lemma: If $f \in L^2(\Omega)$ then $\exists u \in W_0^{1,2}(\Omega)$ s.t. $\forall v \in W_0^{1,2}(\Omega)$ $\int_\Omega \nabla u . \nabla v + g(u) v = \int_\Omega f v$ (weak S.)

Proof: by fixed point $v \in L^2(\Omega)$ - look for $u \in W_0^{1,2}(\Omega)$

winter $\Rightarrow$ $\forall v \in L^2(\Omega)$ $\exists ! \; u \in W_0^{1,2}(\Omega)$ $-\Delta u = f - g(v)$ in Ω, $u = 0$ on $\partial\Omega$

$M: L^2(\Omega) \to L^2(\Omega)$ $v \mapsto u$. To show that M has a fixed point Schauder.

1. M is continuous (Yes - winter semester + Nemytskii)

2. M is compact (Yes $W^{1,2} \hookrightarrow\hookrightarrow L^2$)

3. $\int_\Omega |\nabla u|^2 \le \int_\Omega |f||u| + \int_\Omega |g(v)| \, |u| \le \varepsilon \|u\|_2^2 + C(\varepsilon) (\|f\|_2^2 + \|g(v)\|_2^2)$

$\Rightarrow \|u\|_{1,2}^2 \le C (\|f\|_2^2 + \|g(v)\|_2^2)$

$\|g(v)\|_2^2 \le C \int_\Omega (1 + |v|)^{2\alpha} \overset{\alpha < 1}{\le} \delta \|v\|_2^2 + C(\delta)$

$\|u\|_{1,2}^2 \le C(\delta) (\|f\|_2^2 + 1) + \delta \|v\|_2^2$ $\qquad \|v\|_2 \le R$

$\|u\|_2^2 \le C(\delta) (\|f\|_2^2 + 1) + \delta R^2 \; \boxed{\overset{?}{\le} R^2}$ Set $\delta = \frac{1}{2}$ and

assume $R^2 \ge 2C (\|f\|_2^2 + 1)$ $\Rightarrow$ $\|u\|_2^2 \le R^2$. Hence $M : B_R \to B_R$ where B_R is a ball in $L^2(\Omega)$

Schauder $\Rightarrow$ $\exists u \in W_0^{1,2}$ a fixed point

Uniqueness: u_1, u_2 solutions:

$\int_\Omega \nabla (u_1 - u_2) . \nabla v + \int_\Omega (g(u_1) - g(u_2)) v = 0 \qquad \forall v \in W_0^{1,2}(\Omega)$

$v := u_1 - u_2$, $\quad \|\nabla (u_1 - u_2)\|_2^2 + \int_\Omega (g(u_1) - g(u_2)) (u_1 - u_2) = 0$

a) if g is nondecreasing $\Rightarrow$ $(g(s_1) - g(s_2))(s_1 - s_2) \ge 0$ $\Rightarrow$ uniqueness

b) $\|\nabla (u_1 - u_2)\|_2^2 \le - \int_\Omega (g(u_1) - g(u_2))(u_1 - u_2) = - \int_\Omega \int_0^1 \frac{d}{ds} g(s u_1 + (1-s) u_2)(u_1 - u_2)$

$= - \int_\Omega \int_0^1 g'(s u_1 + (1-s) u_2)(u_1 - u_2)^2$

$C_{\text{Poincaré}} \|u_1 - u_2\|_2^2 \le \|\nabla (u_1 - u_2)\|_2^2 \le - \int_\Omega \int_0^1 g'(\dots)(u_1 - u_2)^2 \le \sup_s -g'(s) \|u_1 - u_2\|_2^2$

if $\sup_s (-g'(s)) < C_{\text{Poincaré}}$ then $\Rightarrow$ $u_1 = u_2$

Another example: $\qquad -\Delta u + e^u = f$ in Ω

$$u = 0 \quad \text{on } \partial\Omega$$

Estimates: ... $\|\nabla u\|_2^2 + \int_\Omega e^u u = \int_\Omega f u \le \varepsilon \|u\|_2^2 + C(\varepsilon) \|f\|_2^2$

$\|\nabla u\|_2^2 + \int_\Omega e^{u_+} u_+ \le \varepsilon \|u\|_2^2 + C(\varepsilon) \|f\|_2^2 - \int_\Omega e^{u_-} u_-$

$\le \varepsilon \|u\|_2^2 + C(\varepsilon) \|f\|_2^2 + \int_\Omega |u| \le 2\varepsilon \|u\|_2^2 + C(\varepsilon, \Omega) (1 + \|f\|_2^2)$

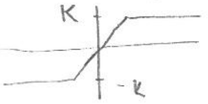
$\|u\|_{1,2}^2 + \int_\Omega e^{u_+} u_+ \le C(1 + \|f\|_2^2)$ $\qquad (\varepsilon \ll 1$ and Poincaré)

Lemma: Let $\Omega \in C^{0,1}$, $f \in L^2$. Then $\exists!\, u \in W_0^{1,2}(\Omega)$ and $\int_\Omega e^u < \infty$, such that $\forall v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$

$$\int_\Omega \nabla u . \nabla v + \int_\Omega e^u v = \int_\Omega f v .$$

Proof. 1. uniqueness .. $\int_\Omega \nabla(u_1 - u_2) . \nabla v + \int_\Omega (e^{u_1} - e^{u_2}) v = 0 \qquad \forall v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$

$v := \frac{(u_1 - u_2)}{|u_1 - u_2|} \min(K, |u_1 - u_2|)$



(such $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$)

$$\nabla v = \nabla(u_1 - u_2)\, \chi_{|u_1 - u_2| \leq K}$$

$$\int_{|u_1 - u_2| \leq K} |\nabla(u_1 - u_2)|^2 + \int_\Omega \underbrace{(e^{u_1} - e^{u_2})(u_1 - u_2)}_{\overset{\vee|}{0}} \frac{\min(K, |u_1 - u_2|)}{|u_1 - u_2|} = 0$$

$$\Rightarrow \int_{|u_1 - u_2| \leq K} |\nabla(u_1 - u_2)|^2 = 0$$

$$K \to \infty \qquad \int_\Omega |\nabla(u_1 - u_2)|^2 = 0 \qquad \Rightarrow \qquad u_1 = u_2$$

2. existence by approximation

$$-\Delta u^n + e^{\min(n, u^n)} = f \text{ in } \Omega \quad , \quad u^n = 0 \text{ on } \partial\Omega$$

by first example $\forall n \; \exists u^n \in W_0^{1,2}(\Omega)$

$$\int_\Omega |\nabla u^n|^2 + \int_\Omega e^{\min(n, u^n_+)} u^n_+ \leq \int_\Omega |f| |u^n| + \int_\Omega |u^n_-|$$

$$\Rightarrow \|u^n\|^2_{1,2} + \int_\Omega e^{\min(n, u^n_+)} u^n_+ \leq c(1 + \|f\|_2^2)$$ uniform (n-independent) estimate

subsequence $\quad u^n \rightharpoonup u \quad$ in $W_0^{1,2}(\Omega)$

$u^n \to u$ in $L^2(\Omega)$

$u^n \to u$ a.e. in Ω

1. $\int_\Omega e^u \overset{\text{Fatou}}{\leq} \liminf_{n \to \infty} \int_\Omega e^{\min(n, u^n)} \leq \liminf \int_\Omega (2 + e^{\min(n, u^n_+)} u^n_+) < \infty$

2. weak formulation $\quad v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$

$$\underbrace{\int_\Omega \underset{\nabla u}{\underset{\downarrow \text{ in } L^2}{\nabla u^n}} . \underset{L^2}{\underset{\Uparrow}{\nabla v}}}_{\downarrow\, n \to \infty} + \int_\Omega \underbrace{e^{\min(n, u^n)} v}_{\overset{\downarrow \text{ a.e}}{e^u v}} = \int_\Omega f v$$

$$\downarrow$$

$$\int_\Omega \nabla u . \nabla v + \int_\Omega e^u v$$, justification of second limit:

Vitali: if $g^n \to g$ a.e. and $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall S \subseteq \Omega \; |S| \leq \delta \; \int_S |g^n| \leq \varepsilon \Rightarrow \int_\Omega g = \lim \int_\Omega g^n$

$$\int_S e^{\min(n, u^n)} |v| \leq \|v\|_\infty \int_S e^{\min(n, u^n)} \leq c\Big(|S| + \int_S e^{\min(n, u^n_+)}\Big)$$

$$= c\Big(|S| + \int_{S \cap \{u^n_+ \leq K\}} e^{\min(n, u^n_+)} + \int_{S \cap \{u^n_+ > K\}} e^{\min(n, u^n_+)}\Big)$$

$$\leq c\Big(|S| + e^K |S| + \int_{S \cap \{u^n_+ > K\}} e^{\min(n, u^n_+)} u^n_+ \cdot \frac{1}{u^n_+}\Big)$$

$$\leq c\Big(|S| + e^K |S| + \frac{1}{K}\Big)$$

$\frac{c}{K} := \frac{\varepsilon}{3}$, $c|S|(1 + e^{\frac{3c}{\varepsilon}}) < \varepsilon$, choose δ : $c\delta(1 + e^{\frac{3c}{\varepsilon}}) < \varepsilon$, then

for $|S| \leq \delta \Rightarrow \int_S e^{\min(n, u^n)} \leq \varepsilon$ and we can use the Vitali theorem for the second limit

Last example: $-\Delta u + b(\nabla u) = f$ in Ω

$u = 0$ on $\partial\Omega$

b is continuous and bounded.

Lemma: $\exists u \in W_0^{1,2}(\Omega)$ s.t. $\forall v \in W_0^{1,2}(\Omega)$ $\int_\Omega \nabla u . \nabla v = \int_\Omega f v - b(\nabla u) v$

Define mapping $M: W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$, $w \mapsto u$, where $-\Delta u = f - b(\nabla w)$ in Ω,

Fixed point: $u = 0$ on $\partial\Omega$.

1. winter semester + Nemytskii $\Rightarrow$ M is continuous

$\|\nabla u\|_2^2 \leq \int_\Omega |f||u| + \underbrace{|b(\nabla w)|}_{\text{bdd}}|u| \leq \varepsilon \|u\|_2^2 + C(1 + \|f\|_2^2)$

2. $\|u\|_{1,2}^2 \leq C(1 + \|f\|_2^2)$

3. compactness: if $\{w^n\}$ bounded in $W_0^{1,2}$, is $\{u^n\}$ precompact in $W_0^{1,2}(\Omega)$?

$\int_\Omega \nabla(u^n - u^m) . \nabla v = -\int_\Omega (b(\nabla w^n) - b(\nabla w^m)) v$

$v := u^n - u^m$ $\|u^n - u^m\|_{1,2}^2 \leq C \|u^n - u^m\|_1$

$\{u^n\}$ is bounded in $W_0^{1,2} \hookrightarrow\hookrightarrow L^1(\Omega) \Rightarrow \{u^n\}$ is cauchy in $L^1(\Omega)$ $\Rightarrow \{u^n\}$ cauchy in $W_0^{1,2}(\Omega)$

$\Rightarrow u^n \to u$ in $W_0^{1,2}(\Omega)$ $\Rightarrow$ M is compact

Schauder $\rightarrow$ $\exists$ a fixed point u

Homework: $\Omega \in C^{0,1}$ $-\Delta u - \frac{1}{1+u} = f$ in Ω 3.4.2019

$u = 0$ on $\partial\Omega$

Show that $\forall f \in L^2(\Omega), f \geq 0$ $\exists! u \in W_0^{1,2}$, $u \geq 0$

3. NONLINEAR ELLIPTIC EQUATIONS - MONOTONE OPERATOR THEORY

3.1 Motivation

Find $\min\limits_{u \in W^{1,p}(\Omega), u = u_0 \text{ on } \partial\Omega} \left(\int_\Omega |\nabla u|^p\right)$ for $u_0 \in W^{1,p}(\Omega)$ given

a) minimum exists (due to convexity - will be proven later)

b) Euler-Lagrange: $\varepsilon > 0$ $\varphi \in W_0^{1,p}(\Omega)$, $\int_\Omega |\nabla u|^p \leq \int_\Omega |\nabla u + \varepsilon \nabla\varphi|^p$

$\Rightarrow 0 \leq \int_\Omega \frac{|\nabla u + \varepsilon\nabla\varphi|^p - |\nabla u|^p}{\varepsilon} \xrightarrow{\varepsilon \to 0+} p \int_\Omega |\nabla u|^{p-2} \nabla u \nabla\varphi$

algebra: $z \in \mathbb{R}^d$, $|z + \varepsilon w|^p - |z|^p = \int_0^1 \frac{d}{dt} |t(z + \varepsilon w) + (1-t) z|^p \, dt$

$= \int_0^1 \frac{d}{dt} \left(\sum_{i=1}^d (t(z_i + \varepsilon w_i) + (1-t) z_i)^2\right)^{\frac{p}{2}} = \varepsilon p \int_0^1 \sum_{i=1}^d (\quad)^{\frac{p-2}{2}} (t(z_i + \varepsilon w_i) + (1-t) z_i) w_i$

$= \varepsilon p \int_0^1 |t(z + \varepsilon w) + (1-t) z|^{p-2} (t(z + \varepsilon w) + (1-t) z) . w$

$$\Rightarrow \int_\Omega \frac{|\nabla u+\varepsilon\nabla\varphi|^p-|\nabla u|^p}{\varepsilon} = p\int_\Omega\int_0^1 |t(\nabla u+\varepsilon\nabla\varphi)+(1-t)\nabla u|^p\,(t(\nabla u+\varepsilon\nabla\varphi)+(1-t)\nabla u).\nabla\varphi\,dt\,dx$$

$$\xrightarrow{\varepsilon\to 0+} p\int_\Omega\int_0^1 |\nabla u|^{p-2}\nabla u.\nabla\varphi\,dt\,dx = p\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\varphi$$

$$\Rightarrow\quad 0\le p\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\varphi \qquad \forall\varphi\in W_0^{1,p}(\Omega)$$

true also for $(-\varphi)\in W_0^{1,p}(\Omega)$

$$\Rightarrow\quad 0 = \cancel{p}\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\varphi \qquad \forall\varphi\in W_0^{1,p}(\Omega)$$

formally, $= -\int_\Omega \operatorname{div}(|\nabla u|^{p-2}\nabla u)\,\varphi \quad\Rightarrow\quad 0=-\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ a.e. in Ω

$$\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u) \qquad (p\text{-Laplacian})$$

Definition: Let $E:\mathbb{R}^N\to\mathbb{R}^N$ be a mapping. We say that

1. E is monotone $\overset{def.}{\iff} \forall x,y\in\mathbb{R}^N \quad (E(x)-E(y)).(x-y)\ge 0$
2. E is strictly monotone $\overset{def.}{\iff} \forall x,y\in\mathbb{R}^N,\ x\ne y \quad (E(x)-E(y)).(x-y)>0$

Example: $E(x):=(\delta+|x|^2)^{\frac{p-2}{2}}x \quad,\ \delta\ge 0$

E is strictly monotone.

Proof. $\left((\delta+|x|^2)^{\frac{p-2}{2}}x-(\delta+|y|^2)^{\frac{p-2}{2}}y\right).(x-y)$

$$=\int_0^1\frac{d}{dt}\left((\delta+|tx+(1-t)y|^2)^{\frac{p-2}{2}}(tx+(1-t)y)\right)dt\,.\,(x-y)$$

$$=\int_0^1\left[(\delta+|tx+(1-t)y|^2)^{\frac{p-2}{2}}(x-y)+\frac{p-2}{2}(\delta+|tx+(1-t)y|^2)^{\frac{p-4}{2}}2(tx+(1-t)y)(x-y)(tx+(1-t)y)\right]dt \quad .(x-y)$$

$$=\int_0^1(\delta+|tx+(1-t)y|^2)^{\frac{p-2}{2}}(x-y)^2+(p-2)(\delta+|tx+(1-t)y|^2)^{\frac{p-4}{2}}((tx+(1-t)y)(x-y))^2$$

$$\ge \begin{cases} (p\ge 2) & (\delta+|tx+(1-t)y|^2)^{\frac{p-2}{2}}(x-y)^2 \overset{x\ne y}{>} 0 \\ (1<p<2) & \int_0^1(\delta+|tx+(1-t)y|^2)^{\frac{p-2}{2}}|x-y|^2-|p-2|(\delta+|tx+(1-t)y|^2)^{\frac{p-4}{2}}|tx+(1-t)y|^2|x-y|^2 \\ & \ge\int_0^1(p-1)(\delta+|tx+(1-t)y|^2)^{\frac{p-2}{2}}|x-y|^2 \overset{x\ne y}{>} 0 \end{cases}$$

Formulation of the problem.

DATA: $\Omega\subseteq\mathbb{R}^d$, $\Omega\in C^{0,1}$, $f:\Omega\to\mathbb{R}$, $A:\Omega\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d$, $B:\Omega\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$,

$\Gamma_D\subseteq\partial\Omega$, $\Gamma_N\subseteq\partial\Omega$, $\Gamma_D\cap\Gamma_N=\emptyset$, $\overline{\Gamma_D\cup\Gamma_N}=\partial\Omega$, $u_0:\Gamma_D\to\mathbb{R}$, $g:\Gamma_N\to\mathbb{R}$

Find $u:\Omega\to\mathbb{R}$

$$-\operatorname{div}(A(x,u,\nabla u))+B(x,u,\nabla u)=f \quad\text{in }\Omega$$

$$u=u_0 \quad\text{on }\Gamma_D$$

$$A(x,u,\nabla u).n=g \quad\text{on }\Gamma_N$$

$$-\operatorname{div}(A(x,u,\nabla u)) = -\sum_{i=1}^d\frac{\partial}{\partial x_i}A_i(x,u,\nabla u)$$

Weak formulation.

Let A and B be Carathéodory, $\exists\, c_2 \in \mathbb{R}$, $c_1 \in L^{p'}(\Omega)$ such that

$$|A(x,u,\xi)| \leq c_2\,(1+|u|^{p-1}+|\xi|^{p-1}) + c_1(x)$$

$$|B(x,u,\xi)| \leq c_2\,(1+|u|^{p-1}+|\xi|^{p-1}) + c_1(x)$$

$p \in (1,\infty)$; $u_0 \in W^{1,p}(\Omega)$, $g \in L^{p'}(\Gamma_N)$, $f \in L^{p'}(\Omega)$ (enough $f \in (W_0^{1,p}(\Omega))^*$) .

We say that $u \in W^{1,p}(\Omega)$ is a weak solution if $\forall \varphi \in W^{1,p}(\Omega)$, $\varphi = 0$ on Γ_D,

$$\int_\Omega A(x,u(x),\nabla u(x))\cdot\nabla\varphi(x) + B(x,u(x),\nabla u(x))\,\varphi(x) = \int_\Omega f(x)\varphi(x) + \int_{\Gamma_N} g(x)\,\varphi(x) .$$

Definition is meaningful:

$u \mapsto A(\cdot,u,\nabla u)$ $\Rightarrow$ growth assumptions on A + Carathéodory by Nemytskii it is continuous mapping from $W^{1,p}(\Omega) \to \underbrace{L^{p'}(\Omega)\times\dots\times L^{p'}(\Omega)}_{d\text{- times}}$

$u \mapsto B(\cdot,u,\nabla u)$, $W^{1,p}(\Omega) \to L^{p'}(\Omega)$

$$\Big|\int_\Omega \underbrace{A(\cdot,u,\nabla u)}_{L^{p'}(\Omega;\mathbb{R}^d)}\cdot\underbrace{\nabla\varphi}_{L^{p}(\Omega,\mathbb{R}^d)}\Big| < \infty \qquad \text{by Hölder}$$

Exercise 1/2 : Show that if u, f, A, B, g, u_0 are smooth and u is a weak solution, then it is a classical solution.

Existence (and uniqueness) of weak solution (for $\Gamma_N = \emptyset$)

Assumption (coercivity):

$\exists\, \alpha > 0$, $\beta \in L^1$ $\forall u \in \mathbb{R}$ $\forall \xi \in \mathbb{R}^d$ for a.a. $x \in \Omega$: $A(x,u,\xi)\cdot\xi + B(x,u,\xi)\cdot u \geq \alpha|\xi|^p - \beta(x)$

Assumption (monotonicity of the leading term):

For a.a. $x \in \Omega$, $\forall u \in \mathbb{R}$, $\forall \xi_1,\xi_2 \in \mathbb{R}^d$: $(A(x,u,\xi_1) - A(x,u,\xi_2))\cdot(\xi_1-\xi_2) \geq 0$

($A(x,u,\xi)$ is monotone w.r.t. ξ)

Assumption (strict monotonicity of the leading term):

$A(x,u,\xi)$ is strictly monotone w.r.t. ξ

Assumption (the whole operator is monotone):

For a.a. $x \in \Omega$, $\forall u_1,u_2 \in \mathbb{R}$ $\forall \xi_1,\xi_2 \in \mathbb{R}^d$: $(A(x,u_1,\xi_1) - A(x,u_2,\xi_2))\cdot(\xi_1-\xi_2) + (B(x,u_1,\xi_1) - B(x,u_2,\xi_2))\cdot(u_1-u_2) \geq 0$

$E: \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$, $\xi = (\xi_1,\dots,\xi_d)$

$$\Updownarrow$$

$(u,\xi) \mapsto (B(x,u,\xi), A_1(x,u,\xi),\dots, A_d(x,u,\xi))$, E is monotone mapping

$(-\operatorname{div}(\delta+|\xi|^2)^{\frac{p-2}{2}}\xi = 0)$ for $p=1$, $\delta = 1$, $\xi = \nabla u$ corresponds to the minimal surface eq

Theorem: Let $\Omega \subseteq \mathbb{R}^d$, $\Omega \in C^{0,1}$, $u_0 \in W^{1,p}(\Omega)$, $p \in (1,\infty)$, A and B are Carátheodory and satisfy the growth assumptions, $f \in (W_0^{1,p}(\Omega))^*$.

Then there exists a weak solution $u \in W^{1,p}(\Omega)$, $u - u_0 \in W_0^{1,p}(\Omega)$, provided that at least one of the following holds:

a) the whole operator is monotone

b) A is monotone w.r.t. ξ and B depends on ξ linearly

c) A is strictly monotone w.r.t. ξ

and, A and B are coercive. Moreover, if the whole operator is strictly monotone, then $\exists ! \, u \in W^{1,p}$.

Proof. Uniqueness. Let u_1, u_2 be two solutions. Then $\forall \varphi \in W_0^{1,p}(\Omega)$

$$\int_\Omega (A(\cdot, u_1, \nabla u_1) - A(\cdot, u_2, \nabla u_2)) \cdot \nabla \varphi + (B(\cdot, u_1, \nabla u_1) - B(\cdot, u_2, \nabla u_2)) \varphi = 0$$

set $\varphi := u_1 - u_2 \in W_0^{1,p}(\Omega)$

$$\int_\Omega \underbrace{(A(\cdot, u_1, \nabla u_1) - A(\cdot, u_2, \nabla u_2)) \cdot \nabla(u_1 - u_2) + (B(\cdot, u_1, \nabla u_1) - B(\cdot, u_2, \nabla u_2)) (u_1 - u_2)}_{=:(M)} = 0$$

& $(M) \geq 0$ $\Rightarrow$ a.e. in Ω, $(M) = 0$ $\Rightarrow$ (strict monotonicity) $u_1 = u_2$ a.e.

Existence.

Step 1. Galerkin approximation (we use fixed point, A,B are Carathéodory & coercive)

Step 2. Uniform estimates (independent of approximation, coercivity is used)

Step 3. Limit passage (monotonicity is used) reflexivity)

Step 1. $W_0^{1,p}$ is separable, $\exists$ linearly independent $\{w_i\}_{i=1}^{\infty}$ dense subset

look for GA $u^n(x) = u_0(x) + \sum_{i=1}^{n} \alpha_i^n w_i(x)$ solving

$$\int_\Omega A(\cdot, u^n, \nabla u^n) \cdot \nabla w_i + B(\cdot, u^n, \nabla u^n) w_i = \langle f, w_i \rangle \qquad i = 1, \dots, n$$

Define $F : \mathbb{R}^n \to \mathbb{R}^n$

$$[F(\alpha)]_i := \int_\Omega A(\cdot, u^n, \nabla u^n) \cdot \nabla w_i + B(\cdot, u^n, \nabla u^n) w_i - \langle f, w_i \rangle, \quad \text{where } u^n = u_0 + \sum_{i=1}^{n} \alpha_i w_i$$

I look for $\alpha \in \mathbb{R}^n$ st. $F(\alpha) = 0$.

Lemma: Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and $\exists R > 0$ s.t. $\forall \alpha \in \mathbb{R}^n, |\alpha| \geq R$: $F(\alpha) \cdot \alpha \geq 0$.

Then $\exists \alpha$, $|\alpha| \leq R$ s.t. $F(\alpha) = 0$.

(without proof, consequence of Browder fixed point theorem)

Use lemma:

1. F is continuous (because continuous dependence of integrand on u^n & u^n on α)

2. $F(\alpha)\cdot\alpha = \sum_{i=1}^{n} {}_{\Omega}\int A(\cdot, u^n, \nabla u^n)\cdot\nabla(\alpha_i w_i) + {}_{\Omega}\int B(\cdot, u^n, \nabla u^n)(\alpha_i w_i) - \langle f, \alpha_i w_i\rangle$

$$= {}_{\Omega}\int A(\cdot, u^n, \nabla u^n)\cdot\nabla(u^n - u_0) + {}_{\Omega}\int B(\cdot, u^n, \nabla u^n)(u^n - u_0) - \langle f, u^n - u_0\rangle$$

$$\overset{\text{Hölder}}{\geq} {}_{\Omega}\int A(\cdot, u^n, \nabla u^n)\cdot\nabla u^n + B(\cdot, u^n, \nabla u^n)\, u^n - \Big[{}_{\Omega}\int |A(\cdot, u^n, \nabla u^n)|\,|\nabla u_0| + |B(\cdot, u^n, \nabla u^n)|\,|u_0| + \|f\|_{(W_0^{1,p})^*}\|u^n - u_0\|_{1,p}\Big]$$

$$\overset{\text{coercivity}}{\geq} c_1\, {}_{\Omega}\int |\nabla u^n|^p - c - \left(\|A\|_{p'}\|\nabla u_0\|_p + \|B\|_{p'}\|u_0\|_p + \|f\|_{(W_0^{1,p})^*}\|u^n - u_0\|_{1,p}\right)$$

$$\overset{\text{growth ass}}{\geq} \frac{c_1}{2}\, {}_{\Omega}\int |\nabla u^n - \nabla u_0|^p - c(1 + \|u_0\|_{1,p}) - c\|u_0\|_{1,p}(1 + \|u^n\|_p^{p-1} + \|\nabla u^n\|_p^{p-1}) - \|f\|_{(W_0^{1,p})^*}\|u^n - u_0\|_{1,p}$$

$$\overset{\text{Poincaré}}{\geq} c_{\text{Poincaré}}\|u^n - u_0\|_{1,p}^p - c\|u_0\|_{1,p}(1 + \|u_0\|_{1,p}^{p-1} + \|u^n - u_0\|_{1,p}^{p-1}) - \|f\|_{(W_0^{1,p})^*}\|u^n - u_0\|_{1,p}$$

$$\geq \|u^n - u_0\|_{1,p}^{p-1}\left(\frac{c_{\text{Poin}}}{2}\|u^n - u_0\|_{1,p} - c\|u_0\|_{1,p}\right) + \|u^n - u_0\|_{1,p}\left(\frac{c_{\text{Poin}}}{2}\|u^n - u_0\|_{1,p}^{p-1} - \|f\|_{(W_0^{1,p})^*}\right)$$

$$- c\,(1 + \|u_0\|_{1,p}^p)$$

$\nearrow \geq 0$

$\exists\, R_0 > 0 : \quad \|u^n - u_0\|_{1,p} \geq R_0 \quad \Rightarrow$

$u^n - u_0 = \sum_{i=1}^{n} \alpha_i w_i$

$\underbrace{\mathbb{R}^n}_{\text{Euclidean norm}} \sim \underbrace{\text{linhul of } \{w_i\}_{i=1}^n}_{W_0^{1,p}\text{-norm}} \quad \Rightarrow \quad K_1(n)\,|\alpha| \leq \|u^n - u_0\|_{1,p} \leq K_2(n)\,|\alpha|$

& $R := \frac{R_0}{K_1(n)}$

if $|\alpha| \geq R \Rightarrow \|u^n - u_0\|_{1,p} \geq R_0 \Rightarrow F(\alpha)\cdot\alpha \geq 0.$

$\Rightarrow \exists\, \alpha,\ F(\alpha) = 0 \quad \Rightarrow \quad \exists\, \{u^n\}$ the Galerkin approximation

Step 2. Uniform estimates

${}_{\Omega}\int A(\cdot, u^n, \nabla u^n)\cdot\nabla w_i + B(\cdot, u^n, \nabla u^n)\, w_i = \langle f, w_i\rangle \qquad \forall\, i = 1, \dots, n$

multiply by α_i and sum w.r.t. i

$(EI^n) \quad {}_{\Omega}\int A(\cdot, u^n, \nabla u^n)\cdot\nabla(u^n - u_0) + B(\cdot, u^n, \nabla u^n)(u^n - u_0) = \langle f, u^n - u_0\rangle$

repeat Step 1: $\quad \|u^n\|_{1,p}^p \leq c\left(1 + \|u_0\|_{1,p}^p + \|f\|_{(W_0^{1,p})^*}^{p'}\right) \leq K_1$

Step 3. Use Nemytskii: $\quad \|A(\cdot, u^n, \nabla u^n)\|_{p'} \leq K_2$

$\|B(\cdot, u^n, \nabla u^n)\|_{p'} \leq K_3$

$W^{1,p}$ is reflexive, $L^{p'}$ is reflexive $(p \in (1, \infty))$

find subsequences $\quad u^n \rightharpoonup u \quad$ in $W^{1,p}(\Omega) \qquad (u - u_0 \in W_0^{1,p}(\Omega))$

$A(\cdot, u^n, \nabla u^n) \rightharpoonup \bar{A} \quad$ in $L^{p'}(\Omega, \mathbb{R}^d)$

$B(\cdot, u^n, \nabla u^n) \rightharpoonup \bar{B} \quad$ in $L^{p'}(\Omega)$

$$\int_\Omega \underbrace{A(\cdot, u^n, \nabla u^n)}_{\rightharpoonup\ \bar{A} \text{ in } L^{p'}} \cdot \underbrace{\nabla w_i}_{L^p} + B(\cdot, u^n, \nabla u^n) w_i = \langle f, w_i \rangle \quad \text{for some } i$$

$$\downarrow \quad n \to \infty$$

$$\int_\Omega \bar{A} \cdot \nabla w_i + \bar{B} w_i = \langle f, w_i \rangle \quad \forall i \in \mathbb{N} \quad \& \ \{w_i\} \text{ is dense in } W_0^{1,p}$$

$$\Rightarrow \int_\Omega \bar{A} \cdot \nabla w + \bar{B} w = \langle f, w \rangle \quad \forall w \in W_0^{1,p}(\Omega)$$

10.4.2019 $\int_\Omega A(u^n, \nabla u^n) \cdot \nabla(u^n - u_0) + B(u^n, \nabla u^n)(u^n - u_0) = \langle f, u^n - u_0 \rangle \qquad (EI)^n$

$$\int_\Omega \bar{A} \cdot \nabla w + \bar{B} w = \langle f, w \rangle \quad \forall w \in W_0^{1,p}(\Omega) \quad (WF)$$

$u^n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, $A(u^n, \nabla u^n) \rightharpoonup \bar{A}$ in $L^{p'}(\Omega, \mathbb{R}^d)$

$B(u^n, \nabla u^n) \rightharpoonup \bar{B}$ in $L^{p'}(\Omega)$

Step 3a: We show that $\lim_{n\to\infty} \int_\Omega A(u^n, \nabla u^n) \cdot \nabla u^n + B(u^n, \nabla u^n) u^n = \int_\Omega \bar{A} \nabla u + \bar{B} u$

we are able to interchange the limit and the product of 2 weakly converging seq.

Proof: $\int_\Omega A(u^n, \nabla u^n) \cdot \nabla u^n + B(u^n, \nabla u^n) u^n = \int_\Omega A(u^n, \nabla u^n) \cdot \nabla(u^n - u_0) + B(u^n, \nabla u^n)(u^n - u_0)$

$$+ \int_\Omega A(u^n, \nabla u^n) \cdot \nabla u_0 + B(u^n, \nabla u^n) u_0$$

$$= \langle f, u^n - u_0 \rangle + \int_\Omega \underbrace{A(u^n, \nabla u^n)}_{\rightharpoonup \text{ in } L^{p'}} \cdot \underbrace{\nabla u_0}_{\in L^p} + \underbrace{B(u^n, \nabla u^n)}_{\rightharpoonup \text{ in } L^{p'}} \underbrace{u_0}_{\in L^p}$$

$$\xrightarrow{n\to\infty} \langle f, u - u_0 \rangle + \int_\Omega \bar{A} \cdot \nabla u_0 + \bar{B} u_0$$

$\int_\Omega \bar{A} \nabla u + \bar{B} u = \int_\Omega \bar{A} \cdot (\nabla u - \nabla u_0) + \bar{B}(u - u_0) + \int_\Omega \bar{A} \nabla u_0 + \bar{B} u_0$, set $w = u - u_0$ in (WF)

$$= \langle f, u - u_0 \rangle + \int_\Omega \bar{A} \cdot \nabla u_0 + \bar{B} u_0$$

Step 3b: We will show that $\lim_{n\to\infty} \int_\Omega A(u^n, \nabla u^n) \cdot \nabla u^n = \int_\Omega \bar{A} \cdot \nabla u$

Proof: $\int_\Omega A(u^n, \nabla u^n) \cdot \nabla u^n = \int_\Omega A(u^n, \nabla u^n) \nabla u^n + \int_\Omega B(u^n, \nabla u^n) u^n - \int_\Omega B(u^n, \nabla u^n) u^n$, use 3a,

$$\xrightarrow{n\to\infty} \int_\Omega \bar{A} \nabla u + \bar{B} u - \lim_{n\to\infty} \int_\Omega \underbrace{B(u^n, \nabla u^n)}_{\rightharpoonup \text{ in } L^{p'}} \underbrace{u^n}_{\to \text{ in } L^p \text{ (strongly!)}} = \int_\Omega \bar{A} \cdot \nabla u$$

Step 3c. I want to identify $\bar{A}$ & $\bar{B}$, monotonicity comes into game.

Case 1. The whole operator is monotone

Take arbitrary $V \in L^p(\Omega, \mathbb{R}^d)$, $v \in L^p(\Omega)$

monotonicity in all Ω

$$0 \le \int_\Omega (A(u^n, \nabla u^n) - A(v, V)) \cdot (\nabla u^n - V) + (B(u^n, \nabla u^n) - B(v, V))(u^n - v)$$

$$\xrightarrow{n\to\infty} \lim_{n\to\infty} \int_\Omega A(u^n, \nabla u^n) \cdot \nabla u^n + B(u^n, \nabla u^n) u^n + \lim_{n\to\infty} \int_\Omega - A(u^n, \nabla u^n) \cdot V - A(v, V) \cdot (\nabla u^n - V) - B(u^n, \nabla u^n) v - B(v, V)(u^n - v)$$

3a + weak conv.

$$= \int_\Omega \bar{A} \cdot \nabla u + \bar{B} u - \int_\Omega \bar{A} \cdot V + A(v, V)(\nabla u - V) + \bar{B} v + B(v, V)(u - v)$$

$$= \int_\Omega (\bar{A} - A(v, V)) \cdot (\nabla u - V) + (\bar{B} - B(v, V)) \cdot (u - v)$$

Minty knocks: $V := \nabla u - \varepsilon W$ $\quad \varepsilon > 0 \quad W \in L^p(\Omega; \mathbb{R}^d)$

$v := u - \varepsilon v \qquad v \in L^p(\Omega)$

$$(\tfrac{1}{\varepsilon}\cdot)\; 0 \le \int_\Omega (\bar{A} - A(u-\varepsilon v, \nabla u - \varepsilon W)).W + (\bar{B} - B(u-\varepsilon v, \nabla u - \varepsilon W))\, v$$

$$\xrightarrow[\varepsilon\to 0+]{\text{Nemytskii}} \int_\Omega (\bar{A} - A(u,\nabla u)).W + (\bar{B} - B(u,\nabla u))\, v$$

set $W := -\dfrac{\bar{A} - A(u,\nabla u)}{1 + |\bar{A} - A(u,\nabla u)|}$, $v := -\dfrac{\bar{B} - B(u,\nabla u)}{1 + |\bar{B} - B(u,\nabla u)|}$

$$\Rightarrow \int_\Omega \frac{|\bar{A} - A(u,\nabla u)|^2}{1 + |\bar{A} - A(u,\nabla u)|} + \frac{|\bar{B} - B(u,\nabla u)|^2}{1 + |\bar{B} - B(u,\nabla u)|} \le 0 \qquad \& \text{ integrand } \ge 0$$

$\Rightarrow \quad \bar{A} = A(u,\nabla u) \;,\quad \bar{B} = B(u,\nabla u)$ a.e. in Ω

Case 2. A - monotone , B - linear w.r.t. ∇u

Identification of $\bar{A}$ (use step 3b) $\quad V \in L^p(\Omega; \mathbb{R}^d)$ arbitrary

$$0 \le \int_\Omega (A(u^n, \nabla u^n) - A(u^n, V)).(\nabla u^n - V)$$

$$= \underbrace{\int_\Omega A(u^n,\nabla u^n).\nabla u^n}_{3b} \underbrace{- \int_\Omega A(u^n, \nabla u^n).V}_{\text{weak conv.}} + A(u^n, V).(\nabla u^n - V)$$

$$\xrightarrow{n\to\infty} \int_\Omega \bar{A}.\nabla u - \int_\Omega \bar{A}.V - \int_\Omega A(u,V).(\nabla u - V) - \lim_{n\to\infty} \int_\Omega (A(u^n,V) - A(u,V)).(\nabla u^n - V)$$

$$= \int_\Omega (\bar{A} - A(u,V)).(\nabla u - V) - \lim_{n\to\infty} \int_\Omega (A(u^n,V) - A(u,V)).(\nabla u^n - V)$$

$$\left|\int_\Omega (A(u^n,V) - A(u,V)).(\nabla u^n - V)\right| \le \|\nabla u^n - V\|_p \cdot \left(\int_\Omega |A(u^n,V) - A(u,V)|^{p'}\right)^{1/p'}$$

$$\le c(V) \cdot \left(\int_\Omega |A(u^n,V) - A(u,V)|^{p'}\right)^{1/p'}$$

$u^n \to u$ a.e. $\Rightarrow A(u^n,V) - A(u,V) \to 0$ a.e.

Vitali $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall U \subseteq \Omega \; |U| \le \delta \; \forall n \; \int_U |A(u^n,V) - A(u,V)|^{p'} \le \varepsilon$

Use growth assumptions on A

$$\int_U |A(u^n,V) - A(u,V)|^{p'} \le \int_U c(|u^n|^p + |u|^p + |V|^p + G) \qquad G \in L^1(\Omega) \;\&\; A(u,v) \sim |u|^{p-1} + |V|^{p-1}$$

$$\le \int_U c|u^n|^p + c \int_\Omega (|u|^p + |V|^p + G)$$

$c(|u|^p + |V|^p + G) \in L^1(\Omega) \Rightarrow \forall \varepsilon \; \exists \delta$ s.t. $|U| \le \delta \Rightarrow \int_U c(|u|^p + |V|^p + G) \le \frac{\varepsilon}{2}$

u^n bounded in $W^{1,p}$ $\Rightarrow$ u^n bounded in L^{p+z} , $z > 0$

$$\Rightarrow c\int_U |u^n|^p . 1 \le c\left(\int_\Omega |u^n|^{p+z}\right)^{\frac{p}{p+z}} . |U|^{\frac{z}{z+p}} \le c\|u^n\|^p_{p+z} |U|^{\frac{z}{z+p}} \le c\|u^n\|^p_{1,p} |U|^{\frac{z}{z+p}} \le c|U|^{\frac{z}{z+p}}$$

if $\delta^{\frac{z}{z+p}} < \frac{\varepsilon}{2c} \Rightarrow \int_\Omega |A(u^n,V) - A(u,V)|^{p'} < \varepsilon \qquad \Rightarrow$ Vitali ok

$\Rightarrow 0 \le \int_\Omega (\bar{A} - A(u,V)).(\nabla u - V) \qquad \forall V \in L^p(\Omega, \mathbb{R}^d)$, set $V := \nabla u - \varepsilon W$

$0 \le \int_\Omega (\bar{A} - A(u, \nabla u - \varepsilon W)).(\nabla u - \nabla u + \varepsilon W) \xrightarrow{\varepsilon\to 0+} \int_\Omega (\bar{A} - A(u,\nabla u)).W$, set $W := \dfrac{-(\bar{A} - A(u,\nabla u))}{1 + |\bar{A} - A(u,\nabla u)|}$

$\Rightarrow \bar{A} = A(u,\nabla u)$

homework: if B is linear w.r.t. ∇u , $B(u,\xi) = \sum_{i=1}^{d} b_i(x,u).\xi_i \Rightarrow \left\{ \begin{array}{l} u^n \to u \text{ in } L^p,\; u^n \rightharpoonup u \text{ in } W^{1,p}_0 \\ B(u^n,\nabla u^n) \to B(u,\nabla u) \text{ in } L^{p'} \end{array} \right.$

Case 3. A is strictly monotone but B is general

We show that $\nabla u^n \to \nabla u$ a.e.

homework: $\left.\begin{array}{l} u^n \to u \text{ a.e} \\ \nabla u^n \to \nabla u \text{ a.e.} \\ u^n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \end{array}\right\} \Rightarrow \begin{array}{l} B(u^n,\nabla u^n) \rightharpoonup B(u,\nabla u) \text{ in } L^{p'} \\ B(u^n,\nabla u^n) \to B(u,\nabla u) \text{ in } L^q \ \forall q < p' \end{array}$

We know $\bar{A} = A(u, \nabla u)$

$$0 \leq \int_\Omega (A(u^n,\nabla u^n) - A(u^n,\nabla u)) \cdot (\nabla u^n - \nabla u) \to \int_\Omega (\bar{A} - A(u,\nabla u)) \cdot (\nabla u - \nabla u) = 0$$

$$\underbrace{(A(u^n,\nabla u^n) - A(u^n,\nabla u)) \cdot (\nabla u^n - \nabla u) \to 0}_{\downarrow} \text{ strongly in } L^1(\Omega)$$

$\forall \varepsilon > 0 \ \exists \Omega_\varepsilon, \ |\Omega \setminus \Omega_\varepsilon| \leq \varepsilon$, uniformly in Ω_ε

$u^n \to u$ uniformly

$\Rightarrow (A(u,\nabla u^n) - A(u,\nabla u)) \cdot (\nabla u^n - \nabla u)$ uniformly

$x \in \Omega_\varepsilon \quad (A(u(x), \nabla u^n(x)) - A(u(x),\nabla u(x))) \cdot (\nabla u^n(x) - \nabla u(x)) \to 0$

Assume $\nabla u^n(x) \not\to \nabla u(x)$. Then because A is STRICTLY MONOTONE

$\lim (A(\nabla u^n(x)) - A(u(x),\nabla u(x))) \cdot (\nabla u^n(x) - \nabla u(x)) > 0$, contradiction

Example: $-\operatorname{div}(\operatorname{arctg}(1+|\nabla u|^2)\nabla u) + u^{123} = f$ in $B_1(0) \subseteq \mathbb{R}^3$

$\operatorname{arctg}(1+|\nabla u|^2) \nabla u \cdot n = 0$ on $\partial B_1(0)$

Let $f = f_1 + f_2$, where $f_1 \in L^{\frac{124}{123}}(\Omega)$, $f_2 \in (W^{1,2}(\Omega))^*$

$\exists! \ u \in W^{1,2}(\Omega) \cap L^{124}(\Omega)$ s.t. $\forall v \in W^{1,2}(\Omega) \cap L^{124}(\Omega)$

$$\int_\Omega \operatorname{arctg}(1+|\nabla u|^2)\nabla u \cdot \nabla v + u^{123} v = \int_\Omega f_1 v + \langle f_2, v \rangle_{W^{1,2}(\Omega)}$$

$A(u,\nabla u) = \operatorname{arctg}(1+|\nabla u|^2)\nabla u$

$B(u,\nabla u) = u^{123}$

A priori estimates.

Set $v := u$

$$c_1 \|\nabla u\|_2^2 + \|u\|_{124}^{124} \leq \int_\Omega \operatorname{arctg}(1+|\nabla u|^2)|\nabla u|^2 + |u|^{124} \leq$$

$$\leq \int_\Omega |f_1||u| + \|f_2\|_{(W^{1,2})^*} \|u\|_{1,2}$$

$$\leq \varepsilon . \|u\|_{124}^{124} + c(\varepsilon) \|f\|_{\frac{124}{123}}^{\frac{124}{123}} + c\|f_2\| (\|\nabla u\|_2 + \|u\|_2)$$

$$\leq \varepsilon . \|u\|_{124}^{124} + c(\varepsilon) \|f\|_{\frac{124}{123}}^{\frac{124}{123}} + \hat{c}\|f_2\| (\|\nabla u\|_2 + \|u\|_{124})$$

$$\leq 2\varepsilon \|u\|_{124}^{124} + \varepsilon \|\nabla u\|_2^2 + c(\varepsilon) \left(\|f_1\|_{\frac{124}{123}}^{\frac{124}{123}} + \|f_2\|^2 + \|f_2\|^{\frac{124}{123}} \right)$$

$$\Rightarrow \|u\|_{1,2} + \|u\|_{124} \leq c\left(\Omega, \|f_1\|_{\frac{124}{123}}, \|f_2\|_{(W^{1,2})^*}\right)$$

Galerkin $\{u_i\}_{i=1}^{\infty}$ dense in $W^{1,2}(\Omega) \cap L^{124}(\Omega)$

$u^n = \sum_{i=1}^{n} \alpha_i^n u_i$, $\int_\Omega A(\nabla u^n) \nabla u_i + B(u^n) u_i = \int_\Omega f_1 u_i + \langle f_2, u_i \rangle$ $i = 1, \dots, n$

$u^n \rightharpoonup u$ in $W^{1,2}$

$u^n \rightharpoonup u$ in L^{124}

$A(\nabla u^n) \rightharpoonup \bar{A}$ in L^2 $\quad (A \sim \operatorname{arctg}(1+|\nabla u|^2) \nabla u \sim \nabla u)$

$B(u^n) = (u^n)^{123} \rightharpoonup \bar{B}$ in $L^{\frac{124}{123}}$

$\int_\Omega \bar{A} \cdot \nabla w + \bar{B} w = \int_\Omega f_1 w + \langle f_2, w \rangle \qquad \forall w \in W^{1,2}(\Omega) \cap L^{124}$

since $u \in W^{1,2}(\Omega) \cap L^{124}$ it can be set $w := u$

$\Rightarrow \lim \int A(\nabla u^n) \cdot \nabla u^n + B(u^n) u^n \overset{\text{Step 3a}}{=} \int_\Omega \bar{A} \cdot \nabla u + \bar{B} u$

because $(A(\xi_1) - A(\xi_2)) \cdot (\xi_1 - \xi_2) + (B(u_1) - B(u_2))(u_1 - u_2) \geq 0$ (check at home)

we use Minty to get $\bar{A} = A(\nabla u)$, $\bar{B} = B(u)$

<u>Example</u>. Let $W_0^{1,p}(\Omega)$ be equipped with $\|u\|_{W_0^{1,p}} := \|\nabla u\|_p$, Ω open, bounded, $p \in (1, \infty)$

Then $\forall F \in (W_0^{1,p}(\Omega))^* \ \exists f \in L^{p'}(\Omega, \mathbb{R}^d)$ s.t. $\|f\|_{p'}^{p'} = \|F\|_{(W_0^{1,p})^*}^{p'}$

$\forall \varphi \in C_0^\infty(\Omega) \quad \int_\Omega f \cdot \nabla \varphi = - \langle F, \varphi \rangle \quad \Leftrightarrow \quad \operatorname{div} f = F$ in weak sense

Use theorem $\forall F \in (W_0^{1,p}(\Omega))^* \ \exists! \, u \in W_0^{1,p}(\Omega)$

$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w = \langle F, w \rangle \qquad \forall w \in W_0^{1,p}(\Omega)$

$\|\nabla u\|_p^p \leq \|F\|_{(W^{1,p})^*} \|u\|_{1,p} = \|F\| \, \|\nabla u\|_p$

$\|\nabla u\|_p^p \leq \|F\|^{p'}$

$\|F\|_{(W_0^{1,p})^*} = \sup_{\substack{w \in W_0^{1,p} \\ \|\nabla w\|_p \leq 1}} \langle F, w \rangle = \sup \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w \leq \|\nabla w\|_p \, \|\nabla u\|_p^{p-1} \leq \|\nabla u\|_p^{p-1}$

$\Rightarrow \|\nabla u\|_p^p = \|F\|^{p'} \qquad f := - |\nabla u|^{p-2} \nabla u \qquad \Rightarrow \quad \operatorname{div} f = F$

$\|f\|_{p'}^{p'} = \|\nabla u\|_p^p = \|F\|_{(W_0^{1,p})^*}^{p'}$

Homework: $\forall F \in (W^{1,p}(\Omega))^* \ \exists f \in L^{p'}(\Omega; \mathbb{R}^d)$, $g \in L^{p'}(\Omega) : F = \operatorname{div} f + g$

($\|F\|^{p'} = \|f\|_{p'}^{p'} + \|g\|_{p'}^{p'}$, then the representation is unique)

4. MINIMIZATION OF (CONVEX) FUNCTIONALS AND ITS RELATION TO MONOTONE OPERATOR THEORY

Given $F: \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, given $f: \Omega \to \mathbb{R}$

$$\min_{u \in ?} \int_\Omega F(u, \nabla u) - \langle f, u \rangle$$

Assumptions: 1. F is Carathéodory

2. $F(\xi) \geq c_1 |\xi|^p - c_2(x)$ $\quad c_1 > 0, \quad c_2(x) \in L^1(\Omega)$

3. $f \in (W_0^{1,p}(\Omega))^*$ $\quad (\Rightarrow \quad u \in W_0^{1,p}(\Omega))$

Theorem: Let 1.-3. hold. Let F be convex w.r.t. ξ. Then $\exists\, u \in W_0^{1,p}$ $\forall\, v \in W_0^{1,p}$

$$\int_\Omega F(u, \nabla u) - \langle f, u \rangle \leq \int_\Omega F(v, \nabla v) - \langle f, v \rangle.$$

↑ fundamental theorem in calculus of variations

Proof:

$\exists\, u^n \in W_0^{1,p}(\Omega)$ $\quad I = \inf_{u \in W_0^{1,p}} \int_\Omega F(u, \nabla u) - \langle f, u \rangle = \lim_{n \to \infty} \int_\Omega F(u^n, \nabla u^n) - \langle f, u^n \rangle$

$\exists\, n_0; \forall\, n > n_0: \int_\Omega F(u^n, \nabla u^n) - \langle f, u^n \rangle \leq I + 1 < \infty$

2. + 3. $\Rightarrow c_1 \|\nabla u^n\|_p^p - \|f\|_{(W_0^{1,p})^*} \|u^n\|_{1,p} - \int_\Omega c_2(x) \leq I + 1$

$\Rightarrow \|u^n\|_{1,p} \leq c(\Omega, c_2, c_1, \|f\|)$ $\quad$ (Young, Poincaré)

reflexivity $\Rightarrow \exists$ subsequence $u^n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$

compact embedding $u^n \to u$ in $L^p(\Omega)$

Theorem: Let $z^n \to z$ in $L^1(\Omega; \mathbb{R}^M)$, $\xi^n \rightharpoonup \xi$ in $L^1(\Omega; \mathbb{R}^N)$

Let $F(x, z, \xi)$ be Carathéodory and convex w.r.t. ξ $\left[(\forall z \in \mathbb{R}^M, \xi_1, \xi_2 \in \mathbb{R}^N, \lambda \in (0,1): \right.$
$\left. F(x, z, \lambda \xi_1 + (1-\lambda)\xi_2) \leq \lambda F(x, z, \xi_1) + (1-\lambda) F(x, z, \xi_2)) \right]$

and $F(x, z, \xi) \geq c(x) \in L^1(\Omega)$.

Then $\int_\Omega F(x, z(x), \xi(x)) \leq \liminf_{n \to \infty} \int_\Omega F(x, z^n(x), \xi^n(x))$.

This property is called weak lower semicontinuity of convex functionals.

use WLSC: $I = \lim_{n \to \infty} \int_\Omega F(u^n, \nabla u^n) - \langle f, u^n \rangle \geq \int_\Omega F(u, \nabla u) - \langle f, u \rangle \geq I$

$\Rightarrow u$ is a minimizer

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Proof: (only if $\frac{\partial F(x,z,\xi)}{\partial \xi}: \Omega \times \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^n$ is Carathéodory)

Lemma: Let $F: \mathbb{R}^M \to \mathbb{R}$ and $A: \mathbb{R}^M \to \mathbb{R}^M$ be continuous, and $A(\xi) = \frac{\partial F(\xi)}{\partial \xi}$

Then: 1. F is (strictly) convex $\Leftrightarrow$ A is (strictly) monotone

2. $F(\xi_1) - F(\xi_2) \geq A(\xi_2) \cdot (\xi_1 - \xi_2)$ $\quad$ for F convex

Proof of lemma: $u, w \in \mathbb{R}^M$ arbitrary, define

$$\varphi_w(t) := F(u + t w) \qquad t \in \mathbb{R}$$

$$\Rightarrow \varphi_w'(t) = \frac{\partial F(u+tw)}{\partial \xi} . w = A(u+tw). w$$

1. „⇒" F is (strictly) convex $\Rightarrow$ φ_w is (strictly) convex (if $w \neq 0$)

$\varphi_w'(1) - \varphi_w'(0) \geq 0$ strict "$>$" if $w \neq 0$ and F strictly convex

$(A(u+w) - A(u)). w \geq 0$ or >0 if $w \neq 0$

$w := v - u \Rightarrow (A(v) - A(u)).(v-u) \geq 0$ (or >0 if $u \neq v$ and F strictly convex)

„⇐" take $t_1 \neq t_2$

$$\varphi_w'(t_1) - \varphi_w'(t_2) = (A(u + t_1 w) - A(u + t_2 w)). w$$

$$= \frac{(A(u+t_1 w) - A(u+t_2 w)) \circ (u + t_1 w - (u + t_2 w))}{t_1 - t_2} \overset{A \text{ mon.}}{\geq} 0 \quad \left(\overset{A \text{ strictly mon.}}{>0} \quad \& \, w \neq 0 \right)$$

$$F(u+w) - F(u) = \varphi_w(1) - \varphi_w(0) = \int_0^1 \varphi_w'(t)\, dt \geq \int_0^1 \varphi_w'(0)\, dt = \varphi_w'(0) = A(u). w$$

set $w := v - u$ $\qquad \downarrow > \int_0^1 \varphi_w'(0)\, dt$ if $w \neq 0$

$F(v) - F(u) \geq A(u).(v-u)$ ($>$ if $v \neq u$) $\Leftrightarrow$ ②

Let $\xi_1, \xi_2 \in \mathbb{R}^n$, $\lambda \in (0,1)$, $z := \lambda \xi_1 + (1-\lambda)\xi_2$

we want $F(z) \leq \lambda F(\xi_1) + (1-\lambda) F(\xi_2)$

use ② with $u := z$, $v := \xi_1$ $\Rightarrow$ $F(\xi_1) - F(z) \geq A(z).(\xi_1 - z)$ $\quad /.\lambda$

$u := z$, $v := \xi_2$ $\quad F(\xi_2) - F(z) \geq A(z).(\xi_2 - z)$ $\quad /.(1-\lambda)$ $\quad \}+$

$$+ \Rightarrow \lambda F(\xi_1) + (1-\lambda) F(\xi_2) - F(z) \geq \lambda A(z).(\xi_1 - z) + (1-\lambda) A(z)(\xi_2 - z)$$

$$= \lambda A(z).(\xi_1 - \lambda\xi_1 - (1-\lambda)\xi_2) + (1-\lambda) A(z).(\xi_2 - \lambda \xi_1 - (1-\lambda)\xi_2)$$

$$= \lambda(1-\lambda) A(z).(\xi_1 - \xi_2) + \lambda(1-\lambda) A(z).(\xi_2 - \xi_1) = 0$$

Continuation of the proof of W-L-S

Step 2: We have $F(x, z^n(x), \xi^n(x)) - F(x, z^n(x), \xi(x)) \geq A(x, z^n(x), \xi(x)).(\xi^n(x) - \xi(x))$ a.e. in Ω

$$A(x, z, \xi) = \frac{\partial F(x,z,\xi)}{\partial \xi}$$

$\forall \varepsilon > 0 \; \exists \Omega_\varepsilon$, $|\Omega \setminus \Omega_\varepsilon| \leq \varepsilon$: $z^n \to z$ uniformly in Ω_ε

$|\xi| \leq \frac{\tilde{c}}{\varepsilon}$ in Ω_ε

$$\int_\Omega F(\cdot, z^n, \xi^n) = \int_\Omega \underbrace{F(\cdot, z^n, \xi^n) - c(x)}_{\geq 0} + \int_\Omega c(x) \geq \int_{\Omega_\varepsilon} F(\cdot, z^n, \xi^n) - c(x) + \int_\Omega c(x)$$

$$= \int_{\Omega_\varepsilon} F(\cdot, z^n, \xi) - c(x) + \int_{\Omega_\varepsilon} F(\cdot, z^n, \xi^n) - F(\cdot, z^n, \xi) + \int_\Omega c(x)$$

$$\geq \int_{\Omega_\varepsilon} F(\cdot, z^n, \xi) - c(x) + \int_{\Omega_\varepsilon} A(\cdot, z^n, \xi).(\xi^n - \xi) + \int_\Omega c(x) \quad , \text{ take liminf}$$

$$\liminf \int_\Omega F(\cdot, z^n, \xi^n) \geq \liminf \Big(\int_{\Omega_\varepsilon} F(\cdot, z^n, \xi) - c(x) + \int_{\Omega_\varepsilon} A(\cdot, z^n, \xi)\cdot(\xi^n - \xi) + \int_\Omega c(x) \Big)$$

$$= \liminf \int_{\Omega_\varepsilon} \underbrace{F(\cdot, z^n, \xi) - c(x)}_{\geq 0} + \lim \int_{\Omega_\varepsilon} A(\cdot, z^n, \xi)\cdot(\xi^n - \xi) + \int_\Omega c(x)$$

Fatou

$$\geq \int_{\Omega_\varepsilon} F(\cdot, z, \xi) - c(x) + \int_\Omega c(x) + \lim \int_{\Omega_\varepsilon} A(\cdot, z^n, \xi)\cdot \underbrace{(\xi^n - \xi)}_{\substack{\downarrow \text{ in } L^1(\Omega_\varepsilon)\\ 0}}$$

(A is continuous wrt. z, ξ and $z^n \to z$ uniformly, ξ bounded in Ω_ε) $\Rightarrow A(\cdot, z^n, \xi) \to A(\cdot, z, \xi)$ in $L^\infty(\Omega_\varepsilon)$

$$= \int_{\Omega_\varepsilon} F(\cdot, z, \xi) + \int_{\Omega \setminus \Omega_\varepsilon} c(x) \xrightarrow{\text{monotone conv.}} \int_\Omega F(\cdot, z, \xi) \quad \text{as } \varepsilon \to 0+$$

<u>Example.</u> $F(u, \xi) = a(u)\,|\xi|^2$ $\quad a \in C^1(\mathbb{R})$, $\quad 0 < c_1 \leq a(s) \leq c_2 < \infty$

$f \in L^2(\Omega)$

Minimize $\displaystyle\min_{u \in W_0^{1,2}(\Omega)} F(u, \nabla u) - f.u$, $\quad F \geq c_1 |\nabla u|^2$, $\quad F$ is convex wrt. ξ

use theorem, $\exists u \in W_0^{1,2}(\Omega)$ $\forall v \in W_0^{1,2}(\Omega)$ $\int_\Omega a(u)\,|\nabla u|^2 - f.u \leq \int_\Omega a(v).\,|\nabla v|^2 - f.v$

set $v = u + \varepsilon \varphi$; $\varphi \in C_0^\infty(\Omega)$ or $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$

$$\int_\Omega a(u)\,|\nabla u|^2 - f.u \leq \int_\Omega a(u+\varepsilon\varphi).\left(|\nabla u|^2 + \varepsilon^2|\nabla\varphi|^2 + 2\varepsilon\,\nabla u.\nabla\varphi\right) - fu - \varepsilon f\varphi \qquad /:\varepsilon$$

$$\int_\Omega f\varphi \leq \int_\Omega a(u+\varepsilon\varphi)(\varepsilon|\nabla\varphi|^2 + 2\nabla u.\nabla\varphi) + \frac{1}{\varepsilon}\int_\Omega (a(u+\varepsilon\varphi) - a(u))\,|\nabla u|^2$$

$$\xrightarrow{\varepsilon \to 0+} \int_\Omega 2a(u)\,\nabla u.\nabla\varphi + a'(u)\,\varphi\,|\nabla u|^2$$

$\Rightarrow \forall \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ $\quad \int_\Omega 2a(u)\,\nabla u.\nabla\varphi + a'(u)\varphi|\nabla u|^2 = \int_\Omega f\varphi$

$\Rightarrow u$ solves in weak sense $\quad -\operatorname{div}(a(u)\nabla u) + a'(u)\,|\nabla u|^2 = f$ in Ω

$$-\operatorname{div}(A(u, \nabla u)) + B(u, \nabla u) = f$$

A satisfies the assumptions of existence theorem for monotone operator with $p = 2$

BUT !!! $\quad$ B does not, because $B(u, \xi) \sim |\xi|^2$ (we would need $B(u, \xi) \sim |\xi|$)

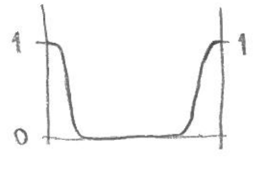
<u>Remark.</u> Being a minimizer is much stronger than being a weak solution.

(In case that F depends on u)

<u>Example (minimization with constraint)</u>

$$\min_{S} \int_\Omega \frac{|\nabla u|^2}{2} - fu$$

$S =: \begin{cases} u \in W^{1,2}(\Omega) \\ u = 1 \text{ on } \partial\Omega \\ u \geq 0 \text{ in } \Omega \end{cases}$



$I := \inf_{u \in S} \int_\Omega \frac{|\nabla u|^2}{2} - fu = \lim_{n \to \infty} \int_\Omega \frac{|\nabla u^n|^2}{2} - fu^n$ $\quad u^n$ is a bounded sequence in S

$u^n \rightharpoonup u$ in $W^{1,2}$ $\quad u^n \to u$ in L^2

$u = 1$ on $\partial\Omega$, $\quad u \geq 0$ in Ω

$\Rightarrow u \in S$

$\overset{\text{WLS}}{\Rightarrow} \geq \int_\Omega \frac{|\nabla u|^2}{2} - fu$

$\Rightarrow u$ is a minimizer

Uniqueness : $I = \int_\Omega \frac{|\nabla u_1|^2}{2} - f u_1 = \int_\Omega \frac{|\nabla u_2|^2}{2} - f u_2$, $u_1 \neq u_2$

$\frac{u_1+u_2}{2} \in S$ $\quad \int_\Omega \left|\nabla \left(\frac{u_1+u_2}{2}\right)\right|^2 - f\left(\frac{u_1+u_2}{2}\right) < I$ a contradiction

$v \in S \quad (1-\lambda) u + \lambda v \in S \quad \lambda \in (0,1)$

$$\int_\Omega \frac{|\nabla u|^2}{2} - f u \leq \int_\Omega \frac{|\nabla((1-\lambda)u + \lambda v)|^2}{2} - f((1-\lambda)u + \lambda v)$$

$$\int_\Omega f(-u + u(1-\lambda) + \lambda v) \leq \int_\Omega \left(\frac{(1-\lambda)^2-1}{2}\right)|\nabla u|^2 + \frac{\lambda^2}{2}|\nabla v|^2 + \lambda(1-\lambda)\nabla u \cdot \nabla v \qquad / : \lambda$$

$$\int_\Omega f(v-u) \leq \int_\Omega (1-\lambda)\nabla u \cdot \nabla v + \frac{\lambda}{2}|\nabla v|^2 + \left(\frac{-2+\lambda}{2}\right)|\nabla u|^2 \qquad / \lambda \to 0+$$

$$\int_\Omega f(v-u) \leq \int_\Omega \nabla u \cdot (\nabla v - \nabla u) \qquad \forall v \in S$$

choose $v = u + \varphi$ with $\varphi \geq 0$ and $\varphi \in W_0^{1,2}$

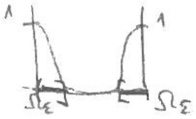
$$\int_\Omega f\varphi \leq \int_\Omega \nabla u \cdot \nabla \varphi \qquad \forall \varphi \geq 0 \quad \varphi \in C_0^\infty(\Omega) \qquad \left(\text{formally } \int_\Omega f\varphi \leq -\int_\Omega \Delta u \varphi \ \forall \varphi\right)$$

$\Rightarrow \quad f \leq -\Delta u$ in weak sense

Assume that $\Omega_\varepsilon \subseteq \Omega$ is open and $u > \varepsilon_0$ in Ω_ε

set $v := u + \varepsilon\varphi \quad \varphi \in C_0^\infty(\Omega_\varepsilon) \quad |\varphi| \leq 1$ and $\varepsilon < \varepsilon_0$ (φ can be negative)

$\int_{\Omega_\varepsilon} f\varphi \leq \int_{\Omega_\varepsilon} \nabla u \cdot \nabla \varphi \quad \forall \varphi \in C_0^\infty(\Omega_\varepsilon)$, $|\varphi| \leq 1 \quad \Rightarrow \quad -\varphi \in$ - " -



$\Rightarrow \int_{\Omega_\varepsilon} f\varphi = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla \varphi \quad \Rightarrow \quad f = -\Delta u$ in Ω_ε

Formally either $u = 0$ or $u > \varepsilon \quad \Rightarrow \quad -\Delta u = f$

$u \geq 0$, $-\Delta u - f \geq 0$, $u \cdot (-\Delta u - f) = 0$ in Ω

MONOTONE OPERATOR THEORY (2) – IN CASE THE POTENTIAL EXISTS

$$\left.\begin{aligned} -\operatorname{div} A(u,\nabla u) + B(u,\nabla u) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \text{(MO 2)}$$

The aim is : Is there some $F(u,\nabla u)$ for which minimization of $F(u,\nabla u) - fu$ gives a solution to (MO 2) ?

Here A, B are Carathéodory.

~~Lemma~~: Heuristic :

$\varphi(t) := \int_\Omega F(u+tw, \nabla u + t\nabla w) - f(u+tw)$

u minimizer $\Rightarrow \varphi'(0) = 0$

$$\varphi'(0) = \int_\Omega \frac{\partial F}{\partial \xi}(u,\nabla u) \cdot \nabla w + \frac{\partial F}{\partial u}(u,\nabla u) w - fw$$

We need at least $\quad A(u,\xi) = \frac{\partial F}{\partial \xi}(u,\xi)$; $\quad B(u,\xi) = \frac{\partial F}{\partial u}(u,\xi)$

Lemma: Let $A(u,\xi)$ and $B(u,\xi)$ be C^1. Then the following is equivalent:

1. $\exists F$ such that $\frac{\partial F}{\partial \xi}(u,\xi) = A(u,\xi)$; $\frac{\partial F}{\partial u}(u,\xi) = B(u,\xi)$
2. $\forall i,j$ $\frac{\partial A_i}{\partial \xi_j}(u,\xi) = \frac{\partial A_j}{\partial \xi_i}$, $\frac{\partial B}{\partial \xi_i}(u,\xi) = \frac{\partial A_i}{\partial u}(u,\xi)$

Proof:

2. is necessary : if F exists then $\frac{\partial}{\partial \xi_i}\left(\frac{\partial F}{\partial \xi_j}\right) = \frac{\partial}{\partial \xi_j}\left(\frac{\partial F}{\partial \xi_i}\right) \Leftrightarrow \frac{\partial A_i}{\partial \xi_j} = \frac{\partial A_j}{\partial \xi_i}$

$\frac{\partial}{\partial \xi_j}\left(\frac{\partial F}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial F}{\partial \xi_j}\right) \Leftrightarrow \frac{\partial}{\partial \xi_j} B = \frac{\partial}{\partial u} A_j$

2. is sufficient.

~~I define $F(u,\xi) := \int_0^1 A(u,t\xi)\cdot\xi \, dt$~~ , I want to show that $\frac{\partial F}{\partial \xi} = A$, $\frac{\partial F}{\partial u} = B$

~~$\frac{\partial F}{\partial u}(u,\xi) = \int_0^1 \frac{\partial A}{\partial u}(u,t\xi)\cdot\xi \, dt$~~ $= \sum_i \int_0^1 \frac{\partial A_i}{\partial u}(u,t\xi)\cdot\xi_i \, dt$

~~$= \sum_i \int_0^1 \frac{\partial B}{\partial \xi_i}(u,t\xi)\cdot\xi_i \, dt$~~

~~$\frac{d}{dt} B(u,t\xi) = \frac{\partial B}{\partial t\xi}(u,t\xi)\cdot\xi$~~

I define $F(u,\xi) := \int_0^1 A(tu,t\xi)\cdot\xi \, dt + \int_0^1 B(tu,t\xi)\, u \, dt$

$\frac{\partial F}{\partial u}(u,\xi) = \int_0^1 \frac{\partial A}{\partial tu}(tu,t\xi)\cdot\xi t \, dt + \int_0^1 \frac{\partial B}{\partial tu}(tu,t\xi)\, tu \, dt + \int_0^1 B(tu,t\xi)\, dt$

$\left[\frac{d}{dt} B(tu,t\xi) = \frac{\partial}{\partial(tu)} B(tu,t\xi)\cdot u + \frac{\partial}{\partial(t\xi)} B(tu,t\xi)\cdot\xi\right]$ $- \frac{\partial}{\partial(tu)} A(tu,t\xi)\cdot t\xi$

$= \int_0^1 \frac{\partial A}{\partial(tu)}(tu,t\xi)\cdot t\xi + t\frac{d}{dt} B(tu,t\xi) - \frac{\partial}{\partial(t\xi)} B(tu,t\xi)\cdot t\xi \, dt + \int_0^1 B(tu,t\xi)\, dt$

$= \int_0^1 t \frac{d}{dt} B(tu,t\xi)\cdot dt + \int_0^1 B(tu,t\xi)\, dt$

$= B(u,\xi) - \int_0^1 B(tu,t\xi)\, dt + \int_0^1 B(tu,t\xi)\, dt = B(u,\xi)$

Theorem: Let A, B be Carathéodory and $\frac{\partial A_j}{\partial \xi_i} = \frac{\partial A_i}{\partial \xi_j}$ and $\frac{\partial B}{\partial \xi_j} = \frac{\partial A_j}{\partial u}$.

Let $A(u,0)=0$, A be monotone w.r.t. ξ and $|A(u,\xi)| \leq c(1+|\xi|)^{p-1}$, $A(u,\xi)\cdot\xi \geq c_1|\xi|^p - c_2$,

and $|B(u,\xi)| \leq c(1+\xi)^p + c|u|^p$

Then $\forall f \in (W_0^{1,p}(\Omega))^* \; \exists u \in W_0^{1,p}(\Omega)$ such that

$\int_\Omega A(u,Du)\cdot D\varphi + B(u,Du)\varphi = \langle f, \varphi\rangle \quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$

Proof: 1. we know $\exists F$ such that $\frac{\partial F}{\partial u} = B$, $\frac{\partial F}{\partial \xi} = A$

2. A is monotone w.r.t ξ $\Rightarrow$ F is convex w.r.t ξ

3. to check that F is coercive , $F(u,\xi) \geq c_1|\xi|^p - c_2$

assume 3. is true.

$\min_{u \in W_0^{1,2}(\Omega)} \int_\Omega F(u,Du) - \langle f,u\rangle$ F coercive & convex $\Rightarrow \exists$ minimizer

$\uparrow$

$F(0) < \infty \Rightarrow$ infimum $< \infty$

Take $\varphi \in C_0^\infty(\Omega)$, $\varepsilon > 0$

$$\int_\Omega F(u,\nabla u) - \langle f, u\rangle \le \int_\Omega F(u+\varepsilon\varphi, \nabla u + \varepsilon \nabla\varphi) - \langle f, u+\varepsilon\varphi\rangle \qquad /:\varepsilon \ \& \ \varepsilon \to 0+$$

$$\langle f,\varphi\rangle \le \int_\Omega \frac{\partial F}{\partial u}(u,\nabla u)\varphi + \frac{\partial F}{\partial \xi}(u,\nabla u)\cdot\nabla\varphi$$

$$= \int_\Omega A(u,\nabla u)\cdot\nabla\varphi + B(u,\nabla u)\varphi$$

Proof of 3. We want $F(u,\xi) \ge c_1|\xi|^p - c_2$ 24.4.2019

$$F(u,\xi) - F(0,0) = F(u,\xi) - F(u,0) + F(u,0) - F(0,0)$$

$$= \int_0^1 \frac{d}{dt}F(u,t\xi) + \frac{d}{dt}F(tu,0)\,dt$$

$$= \int_0^1 A(u,t\xi)\cdot\xi + \frac{1}{t}B(tu,0)\,tu\,dt \overset{B(u,0)u\ge 0}{\ge} \int_0^1 \underbrace{A(u,t\xi)\cdot\xi}_{\ge 0 \text{ because } A(u,0)=0}\,dt \qquad (= (A(u,\xi)-A(u,0))(\xi-0)\ge 0)$$

$$\ge \int_{1/2}^1 \frac{1}{t} A(u,t\xi)\cdot t\xi\,dt$$

$$\ge \int_{1/2}^1 \frac{1}{t}(c_1|t\xi|^p - c_2)\,dt \ge \tilde{c}_1|\xi|^p - \tilde{c}_2$$

Theorem. Let F, A, B be as in previous and satisfy the same assumption, in addition

$$(A(u_1,\xi_1) - A(u_2,\xi_2))\cdot(\xi_1-\xi_2) + (B(u_1,\xi_1) - B(u_2,\xi_2))\cdot(u_1-u_2) \ge 0 .$$

Then every weak solution is a minimizer.

Proof. 1, F is convex w.r.t. all variables (thanks to ↙).

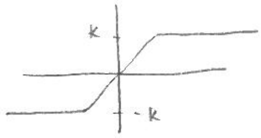
$$\Rightarrow F(u_2,\xi_2) - F(u_1,\xi_1) \ge \frac{\partial F}{\partial \xi}(u_1,\xi_1)\cdot(\xi_2-\xi_1) + \frac{\partial F}{\partial u}(u_1,\xi_1)\cdot(u_2-u_1)$$

$$= A(u_1,\xi_1)\cdot(\xi_2-\xi_1) + B(u_1,\xi_1)(u_2-u_1) \qquad \forall u_1,u_2\in\mathbb{R};\ \xi_1,\xi_2\in\mathbb{R}^d$$

$v \in C_0^\infty(\Omega)$

set $u_2 := v$ $\quad u_1 := T_k(u)$ $\quad$ where $T_k(s) = \operatorname{sign} s \min\{|s|,k\}$

$\xi_2 := \nabla v$ $\quad \xi_1 := \nabla T_k(u)$ $\quad$ u is a weak solution



$$\int_\Omega F(v,\nabla v) - F(T_k(u),\nabla T_k(u)) \ge \int_\Omega A(T_k(u),\nabla T_k(u))\cdot(\nabla v - \nabla T_k(u)) + B(T_k(u),\nabla T_k(u))(v - T_k(u))$$

$$= \int_\Omega A(T_k(u),\nabla T_k(u))\cdot\nabla v + B(T_k(u),\nabla T_k(u))v - \int_\Omega A(T_k(u),\nabla T_k(u))\cdot\nabla T_k(u) + B(T_k(u),\nabla T_k(u))T_k(u)$$

[(W-F) $\int_\Omega A(u,\nabla u)\cdot\nabla v + B(u,\nabla u)v = \int_\Omega fv \quad \forall v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$]

$$\int_\Omega F(v,\nabla v) \ge \liminf_{k\to\infty}\Big[\int_\Omega F(T_k(u),\nabla T_k(u)) + \int_\Omega A(T_k(u),\nabla T_k(u))\cdot\nabla v + B(T_k(u),\nabla T_k(u))v$$

$$- \int_\Omega A(T_k(u),\nabla T_k(u))\cdot\nabla T_k(u) + B(T_k(u),\nabla T_k(u))T_k(u)\Big]$$

Fatou+Němytskii

$$\ge \int_\Omega F(u,\nabla u) + \int_\Omega A(u,\nabla u)\cdot\nabla v + B(u,\nabla u)v$$

$$+ \liminf_{k\to\infty}\Big[-\int_\Omega (A(T_k(u),\nabla T_k(u)) - A(u,\nabla u))\cdot\nabla T_k(u) + (B(T_k(u),\nabla T_k(u)) - B(u,\nabla u))T_k(u)$$

$$- \int_\Omega A(u,\nabla u)\cdot\nabla T_k(u) + B(u,\nabla u)T_k(u)\Big]$$

$$\overset{(W-F)}{=} \int_\Omega F(u,\nabla u) + \int_\Omega fv + \liminf_{k\to\infty}\Big(-\int_\Omega fT_k(u)\Big) + \liminf_{k\to\infty}\Big[-\int_{|u|>k} \quad -''- \quad\Big]$$

$$= \int_\Omega F(u,\nabla u) + \int_\Omega f v - \int_\Omega f u + \liminf_{K\to\infty} \Big[- \int_{|u|>K} \big(B(\operatorname{sign} u \cdot K, 0) - B(u,\nabla u)\big) \operatorname{sign} u \cdot K \Big]$$

$$\underset{\substack{\uparrow \\ \text{growth assumption } + \{|u|>K \to 0\}}}{\geq} \int_\Omega F(u,\nabla u) + \int_\Omega f v - \int_\Omega f u \qquad \Rightarrow \quad u \text{ is a minimizer}$$

MONOTONE OPERATOR (3) - DUAL APPROACH

Recall winter semester: A - elliptic symmetric matrix, we showed that ① <=> ② <=> ③.

① u is a weak solution to $\quad -\operatorname{div} A\nabla u = f \quad$ in Ω

$$u = u_0 \quad \text{on } \partial\Omega$$

② u is a minimizer, $u \in W^{1,2}(\Omega)$, $u = u_0$ on $\partial\Omega$, $\forall v \in W^{1,2}(\Omega)$, $v = u_0$ on $\partial\Omega$

$$\int_\Omega \tfrac{1}{2} A\nabla u \cdot \nabla u - f u \leq \int_\Omega \tfrac{1}{2} A\nabla v \cdot \nabla v - f v$$

③ dual formulation: $\xi \in L^2(\Omega, \mathbb{R}^d)$ is a minimizer to a dual formulation and $A\nabla u = \xi$

(dual form.: $S := \{\xi \in L^2(\Omega, \mathbb{R}^d), \forall v \in W_0^{1,2} : \int_\Omega \xi \cdot \nabla v = \int_\Omega f v\}$, $\min_{\xi \in S} \int_\Omega \frac{1}{2} A^{-1}\xi \cdot \xi - \nabla u_0 \cdot \xi$)

Theorem: Let $F : \mathbb{R}^d \to \mathbb{R}$, $A : \mathbb{R}^d \to \mathbb{R}^d$, $\frac{\partial F}{\partial \xi}(\xi) = A(\xi)$, F be strictly convex (A be strictly monotone) and $A(0) = 0$, $F(0) = 0$.

Let $|F(\xi)| \leq c_2(1 + |\xi|^p)$, $|A(\xi)| \leq c_2(1 + |\xi|^{p-1})$, $F(\xi) \geq c_1|\xi|^p - c_2$, $A(\xi)\cdot\xi \geq c_1|\xi|^p - c_2$.

Let $u_0 \in W^{1,p}(\Omega)$, $g \in L^{p'}(\Gamma_N)$, where $\Gamma_N \subseteq \partial\Omega$ but $|\partial\Omega \setminus \Gamma_N| > 0$, $\partial\Omega \setminus \Gamma_N =: \Gamma_D$, $f \in (W^{1,p}(\Omega))^*$; $p \in (1,\infty)$.

Then the following is equivalent:

① u is a weak solution to $\quad -\operatorname{div} A\nabla u = f$ in Ω, $u = u_0$ on Γ_D, $A\nabla u \cdot n = g$ on Γ_N

that means; $u \in W^{1,p}(\Omega)$, $\operatorname{tr} u = u_0$ on Γ_D, $\forall v \in W^{1,p}(\Omega)$, $\operatorname{tr} v = 0$ on Γ_D, $\int_\Omega A\nabla u \cdot \nabla v = \langle f, v\rangle - \int_{\Gamma_N} g v$

② u is a minimizer to "primar formulation"

$\overset{\text{def}}{\Longleftrightarrow} \quad V := \{v \in W^{1,p}(\Omega), v = u_0 \text{ on } \Gamma_D\}, \quad u \in V, \ \forall v \in V$

$$\int_\Omega F(u) - \langle f, u\rangle - \int_{\Gamma_N} g u \leq \int_\Omega F(v) - \langle f, v\rangle - \int_{\Gamma_N} g v$$

③ $\xi := A(\nabla u)$, ξ is a minimizer to "dual formulation"

$\overset{\text{def}}{\Longleftrightarrow} \quad K := \{\xi \in L^{p'}(\Omega; \mathbb{R}^d), \forall v \in W^{1,p}(\Omega), v = 0 \text{ on } \Gamma_D ; \int_\Omega \xi \cdot \nabla v = \langle f, v\rangle + \int_{\Gamma_N} g v\}$

$$\xi \in K, \forall \tilde{\xi} \in K : \int_\Omega F^*(\xi) - \nabla u_0 \cdot \xi \leq \int_\Omega F^*(\tilde{\xi}) - \nabla u_0 \cdot \tilde{\xi},$$

where $F^* : \mathbb{R}^d \to \mathbb{R}$ is convex conjugate to F $\overset{\text{def}}{\Longleftrightarrow}$ $F^*(\xi) := \sup_{z \in \mathbb{R}^d} (\xi \cdot z - F(z))$

Remark: $F^*(\xi) + F(z) \geq \xi \cdot z$ - Young inequality

Example: $F(t) = \frac{t^p}{p}$, then $F^*(t) = \frac{t^{p'}}{p'}$

($\sup_{z\in\mathbb{R}} (\xi \cdot z - \frac{z^p}{p})$, $\frac{\partial}{\partial z}$, $\xi = z^{p-1}$, $z = \xi^{\frac{1}{p-1}}$, $\sup_{z\in\mathbb{R}} (\xi \cdot z - \frac{z^p}{p}) = \xi \cdot \xi^{\frac{1}{p-1}} - \frac{\xi^{p'}}{p} = \frac{\xi^{p'}}{p'}$)

Application to p-Laplacian

$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = -\Delta_p u = f$ in Ω $\qquad A(\xi) = |\xi|^{p-2}\xi$

$u = u_0$ on $\partial\Omega$ $\qquad F(\xi) = \frac{|\xi|^p}{p}$, $\quad F^*(\xi) = \frac{|\xi|^{p'}}{p'}$

① $u \in W^{1,p}$, $u = u_0$ on $\partial\Omega$, $\forall v \in W_0^{1,p}(\Omega)$ $\quad \int_\Omega |\nabla u|^{p-2} \nabla v = \langle f, v\rangle$

② - " -, $V = \{ v \in W^{1,p}(\Omega),\ v = u_0 \text{ on } \partial\Omega\}$; $\forall v \in V$ $\quad \int_\Omega \frac{|\nabla u|^p}{p} - \langle f, u\rangle \leq \int_\Omega \frac{|\nabla v|^p}{p} - \langle f, v\rangle$

③ $K := \{ \xi \in L^{p'}(\Omega; \mathbb{R}^d),\ \forall v \in W_0^{1,p}(\Omega): \ \int_\Omega \xi . \nabla v = \langle f, v\rangle \}$

$\xi = A\nabla u \Rightarrow \xi \in K$ and $\forall \tilde{\xi} \in K$ $\quad \int_\Omega \frac{\xi^{p'}}{p'} - \nabla u_0 . \xi \leq \int_\Omega \frac{\tilde{\xi}^{p'}}{p'} - \nabla u_0 . \tilde{\xi}$

Proof. ① <=> ②

"=>" u is a weak solution, $u \in V$. Let $v \in V$ be arbitrary.

F is convex $\Rightarrow$ $F(\nabla v) - F(\nabla u) \geq A(\nabla u) . (\nabla v - \nabla u)$

$\int_\Omega F(\nabla v) - F(\nabla u) \geq \int_\Omega A(\nabla u) . \nabla (\underbrace{v-u}_{=0 \text{ on } \Gamma_D}) \overset{W\text{-}F}{=} \langle f, v-u\rangle + \int_{\Gamma_N} g(v-u) \Rightarrow u$ is a minimizer

"<=" u is a minimizer, $v \in W^{1,p}(\Omega)$, $\operatorname{tr} v = 0$ on Γ_D arbitrary. $(u + tv \in V)$

$\varphi(t) := \int_\Omega F(\nabla u + t\nabla v) - \langle f, u+tv\rangle - \int_{\Gamma_N} g(u+tv)$, u is a minimizer $\Rightarrow \varphi$ has minimum in $t=0$

$0 = \varphi'(0) = \int_\Omega \underbrace{A(\nabla u)}_{\frac{\partial F}{\partial \xi}} . \nabla v - \langle f, v\rangle - \int_{\Gamma_N} g v \quad \Leftrightarrow$ weak formulation

③ <=> ② or ①

Properties of F^*: (P1) $F^*(0) = 0$, F^* is strictly convex

(P2) $\frac{\partial F^*(\xi)}{\partial \xi} = A^{-1}(\xi)$, where A^{-1} is inverse to A

(P3) $|F^*(\xi)| \leq c(1 + |\xi|^{p'})$; $\quad F^*(\xi) \geq \tilde{c}_1 |\xi|^{p'} - \tilde{c}_2$

dual formulation has unique! minimizer

$\min_{\xi \in K} \int_\Omega F^*(\xi) - \nabla u_0 . \xi \qquad (K := \{ \xi \in L^{p'}(\Omega; \mathbb{R}^d);\ \forall v \in W^{1,p}(\Omega),\ v = 0 \text{ on } \Gamma_D,\ \int_\Omega \xi . \nabla v = \langle f, v\rangle + \int_{\Gamma_N} g v\})$

$-\operatorname{div} \xi = f$ in Ω, $\xi . n = g$ on Γ_N

F^* convex + (P3) $\Rightarrow \exists$ a minimum

F^* strictly convex $\Rightarrow \exists!$ minimizer

$\xi_1 \neq \xi_2$ - minimizers, $\quad \xi = \frac{\xi_1 + \xi_2}{2} \in K$

$$\int_\Omega F^*(\xi) - \nabla u_0 . \xi \overset{\text{strict convexity}}{<} \int_\Omega \tfrac{1}{2} F^*(\xi_1) + \tfrac{1}{2} F^*(\xi_2) - \tfrac{1}{2}\nabla u_0 . \xi_1 - \tfrac{1}{2}\nabla u_0 . \xi_2$$

$$= \tfrac{1}{2}\left(\int_\Omega F^*(\xi_1) - \nabla u_0 . \xi_1 + \int_\Omega F^*(\xi_2) - \nabla u_0 . \xi_2 \right) = \text{minimum}$$

Define $\xi := A(\nabla u)$ and show that ξ is a minimizer

a) $\underset{\uparrow}{\xi \in K}$? $\quad {}_{\Omega}\int \xi . \nabla v = {}_{\Omega}\int A(\nabla u) . \nabla v = \langle f, v \rangle + {}_{\Gamma_N}\int g v$

yes because u is a weak solution

b) F^* is convex, $\tilde{\xi} \in K$

$$(F^*(\tilde{\xi}) - F^*(\xi)) - (\nabla u_0 . \tilde{\xi} - \nabla u_0 . \xi) \geq \frac{\partial F^*}{\partial \xi}(\hat{\xi} - \xi) - \nabla u_0 . (\hat{\xi} - \xi)$$

$$\overset{(P2)}{=} A^{-1}(\xi) . (\tilde{\xi} - \xi) - \nabla u_0 . (\tilde{\xi} - \xi) \overset{\xi = A(\nabla u)}{=} \nabla u . (\hat{\xi} - \xi) - \nabla u_0 . (\hat{\xi} - \xi) = (\underset{K}{\underset{\in}{\tilde{\xi}}} - \underset{K}{\underset{\in}{\xi}}) . \nabla(\underbrace{u - u_0}_{= 0 \text{ on } \Gamma_D})$$

$${}_{\Omega}\int (F^*(\tilde{\xi}) - \nabla u_0 . \tilde{\xi}) - (F^*(\xi) - \nabla u_0 . \xi) \geq {}_{\Omega}\int (\hat{\xi} - \xi) . \nabla(u - u_0)$$

$$= \langle f, u - u_0 \rangle + {}_{\Gamma_N}\int g(u - u_0) - \langle f, u - u_0 \rangle - {}_{\Gamma_N}\int g(u - u_0) = 0 \qquad \Rightarrow \xi \text{ is a minimizer}$$

Proof of properties of F^*:

Convexity of F^*: $\lambda \in [0,1]$, $\xi_1, \xi_2 \in \mathbb{R}^d$

$$F^*(\lambda \xi_1 + (1-\lambda)\xi_2) \overset{def}{=} \sup_{z \in \mathbb{R}^d} [(\lambda \xi_1 + (1-\lambda)\xi_2) . z - F(z)]$$

$$= \sup_{z \in \mathbb{R}^d} [(\lambda \xi_1 . z - \lambda F(z)) + ((1-\lambda)\xi_2 . z - (1-\lambda)F(z))]$$

$$\leq \lambda \sup_{z \in \mathbb{R}^d} [\xi_1 . z - F(z)] + (1-\lambda) \sup_{z \in \mathbb{R}^d} [\xi_2 . z - F(z)] \overset{def}{=} \lambda F^*(\xi_1) + (1-\lambda) F^*(\xi_2)$$

$F^*(\xi) + F(z) \geq \xi . z$ (from definition of F^*)

$F^*(\xi) = \sup_{z \in \mathbb{R}^d} (\xi . z - F(z))$, $\xi . z - F(z) \to -\infty$ as $|z| \to \infty$ (p-growth of F)

$\Rightarrow$ supremum is attained if $\frac{\partial}{\partial z}(\xi . z - F(z)) = 0 \iff \xi = \frac{\partial F}{\partial z} = A(z) \iff z = A^{-1}(\xi)$

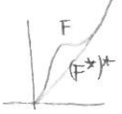
$F^*(\xi) = \xi . A^{-1}(\xi) - F(A^{-1}(\xi))$

$F^*(A(z)) = A(z) . z - F(z)$

Apply to Young inequality: $F^*(\xi) + F(z) - \xi . z \geq 0$, and $= 0 \iff z = A^{-1}(\xi)$

$\Rightarrow \frac{\partial}{\partial \xi}(F^*(\xi) + F(z) - \xi . z) = 0$ if $z = A^{-1}(\xi)$

$\frac{\partial F^*}{\partial \xi}(\xi) = z = A^{-1}(\xi)$



$$\left[\begin{array}{ll} (F^*)^* = F & \text{for F convex and } \frac{F(\xi)}{|\xi|} \to \infty \text{ as } |\xi| \to \infty \\ (F^*)^* \leq F & \text{for F non-convex and } \frac{F(\xi)}{|\xi|} \to \infty \text{ as } |\xi| \to \infty, \text{ then } (F^*)^* \text{ is } \underline{\text{convex envelope}} \end{array}\right.$$

Homework:

$K := \{ \xi \in L^p(\Omega; \mathbb{R}^d), \ {}_{\Omega}\int \xi . \nabla v = 0 \ \forall v \in W_0^{1,p'}(\Omega) \}$

$a_{ij}(x) \in L^\infty(\Omega)$, $a_{ij}(x) z_i z_j \geq c_1 |z|^2$, $\quad \Omega = B_1(0) \subseteq \mathbb{R}^2$

1. Prove that $\exists!$ minimizer to ${}_{\Omega}\int \frac{|\xi|_a^{p'}}{p'} - \xi_1 x_2 \, dx_1 dx_2$, $\quad |\xi|_a^2 := \sum_{i,j=1}^{2} a_{ij}(x) \xi_i \xi_j$

2. Show that it is a dual formulation of some elliptic non-linear PDE with some data

PARABOLIC EQUATIONS (NONLINEAR VERSION)

$$\partial_t u - \operatorname{div} A(u,\nabla u) + B(u,\nabla u) = f \quad \text{in } \Omega \times (0,T)$$

$$u = u_D \quad \text{on } \partial\Omega \times (0,T)$$

$$u(0) = u_0 \quad \text{in } \Omega$$

Assumptions on A and B are the same as in the elliptic setting, i.e.

- A, B are Carathéodory
- $|A(u,\xi)| + |B(u,\xi)| \le c(1 + |u|^{p-1} + |\xi|^{p-1})$ growth
- $A(u,\xi)\cdot\xi + B(u,\xi)\cdot u \ge c_1|\xi|^p - c_2(|u|^q + 1)$ with $q \le \max(2, p-\varepsilon)$ coercivity

Theorem: Let $\Omega \in C^{0,1}$, $f \in L^{p'}(0,T;(W_0^{1,p}(\Omega))^*)$, $u_0 \in L^2(\Omega)$, $u_D = 0$.

Assume that $A(u,\xi)$ is strictly monotone w.r.t. ξ. Then $\exists\, u$,

$u \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$, $\partial_t u \in L^{p'}(0,T;V^*)$ where $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$

such that $\forall v \in V$ and a.a. $t \in (0,T)$ and $u(0) = u_0$,

$$\underbrace{\langle \partial_t u, v\rangle_V}_{} + \int_\Omega A(u,\nabla u)\cdot\nabla v + B(u,\nabla u)v = \langle f, v\rangle.$$

Gelfand triple: $V \overset{\text{dense}}{\hookrightarrow} L^2 \hookrightarrow V^*$

note $u \in L^p(0,T;V)$, $\partial_t u \in L^{p'}(0,T;V^*) \Rightarrow u \in C([0,T];L^2(\Omega))$

Aubin-Lions lemma: Let $V_1 \hookrightarrow\hookrightarrow V_2 \hookrightarrow V_3$ Banach spaces, V_1, V_2 reflexive, $p \in [1,\infty)$.

Then the space $U := \{u \in L^p(0,T;V_1),\ \partial_t u \in L^1(0,T;V_3)\} \hookrightarrow\hookrightarrow L^p(0,T; V_2)$.

Example of application: u^n bounded in $L^2(0,T;W_0^{1,2})$ and $\partial_t u^n$ bounded in $L^1(0,T;(W_0^{100,2})^*)$, then $u^n \to u$ in $L^2(0,T;L^2)$ (for subsequence)

$V_1 = W_0^{1,2}(\Omega) \hookrightarrow\hookrightarrow V_2 = L^2 \hookrightarrow V_3 = (W_0^{100,2})^*$

Homework (up to 10% of exam): Find/create difficult pde HW with nice but tricky proof.

14.5.
2019

Remark: In the Aubin-Lions lemma, $\partial_t u \in M(0,T;V_3)$ is enough.

Proof (of A-L):

Ehrling lemma: Let $V_1 \hookrightarrow\hookrightarrow V_2 \hookrightarrow V_3$. Then $\forall \varepsilon > 0\ \exists C \ge 0\ \forall u \in V_1 : \|u\|_{V_2} \le \varepsilon\|u\|_{V_1} + C(\varepsilon)\|u\|_{V_3}$.

Proof of EL: By contradiction. $\exists \varepsilon > 0\ \forall n \in \mathbb{N}\ \exists u^n \in V_1 : \|u^n\|_{V_2} > \varepsilon\|u^n\|_{V_1} + n\|u^n\|_{V_3}$

$u^n \ne 0$, $v^n = \frac{u^n}{\|u^n\|_{V_2}}$, $1 = \|v^n\|_{V_2} > \varepsilon\|v^n\|_{V_1} + n\|v^n\|_{V_3}$

$\{v^n\}_{n\in\mathbb{N}}$ is bounded in V_1, $v^n \to 0$ in V_3

$V_1 \hookrightarrow\hookrightarrow V_2$: $\exists\, v^{n_k}$: $v^{n_k} \to v$ in V_2 , $1 = \|v^{n_k}\|_{V_2} \to \|v\|_{V_2} \Rightarrow v \neq 0$

$v^n \to 0$ in V_3 $\Rightarrow$ $v = 0$, a contradiction.

Proof of A-L -continuation. Goal:

If $M \subseteq U$ is bounded $\Leftrightarrow \exists\, c^* \; \forall u \in M \; {}_0\int^T \|u\|_{V_1}^p + \|\partial_t u\|_{V_3} \leq c^*$,

then M is precompact in $L^p(0,T;V_2) \Leftrightarrow \forall \varepsilon \; \exists \{w_k\}_{k=1}^N \; \forall u \in M \; \exists\, k = 1,..,N \; {}_0\int^T \|u - w_k\|_{V_2}^p \leq \varepsilon$

How to prove the goal:

1. Mollification w.r.t. t and use of Arzelà-Ascoli
2. Mollification is "close" to origin
3. Combine it with Ehring

1. A-A for Banach valued functions, consequence: $X \hookrightarrow\hookrightarrow Y \Rightarrow C^1(0,T;X) \hookrightarrow\hookrightarrow C(0,T;Y)$

$u \in M$ → extension $\tilde{u}(t) = \begin{cases} u(t) & t \in (0,T) \\ u(2T-t) & t \in (T,2T) \end{cases}$

$${}_0\int^{2T} \|\tilde{u}\|_{V_1}^p + \|\partial_t u\|_{V_3} = 2\, {}_0\int^T \|u\|_{V_1}^p + \|\partial_t u\|_{V_3}$$

$\forall\; 0 < \delta < T$ and $t \in (0,T)$, $u_\delta(t) := {}_0\int^\delta \tilde{u}(t+s)\, \varphi_\delta(s)\, ds$,

where $\varphi \in C_0^\infty(0,1)$, $\varphi \geq 0$ and ${}_0\int^1 \varphi(t)\, dt = 1$ and $\varphi_\delta(t) = \frac{1}{\delta}\varphi(\frac{t}{\delta})$

$$u_\delta(t) = {}_0\int^\delta \tilde{u}(t+s)\, \varphi_\delta(s)\, ds = {}_{\mathbb{R}}\int \tilde{u}(t+s)\, \varphi_\delta(s)\, ds = {}_{\mathbb{R}}\int \tilde{u}(s)\, \varphi_\delta(s-t)\, ds$$

$$\left.\begin{aligned} \|u_\delta(t)\|_{V_1}^p &\leq {}_0\int^{2T} \|\tilde{u}\|_{V_1}^p \frac{c}{\delta} \\ \|\partial_t u_\delta(t)\|_{V_1}^p &\leq {}_{\mathbb{R}}\int \|\tilde{u}(s)\|_{V_3}^p \,|\varphi_\delta'(s-t)|^p \leq {}_0\int^{2T} \|\tilde{u}\|_{V_1}^p \frac{c}{\delta^{2p}} \end{aligned}\right\} \; c(\delta) \qquad c^*$$

$M_\delta := \{u_\delta ; u \in M\}$ $\qquad$ M_δ is bounded in $C^1(0,T;V_1)$

$\forall \tilde{\varepsilon} \; \forall \delta \; \exists N \; \{w_k\}_{k=1}^N \subseteq L^p(0,T;V_1)$ such that $\forall u_\delta \in M_\delta \; \exists K \; {}_0\int^T \|u_\delta - w_K\|_{V_2}^p \leq \tilde{\varepsilon}$ $\qquad (*)$

2. $u(t) - u_\delta(t) \overset{def}{=} u(t) - {}_0\int^\delta \tilde{u}(t+s)\, \varphi_\delta(s)\, ds$ $\qquad {}_0\int^\delta \varphi_\delta(s)\, ds = 1$
$t \in (0,T)$

$$\begin{aligned} &= {}_0\int^\delta (u(t) - \tilde{u}(t+s))\, \varphi_\delta(s)\, ds \\ &= -{}_0\int^\delta (u(t) - \tilde{u}(t+s)) \frac{d}{ds}\Big({}_s\int^\delta \varphi_\delta(\tau)\, d\tau \Big)\, ds \qquad \text{, IBP, no bdary terms!} \\ &= -\, {}_0\int^\delta \partial_s \tilde{u}(t+s) \Big({}_s\int^\delta \varphi_\delta(\tau)\, d\tau \Big)\, ds \qquad \text{Fubini} \\ &= -\, {}_0\int^\delta {}_0\int^\tau \partial_s u(t+s)\, \varphi_\delta(\tau)\, ds\, d\tau \end{aligned}$$

$$\Rightarrow \|u(t) - u_\delta(t)\|_{V_3} \leq \int_0^\delta \int_0^\tau \|\partial_t u(t+s)\|_{V_3}\, \varphi_\delta(\tau)\, ds\, d\tau$$

$$\Rightarrow L^1 \text{ estimate} : {}_0\int^T \|u(t) - u_\delta(t)\|_{V_3} \leq \int_0^T \int_0^\delta \int_0^\tau \|\partial_t u(t+s)\|_{V_3}\, \varphi_\delta(\tau)\, ds\, d\tau\, dt$$

$$\leq {}_0\int^{2T} \|\partial_t u(t)\|_{V_3}\, dt \int_0^\delta \int_0^\delta \varphi_\delta(\tau)\, ds\, d\tau \leq 2\delta c^*$$

L^∞ - estimate: $\|u(t)-u_\delta(t)\|_{V_3} \le \int_0^\delta\int_0^\delta \|\partial_t u(t+s)\|_{V_3} \varphi_\delta(\tau)\, ds\, d\tau$

$$\le \int_0^{2T} \|\partial_t u\|\, dt \int_0^\delta \varphi_\delta(\tau)\, d\tau \le 2c^*$$

$p\in[1,\infty)$: $\int_0^T \|u(t)-u_\delta(t)\|_{V_3}^p\, dt = \int_0^T \|u(t)-u_\delta(t)\|_{V_3}^1 \|u(t)-u_\delta(t)\|_{V_3}^{p-1}$

$$\le \sup_{t\in(0,T)} \|u(t)-u_\delta(t)\|_{V_3}^{p-1} \int_0^T \|u(t)-u_\delta(t)\|_{V_3} \qquad L^\infty \,\&\, L^1$$

$$\le (2C^*)^{p-1}\, 2\delta c^* = (2C^*)^p \delta$$

3. $u\in M$

$$\int_0^T \|u-w_k\|_{V_2}^p \overset{\text{Ehrling}}{\le} \tilde{\tilde{\varepsilon}} \int_0^T \|u-w_k\|_{V_1}^p + C(\tilde{\tilde{\varepsilon}}) \int_0^T \|u-w_k\|_{V_3}^p \qquad \forall \tilde{\tilde{\varepsilon}}>0$$

$$\le 2^p \tilde{\tilde{\varepsilon}} c^* + C(\tilde{\tilde{\varepsilon}}) \int_0^T \|u-w_k\|_{V_3}^p$$

$$\le 2^p \tilde{\tilde{\varepsilon}} c^* + C(\tilde{\tilde{\varepsilon}}) \int_0^T \|u-u_\delta\|_{V_3}^p + C(\tilde{\tilde{\varepsilon}}) \int_0^T \|u_\delta - w_k\|_{V_3}^p$$

$$\le 2^p \tilde{\tilde{\varepsilon}} c^* + C(\tilde{\tilde{\varepsilon}}) (2c^*)^p \delta + C(\tilde{\tilde{\varepsilon}}) \int_0^T \|u_\delta - w_k\|_{V_3}^p$$

you give me $\varepsilon>0$, I choose $\tilde{\tilde{\varepsilon}}>0$ so that $2^p \tilde{\tilde{\varepsilon}} c^* = \frac{\varepsilon}{3}$

then I choose $\delta>0$ so that $C(\tilde{\tilde{\varepsilon}})(2c^*)^p \delta = \frac{\varepsilon}{3}$

finally I choose $\tilde{\varepsilon}>0$ in (*) so $C(\tilde{\tilde{\varepsilon}})\tilde{\varepsilon} \le \frac{\varepsilon}{3}$

Now, we have

$$\partial_t u - \operatorname{div} A(u,\nabla u) + B(u,\nabla u) = f \qquad \text{in } \Omega\times(0,T)$$

$$u = u_D \qquad \text{on } \partial\Omega\times(0,T)$$

$$u(0) = u_0 \qquad \text{in } \Omega$$

A, B are Carathéodory

$|A(u,\xi)| + |B(u,\xi)| \le c\,(1+|u|^{p-1} + |\xi|^{p-1})$ .. growth

$A(u,\xi)\cdot\xi + B(u,\xi)\, u \ge c_1 |\xi|^p - c_2\,(1+|u|^2 + |u|^{p-\varepsilon})$ .. coercivity

1. A and B are monotone as whole operator

2. A is monotone (w.r.t. ξ) and B is linear w.r.t. ξ

3. A is strictly monotone (w.r.t. ξ)

Theorem: Let $\Omega\in C^{0,1}$, A, B satisfy growth + coercivity and let at least one of 1.-3. hold. Then $\forall u_0\in L^2(\Omega)$ $\forall f\in L^{p'}(0,T;(W_0^{1,p})^*)$ $\exists u\in L^p(0,T;W_0^{1,p}(\Omega))\cap C(0,T;L^2(\Omega))$ and $\partial_t u\in L^{p'}(0,T;(W_0^{1,p})^*)$ and for almost all $t\in(0,T)$ and $\forall w\in W_0^{1,p}(\Omega)\cap L^2(\Omega)$

$$\langle \partial_t u, w\rangle + \int_\Omega A(u,\nabla u)\cdot\nabla w + B(u,\nabla u)\, w = \langle f, w\rangle$$

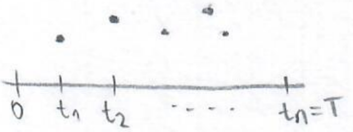
and $u(0)=u_0$.

Note: meaning of $\langle , \rangle$: Gelfand triple $(W_0^{1,p} \cap L^2) \overset{\text{densely}}{\hookrightarrow} L^2(\Omega) \hookrightarrow (\quad)^*$

$\overset{\wedge}{W^{1,p}}$ if $p \geq \frac{2d}{d+2}$ $\quad (\Rightarrow W^{1,p} \hookrightarrow L^2)$

Proof (Rothe method): "$\Leftarrow\Rightarrow$" approximate $\partial_t u$ by $\frac{u(t+\tau)-u(t)}{\tau}$

Choose $n \in \mathbb{N}$, define $\tau = \frac{T}{n}$ and t_k: $t_0 = 0$, $t_{k+1} = t_k + \tau$ Then $t_n = T$.



if I know $u(t_k) =: u_k$, I want to find $u(t_{k+1}) = u_{k+1}$.

1st explicit scheme - will never converge!!! Don't do that!

$$\frac{u_{k+1}-u_k}{\tau} - \operatorname{div} A(u_k, \overset{\swarrow \text{ I'm loosing derivatives}}{\nabla u_k}) + B(u_k, \nabla u_k) = f_k \quad \text{(approximation of } f)$$

2nd implicit scheme - good scheme. Rothe method:

$$\frac{u_{k+1}-u_k}{\tau} - \operatorname{div} A(u_{k+1}, \nabla u_{k+1}) + B(u_{k+1}, \nabla u_{k+1}) = f_{k+1}, \quad u_0 \text{ "="} u_0.$$

1. $\exists$ of $\{u_k\}_{k=1}^{n}$ $\qquad f_k = \fint_{t_{k-1}}^{t_{k+1}} f(t)\,dt$

$u_k \in L^2(\Omega)$ given

find $u_{k+1} \in W_0^{1,p}(\Omega) \cap L^2(\Omega) =: V$

$$\left.\begin{aligned} \forall w \in V \quad & \int_\Omega u_{k+1} w + \tau \int_\Omega A(u_{k+1}, \nabla u_{k+1}) \nabla w + B(u_{k+1}, \nabla u_{k+1}) w \\ & = \tau \langle f_{k+1}, w \rangle + \int_\Omega u_k w \end{aligned}\right\} (W\text{-}F\ (k,n))$$

Easy homework: prove existence of this solution, if $\tau \ll 1$.

2. Uniform estimates (= independent of n)

set $w := u_{k+1}$ in $(W\text{-}F\ (k,n))$

$$\int_\Omega (u_{k+1})^2 - u_k u_{k+1} + \tau \int_\Omega A(u_{k+1}, \nabla u_{k+1}) \cdot \nabla u_{k+1} + B(u_{k+1}, \nabla u_{k+1}) u_{k+1} = \tau \langle f_{k+1}, u_{k+1} \rangle \quad (EI)_k^n$$

$$\int_\Omega (u_{k+1})^2 - u_k u_{k+1} + c_1 \tau \int_\Omega |\nabla u_{k+1}|^p - c\tau \int_\Omega |u_{k+1}|^2 + |u_{k+1}|^{p-\varepsilon} \leq \tau \|f_{k+1}\| \|u_{k+1}\|$$

Poincaré, Young

$$\int_\Omega (u_{k+1})^2 - u_k u_{k+1} + \tilde{c}_1 \tau \|u_{k+1}\|_{1,p}^p \leq \varepsilon \tau \|u_{k+1}\|_{1,p}^p + \tau c(\varepsilon) \|f_{k+1}\|^{p'} + c\tau \|u_{k+1}\|_2^2 + \underline{\varepsilon \tau \|u_{k+1}\|_{1,p}^p}$$

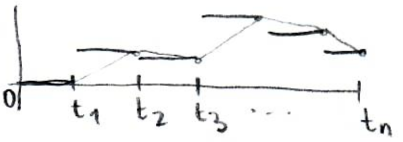
$$\int_\Omega u_{k+1}^2 - u_k u_{k+1} + \tilde{c}_1 \tau \|u_{k+1}\|_{1,p}^p \leq c\tau (\|f_{k+1}\|^{p'} + 1) + c\tau \|u_{k+1}\|_2^2 \quad \underline{+ c(\varepsilon)\tau}$$

$$\sum_{k=0}^{N \leq n-1} \int_\Omega u_{k+1}^2 - u_k u_{k+1} + \tilde{c}_1 \tau \sum_{k=0}^{N} \|u_{k+1}\|_{1,p}^p \leq c\tau \sum_{k=0}^{N} (\|f_{k+1}\|^{p'} + 1) + c\tau \sum_{k=0}^{N} \|u_{k+1}\|_2^2$$

$$\left(u_{k+1}^2 - u_k u_{k+1} = \frac{1}{2}(u_{k+1} - u_k)^2 + \frac{u_{k+1}^2}{2} - \frac{u_k^2}{2} \right)$$

$$\frac{\|u_{N+1}\|_2^2}{2} - \frac{\|u_0\|_2^2}{2} + \sum_{k=0}^{N} \frac{\|u_{k+1} - u_k\|_2^2}{2} + \tilde{c}_1 \tau \sum_{k=0}^{N} \|u_{k+1}\|_{1,p}^p \leq c\tau \left(\sum_{k=0}^{N} \|f_{k+1}\|^{p'} + 1 \right) + c\tau \sum_{k=0}^{N} \|u_{k+1}\|_2^2$$

3. Definition of $u^n, \tilde{u}^n, f^n, A^n, B^n$



$u^n(t) := u_k$ where $t \in (t_{k-1}, t_k]$

$f^n(t) := f_k$ — " —

u^n ... piece-wise constant

$A^n(t) := A(u^n(t), \nabla u^n(t))$

$\tilde{u}^n$ .. piece-wise linear

$B^n(t) := B(u^n(t), \nabla u^n(t))$

$\tilde{u}^n(t) := \frac{1}{\tau}(t - t_{k-1}) u_k + \frac{1}{\tau}(t_k - t) u_{k-1}$ for $t \in (t_{k-1}, t_k]$

for a.a $t \in (0,T)$, $\partial_t \tilde{u}^n(t) = \frac{u_k - u_{k-1}}{\tau}$ $t \in (t_{k-1}, t_k)$

for a.a. $t \in (0,T)$, $\underbrace{\int_\Omega \partial_t \tilde{u}^n(t) w}_{\langle \partial_t \tilde{u}^n(t), w \rangle} + \int_\Omega A^n(t) \cdot \nabla w + B^n(t) w = \langle f^n(t), w \rangle$ $\forall w \in V$

then, $\tau \sum_{k=0}^{N} \|u_{k+1}\|_{1,p}^p = \int_0^{N\tau} \|u^n(t)\|_{1,p}^p \, dt$ energy for k

$$\underbrace{\|u^n(N\tau)\|_2^2}_{g'} + \sum_{k=0}^{N} \frac{\|u_{k+1} - u_k\|_2^2}{2} + c_1 \int_0^{N\tau} \|u^n\|_{1,p}^p \le c \int_0^{N\tau} \|f^n\|^{p'} + 1 + \underbrace{\int_0^{N\tau} \|u^n\|_2^2}_{g},$$

Gronwall lemma:

$$\|u^n(t)\|_2^2 \le c\left(\|u_0\|_2^2 + \int_0^T \|f^n\|^{p'} + 1\right) \le c\left(\|u_0\|_2^2 + \int_0^T \|f\|^{p'} + 1\right) \le c(data)$$

$$\Rightarrow \sup_{t \in (0,T)} \|u^n(t)\|_2^2 + \int_0 \|u^n(t)\|_{1,p}^p \le c(data)$$

growth $\Rightarrow$ $\int_0^T \|A^n\|_{p'}^{p'} + \|B^n\|_{p'}^{p'} \le c(data)$

$$\int_0^T \|\partial_t \tilde{u}^n(t)\|_{V^*}^{p'} = \int_0^T \Big(\sup_{w \in V, \|w\|_V \le 1} \langle \partial_t \tilde{u}^n(t), w \rangle \Big)^{p'} \le \int_0^T \Big(\sup_{\|w\| \le 1} \int_\Omega |A^n||\nabla w| + |B^n||w| + \|f^n\| \|w\| \Big)^{p'}$$

$$\le \int_0^T c\left(\|A^n\|_{p'}^{p'} + \|B^n\|_{p'}^{p'} + \|f^n\|^{p'}\right) \le c(data)$$

4. Existence of the limits

$\langle \partial_t \tilde{u}^n, w \rangle + \int_\Omega A^n \cdot \nabla w + B^n w = \langle f^n, w \rangle$ $\forall w \in V$ and a.a. $t \in (0,T)$

$\sup_{t \in (0,T)} \|u^n(t)\|_2 + \int_0 \int_\Omega |A^n|^{p'} + |B^n|^{p'} + |\nabla u^n|^{p'} + |u^n|^{p'} \le c(data)$

$\int_0^T \|\partial_t \tilde{u}^n\|_{V^*}^{p'} \le c(data)$ $\tilde{u}(0) = u_0$.

$\int_\Omega u_{k+1}^2 - u_k u_{k+1} + \tau \int_\Omega A^n(t) \cdot \nabla u^n + B^n \cdot u^n = \tau \langle f^n, u^n \rangle$ $t \in (t_k, t_{k+1})$

$u^n \rightharpoonup u$ in $L^p(0,T; W_0^{1,p}(\Omega))$ $f^n \to f$ in $L^{p'}(0,T; (W_0^{1,p}(\Omega))^*)$

$u^n \rightharpoonup^* u$ in $L^\infty(0,T; L^2(\Omega))$ $\tilde{u}^n \rightharpoonup \tilde{u}$ in $L^p(0,T; W_0^{1,p}(\Omega))$

$A^n \rightharpoonup \bar{A}$ in $L^{p'}(0,T; L^{p'}(\Omega, \mathbb{R}^d))$ $\tilde{u}^n \rightharpoonup^* \tilde{u}$ in $L^\infty(0,T; L^2(\Omega))$

$B^n \rightharpoonup \bar{B}$ in $L^{p'}(0,T; L^{p'}(\Omega))$ $\partial_t \tilde{u}^n \rightharpoonup \partial_t \tilde{u}$ in $L^{p'}(0,T; W_0^{1,p}(\Omega))$

$\varphi \in C_0^\infty(0,T)$

$$\int_0^T \langle \partial_t \tilde{u}^n, \underbrace{\varphi w}_{\in L^p(0,T; W_0^{1,p}(\Omega))} \rangle + \int_0^T \int_\Omega A^n \cdot \nabla w \varphi + B^n w \varphi = \int_0^T \langle f^n, w\varphi \rangle$$

$$\downarrow$$

$\int_0^T \langle \partial_t \tilde{u}, \varphi w \rangle + \int_0 A \cdot \nabla w \varphi + B w \varphi = \int_0^T \langle f, w\varphi \rangle$ & φ arbitrary for a.a. t,

$$\langle \partial_t \tilde{u}, w\rangle + \int_\Omega A \cdot \nabla w + B \cdot w = \langle f, w\rangle$$

5. Identification of $A, B, \tilde{u}$

1. $\tilde{u} = u$

Recall $\sum_{k=0}^{n-1} \int_\Omega (u_{k+1} - u_k)^2 \leq c(data)$, $\quad u^n(t) = u_k$, $\tilde{u}^n(t) = \frac{t_k - t}{\tau} u_{k-1} + \frac{t - t_{k-1}}{\tau} u_k$

$$u^n(t) - \tilde{u}^n(t) = u_k\left(1 - \frac{t-t_{k-1}}{\tau}\right) - \frac{t_k - t}{\tau} u_{k-1} \qquad \text{for } t \in (t_{k-1}, t_k)$$

$$= (u_k - u_{k-1})\left(1 - \frac{t-t_{k-1}}{\tau}\right) + u_{k-1}\left(1 - \frac{t-t_{k-1}}{\tau} - \frac{t_k - t}{\tau}\right)$$

$$= (u_k - u_{k-1})\left(1 - \frac{t-t_{k-1}}{\tau}\right) + u_{k-1}\underbrace{(\tau - t + t_{k-1} - t_k + t)}_{=0} \cdot \frac{1}{\tau}$$

$$= (u_k - u_{k-1})\left(1 - \frac{t - t_{k-1}}{\tau}\right)$$

$$\| u^n(t) - \tilde{u}^n(t) \|_2^2 \leq \| u_k - u_{k-1} \|_2^2 \left(1 - \frac{t-t_{k-1}}{\tau}\right)^2 \qquad t \in (t_{k-1}, t_k)$$

$$\int_0^T \| u^n(t) - \tilde{u}^n(t)\|_2^2 = \sum_k \int_{t_{k-1}}^{t_k} \| u^n(t) - \tilde{u}^n(t) \|_2^2 \leq \sum_k \int_{t_{k-1}}^{t_k} \| u_k - u_{k-1} \|_2^2 \left(1 - \frac{t-t_{k-1}}{\tau}\right)^2$$

$$\leq \sum_k \int_{t_{k-1}}^{t_k} \| u_k - u_{k-1} \|_2^2 = \tau \sum_k \| u_k - u_{k-1} \|_2^2 \leq \tau\, c(data) = \frac{T}{n} c(data)$$

$\Rightarrow u = \tilde{u}$.

2. $u^n \to u$ in $L^1(0,T; L^1(\Omega))$

Apply Aubin-Lions to $\tilde{u}^n$ $\quad V_1 = W_0^{1,p}$, $V_2 = L^1$, $V_3 = \left(W_0^{d,p}(\Omega)\right)^*$

$\tilde{u}^n \to \tilde{u} = u$ in $L^p(0,T; L^1(\Omega))$

Then

$$\int_0^T \| u^n - u \|_1 \leq \int_0^T \| u^n - \tilde{u}^n \|_1 + \int_0^T \| \tilde{u}^n - \tilde{u} \|_1 \leq c \int_0^T \| u^n - \tilde{u}^n \|_2 + \int_0^T \| \tilde{u}^n - \tilde{u} \|_1 \xrightarrow{\text{Step 1}} 0$$

3. Show that $\limsup_{n\to\infty} \int_0^T \int_\Omega A^n \nabla u^n + B^n u^n \leq \int_0^T \bar{A} \cdot \nabla u + \bar{B} \cdot u$

$$\int_0^T \int_\Omega A^n \cdot \nabla u^n + B^n \cdot u^n = \int_0^T \langle f^n, u^n \rangle - \int_0^T \int_\Omega \partial_t \tilde{u}^n u^n$$

$$\int_0^T \int_\Omega \partial_t \tilde{u}^n u^n = \sum^{n\tau} \int_\Omega (u_{k+1}^2 - u_{k+1} u_k) = \sum \int_\Omega \frac{1}{2}\left(u_{k+1}^2 - u_k^2 + (u_{k+1} - u_k)^2\right)$$

$$= \frac{\| \tilde{u}^n(T) \|_2^2}{2} - \frac{\| u_0 \|_2^2}{2} + \sum \int_\Omega \frac{(u_{k+1} - u_k)^2}{2} \geq \frac{\| \tilde{u}^n(T) \|_2^2}{2} - \frac{\| u_0 \|_2^2}{2}$$

$$\limsup \int_0^T \int_\Omega A^n \cdot \nabla u^n + B^n u^n \leq \limsup \left(- \frac{\| \tilde{u}^n(T) \|_2^2}{2} + \frac{\| u_0 \|_2^2}{2} + \int_0^T \langle \underset{\downarrow}{f^n}, \underset{\downarrow}{u^n} \rangle \right)$$

$$= \int_0^T \langle f, u \rangle + \frac{\| u_0 \|_2^2}{2} - \liminf \frac{\| \tilde{u}^n(T) \|_2^2}{2} \leq \int_0^T \langle f, u \rangle + \frac{\| u_0 \|_2^2}{2} - \frac{\| u(T) \|_2^2}{2}$$

Why $\liminf \| \tilde{u}^n(T) \|_2^2 \geq \| u(T) \|_2^2$???

if $\tilde{u}^n(T) \to u(T)$ in $L^2(\Omega)$ it is clear from the LSC of the norm

we know $\| \tilde{u}^n(T) \|_2 \leq c(data)$, $\Rightarrow$ $\tilde{u}^n(T) \to M$ in $L^2(\Omega)$

$\tilde{u}^n(T) = u_0 + \int_0^T \partial_t \tilde{u}^n$, $w \in V$

$$\int_\Omega \tilde{u}^n(T)\, w = \int_\Omega u_0 w + \int_0^T \langle \partial_t \tilde{u}^n, w\rangle$$

$$\begin{array}{c}\downarrow\end{array}$$

$$\int_\Omega M\, w = \int_\Omega u_0 w + \int_0^T \langle \partial_t \tilde{u}, w\rangle = \int_\Omega u_0 w + \overbrace{\int_0^T \langle \partial_t u, w\rangle}^{\int_\Omega u(T) w - \int_\Omega u_0 w} = \int_\Omega u(T)\, w$$

$$\Rightarrow M = u(T)$$

Test the limit equation by u!

$$\int_0^T \langle \partial_t u, u\rangle + \int_0^T\int_\Omega \bar{A}\cdot\nabla u + \bar{B}u = \int_0^T \langle f, u\rangle$$

$$\int_0^T\int_\Omega \bar{A}\cdot\nabla u + \int_0^T\int_\Omega \bar{B}u = \int_0^T \langle f, u\rangle + \frac{\|u_0\|_2^2}{2} - \frac{\|u(T)\|_2^2}{2} \geq \limsup \int_0^T\int_\Omega A^n\cdot\nabla u^n + B^n u^n$$

Monotonicity

Assume 1., ie $\forall u_1, u_2, \xi_1, \xi_2 : \ (A(u_1,\xi_1) - A(u_2,\xi_2))\cdot(\xi_1-\xi_2) + (B(u_1,\xi_1) - B(u_2,\xi_2))\cdot(u_1-u_2) \geq 0$

$w \in L^p(0,T; L^p(\Omega))$, $\xi \in L^p(0,T; L^p(\Omega;\mathbb{R}^d))$

$$0 \leq \int_0^T\int_\Omega (A(u^n,\nabla u^n) - A(w,\xi))\cdot(\nabla u^n - \xi) + (B(u^n,\nabla u^n) - B(w,\xi))\cdot(u^n - w)$$

$n \to \infty$, weak limit + limsup $\leq$

$$\leq \int_0^T\int_\Omega \bar{A}\cdot\nabla u + \bar{B}u - \bar{A}\xi - A(w,\xi)\cdot(\nabla u - \xi) - \bar{B}w - B(w,\xi)\cdot(u-w)$$

$$= \int_0^T\int_\Omega (\bar{A} - A(w,\xi))\cdot(\nabla u - \xi) + (\bar{B} - B(w,\xi))\cdot(u-w)$$

Set $w = u - \varepsilon v$, $\xi = \nabla u - \varepsilon\bar{\xi}$, $\varepsilon \to 0$

$$0 \leq \int_0^T\int_\Omega (\bar{A} - A(u-\varepsilon v, \nabla u - \varepsilon\bar{\xi}))\cdot\bar{\xi} + (\bar{B} - B(u-\varepsilon v, \nabla u - \varepsilon\bar{\xi}))\cdot v \quad , \ \varepsilon \to 0+$$

$$0 \leq \int_0^T\int_\Omega (\bar{A} - A(u,\nabla u))\cdot\bar{\xi} + (\bar{B} - B(u,\nabla u))\cdot v \qquad \forall v \in L^p(0,T; L^p(\Omega)), \ \bar{\xi} \in L^p(0,T; L^p(\Omega;\mathbb{R}^d))$$

$$\Rightarrow \quad \bar{A} = A(u,\nabla u) \ , \ \bar{B} = B(u,\nabla u) \quad \text{a.e. in } (0,T)\times\Omega$$

Maximum principle for parabolic equation

Problem:

$$\partial_t u - \Delta u = f \geq 0 \quad \text{in } (0,T)\times\Omega$$

$$u = 1 \quad \text{on } (0,T)\times\partial\Omega$$

$$u(0) = u_0 \quad \text{in } \Omega$$

$$u \geq \operatorname{ess\,min}_x (1, u_0(x)) =: m$$

test by $(u-m)_- := \min(0, u-m) \in L^2(0,T; W_0^{1,2}(\Omega))$:

$$\int_0^T \langle \partial_t u, (u-m)_-\rangle\, dt + \underbrace{\int_0^T\int_\Omega \nabla u\cdot\nabla(u-m)_-}_{=\int_0^T\int_\Omega |\nabla(u-m)_-|^2} = \int_0^T\int_\Omega \underbrace{f}_{\geq 0}\ \underbrace{(u-m)_-}_{\leq 0} \leq 0$$

$$\int_0^T \langle \partial_t u, (u-m)_- \rangle \overset{\partial_t u \in L^2}{=} \int_0^T \int_\Omega \partial_t u \, (u-m)_- = \frac{1}{2} \int_0^T \int_\Omega \partial_t ((u-m)_-)^2$$

$$= \frac{1}{2} \int_\Omega (u(T)-m)_-^2 - \frac{1}{2} \int_\Omega (u(0)-m)_-^2 \qquad (u(0)-m)_- = 0 \text{ by definition}$$

$$= \frac{1}{2} \int_\Omega (u(T)-m)_-^2 \geq 0$$

$$\Rightarrow \int_\Omega |\nabla (u-m)_-|^2 = 0$$

$$\Rightarrow (u-m)_- \equiv 0 \qquad \Rightarrow \qquad u \geq m$$

Now, in general:

$$\partial_t u - \operatorname{div} A(u, \nabla u) = f \quad \text{in } (0,T) \times \Omega$$

$$u = u_D \quad \text{on } (0,T) \times \partial\Omega$$

$$u(0) = u_0 \quad \text{in } \Omega$$

$$|A(u,\xi)| \leq c(1+|\xi|)^{p-1}, \quad A(u,\xi) . \xi \geq 0$$

Let $u \in L^p(0,T; W^{1,p}(\Omega))$, $\partial_t u \in L^{p'}(0,T; V^*)$, $u \in L^\infty(0,T; L^2(\Omega))$ be a solution, where $V = W_0^{1,p} \cap L^2$, $f \geq 0$.

Set $m := \min \{ \operatorname{ess\,inf}_{(t,x) \in (0,T) \times \partial\Omega} u_D(t,x), \operatorname{ess\,inf}_{x \in \Omega} u_0(x) \}$, then $u \geq m$ a.e. in $(0,T) \times \Omega$

Proof: WF: $\langle \partial_t u, w \rangle + \int_\Omega A(u, \nabla u) . \nabla w = \int_\Omega f w \quad \forall w \in V.$

Set $w := (u(t)-m)_- := \min(0, u(t)-m)$.

Since $u \geq u_D \geq m$ on $(0,T)$ and $u(t) \in L^2(\Omega) \cap W^{1,p}(\Omega) \Rightarrow w \in V.$

$$\Rightarrow \langle \partial_t u, (u-m)_- \rangle + \int_\Omega \underbrace{A(u,\nabla u) . \nabla (u-m)_-}_{= A(u,\nabla u) . \nabla u \, \chi_{\{u \leq m\}} \geq 0} = \int_\Omega \underset{\geq 0}{f} \underbrace{(u-m)_-}_{\leq 0} \leq 0$$

$$\Rightarrow \langle \partial_t u, (u-m)_- \rangle \leq 0 \quad \Rightarrow \quad \int_0^t \langle \partial_t u, (u-m)_- \rangle \leq 0 \quad (*)$$

Lemma: Let $g \in C^{0,1}$, $g(m) = 0$, $G' := g$, then $\int_0^t \langle \partial_t u, g(u) \rangle = \int_\Omega G(u(t)) - G(u(0))$.

In $(*)$, use lemma with $G(s) := \frac{(s-m)_-^2}{2}$:

$$\int_\Omega (u(t)-m)_-^2 \leq \int_\Omega (u_0-m)_-^2 = 0 \qquad \Rightarrow \qquad u \geq m \quad \text{a.e.}$$

Proof of lemma: $\quad u^\varepsilon := \fint_t^{t+\varepsilon} u$

$$\partial_t u^\varepsilon \to \partial_t u \quad \text{in } L^{p'}(0,T; V^*)$$

$$u^\varepsilon \to u \quad \text{in } L^p(0,T; W^{1,p}(\Omega)) \cap C(0,T; L^2(\Omega))$$

$$\int_0^t \langle \partial_t u, g(u) \rangle = \lim_{\varepsilon \to 0} \int_0^t \langle \partial_t u^\varepsilon, g(u^\varepsilon) \rangle = \lim_{\varepsilon \to 0} \int_0^t \int_\Omega \partial_t u^\varepsilon g(u^\varepsilon)$$

$$= \lim_{\varepsilon \to 0} \int_0^t \frac{d}{dt} \int_\Omega G(u^\varepsilon) = \lim_{\varepsilon \to 0} \int_\Omega G(u^\varepsilon(t)) - G(u^\varepsilon(0)) = \int_\Omega G(u(t)) - G(u(0)).$$

SEMIGROUP THEORY

15.5.2019

Introduction. Exponential function, linear operator.

For $a \in \mathbb{R}$, e^a has several definitions:

$$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!} = \lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n$$

Important property: $f(x) := e^x$, $f(x+y) = f(x) f(y)$ & continuity (e)

For $A \in \mathbb{R}^{d \times d}$ a matrix or for A a bounded linear operator we can define

$e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}$, thanks to the fact that the series is convergent

$$\|e^A\| \le \sum_{k=0}^{\infty} \frac{\|A^k\|}{k!} \le \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!} = e^{\|A\|} < \infty .$$

Our goal is to study equations of type $$\left.\begin{cases} u'(t) = Au(t) & t \ge 0 \\ u(0) = u_0 \end{cases}\right\} (1)$$

where $u_0 \in X$ - a real Banach space and $A : D(A) \subset X \to X$ is a linear (possibly unbounded) operator, which is independent of t, $D(A)$ is linear subspace of X.

We study existence and uniqueness of a solution $u : [0, \infty) \to X$.

For bounded operator, $u(t) = e^{tA} u_0$. What if A is unbounded?

Reminder:

Definition: A linear operator $L : X \to \tilde{X}$ is bounded whenever

$\exists M \ge 0 \ \forall x \in X : \|Lx\|_{\tilde{X}} \le M \|x\|_X$.

Lemma: Let L be linear operator, then the following is equivalent:

1. L is bounded
2. L is continuous
3. L is continuous at 0.

Examples. 1. $\Delta : W^{2,2} \to L^2$ is bounded, $$\|\Delta u\|_2 \le \left(\sum_{i=1}^{d} \left\|\frac{\partial^2 u}{\partial x_i^2}\right\|_2^2\right)^{1/2} \le \overset{\|u\|_{2,2}}{\overset{\|}{\left(\sum_{\alpha, |\alpha| \le k} \|D^\alpha u\|_2^2\right)^{1/2}}}$$

2. $\Delta : W^{2,2} \to W^{2,2}$ is unbounded.

Contradiction. If bounded, $\exists M \ge 0$ s.t. $\|\Delta u\|_{2,2} \le M \|u\|_{2,2} \ \forall u \in W^{2,2}$

For sure $\exists u \in W^{2,2}$ s.t. $\Delta u \notin W^{2,2}$.

Note: In (1), operator A acts from X to X, i.e. for $A = \Delta$, the Banach space X must include smooth functions for A to be bounded.

Instead: we study (1) for unbounded operators.

Semigroup.

Notation: $\mathcal{L}(X) := \{ L: X \to X \text{ is linear bounded operator} \}$ is a Banach space

$$\|L\|_{\mathcal{L}(X)} = \sup_{\substack{x\in X \\ \|x\|_X = 1}} \|Lx\|_X$$

Unbounded operator is the couple $(A, D(A))$, where the domain of A, $D(A) \subset\subset X$ is a subspace, $A: D(A) \to X$ is linear.

Definition: The function $S(t): [0,\infty) \to \mathcal{L}(X)$ is called a semigroup, iff

1. $S(0)$ is identity, $S(0)x = x \quad \forall x \in X$
2. $S(t)S(s) = S(t+s) \quad \forall t,s \geq 0$

If moreover 3. $S(t)x \to x$ as $t \to 0+$ $\forall x \in X$, we call $S(t)$ a C_0-semigroup.

Remarks: Due to property (e) of exponential, a C_0-semigroup is a suitable candidate for generalized exponential via relation "$S(t) = e^{ta}$".

Stronger assumption 3'. $\|S(t) - I\|_{\mathcal{L}(X)} \to 0$ as $t \to 0+$ (uniform continuity) implies $S(t) = e^{tA}$ for some linear continuous operator A.

Lemma 1. Let $S(t)$ be a C_0-semigroup in X. Then

1. $\exists M \geq 1, \omega \geq 0$ s.t. $\|S(t)\|_{\mathcal{L}(X)} \leq M \cdot e^{\omega t} \quad \forall t \geq 0$.
2. $t \mapsto S(t)x$ is continuous mapping from $[0,\infty)$ to X $\forall x \in X$ fixed.

Proof: 1. we claim that $\exists M \geq 1\ \exists \delta > 0$ s.t. $\|S(t)\|_{\mathcal{L}(X)} \leq M \quad \forall t \in [0,\delta]$.

If not, then $\exists \{t_n\}$, $t_n \to 0+$ st. $\|S(t_n)\|_{\mathcal{L}(X)} \to \infty$.

But due to 3. from definition of the semigroup; $S(t_n)x \to x \ \forall x \in X \Rightarrow \|S(t_n)x\|_X$ is bdd.

FA: $\|S(t_n)\|_{\mathcal{L}(X)}$ is bounded $\forall n \in \mathbb{N}$ (Uniform boundedness principle) $\forall x \in X\ \forall n \in \mathbb{N}$

Set $\omega = \frac{1}{\delta} \ln M$ (ie $M = e^{\omega\delta}$), for every $t \geq 0$, $\exists n \in \mathbb{N}_0\ \exists \varepsilon \in [0,\delta)$: $t = n\delta + \varepsilon$.

$$\|S(t)\|_{\mathcal{L}(X)} = \|\underbrace{S(\delta) \dots S(\delta)}_{n\text{-times}} S(\varepsilon)\|_{\mathcal{L}(X)} \leq \|S(\delta)\|^n_{\mathcal{L}(X)} \|S(\varepsilon)\|_{\mathcal{L}(X)} \leq M^n \cdot M = e^{\omega n \delta} \cdot M \leq M \cdot e^{\omega t}.$$

2. Continuity, in $0+$ we have from 3. of def. Let $t > 0$.

Continuity from the right: $\lim_{h\to 0+} S(t+h)x = \lim_{h\to 0+} S(t)S(h)x \overset{3. + \text{linearity}}{=} S(t)x$

From the left (WLOG $h < t$):

$$\lim_{h\to 0+} \|S(t-h)x - S(t)x\|_X \overset{2.\text{def}}{=} \lim_{h\to 0+} \|S(t-h)(x - S(h)x)\|_X \leq \lim_{h\to 0+} \|S(t-h)\|_{\mathcal{L}(X)} \|x - S(h)x\|_X = 0.$$

$\|S(t-h)\|_{\mathcal{L}(X)} \overset{1.}{\leq} Me^{\omega(t-h)} \to Me^{\omega t}$ as $h \to 0+$

$\|x - S(h)x\|_X \to 0$ due to 3.

Definition (Generator of a semigroup). An unbounded operator $(A, D(A))$ is called a generator of a semigroup $S(t)$ iff

$$Ax = \lim_{h\to 0+} \frac{S(h)x - x}{h}$$

$$D(A) = \{ x \in X : \lim_{h\to 0+} \frac{S(h)x-x}{h} \text{ exists in } X \}$$

Remark. A is linear ($S(t) \in \mathcal{L}(X)$) and $D(A) \subset X$ is linear subspace.

Theorem 1 (properties of generator): Let $(A, D(A))$ be a generator of $S(t)$ a C_0-semigroup in X. Then:

1. $x \in D(A) \Rightarrow S(t)x \in D(A) \quad \forall t \geq 0$
2. $x \in D(A) \Rightarrow AS(t)x = S(t)Ax = \frac{d}{dt} S(t)x \quad \forall t \geq 0 \quad (\text{in } t = 0+)$
3. $x \in X, t \geq 0 \Rightarrow x_t := \int_0^t S(s)x\, ds \in D(A), \quad A(x_t) = S(t)x - x.$

Proof. 1.&2. $x \in D(A)$, $t \geq 0$ given.

$$\lim_{s\to 0+} \frac{S(s)S(t)x - S(t)x}{s} \overset{2.}{=} \lim_{s\to 0+} \frac{S(t)S(s)x - S(t)x}{s} \overset{S(t)\in\mathcal{L}(X)}{=} S(t) \lim_{s\to 0+} \frac{S(s)x - x}{s} = S(t)Ax$$

$$\Rightarrow AS(t)x = S(t)Ax$$

$$\Rightarrow \frac{d}{dt} S(t)x = S(t)Ax \quad \forall t \geq 0 \text{ from the right}$$

For $t > 0$ from the left:

$$\lim_{h\to 0+} \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax = \lim_{h\to 0+} \underset{\substack{\downarrow \\ \text{bounded}}}{S(t-h)} \Bigg(\underset{\substack{\downarrow \text{def} \\ Ax}}{\frac{x - S(h)x}{-h}} - \underset{\substack{\downarrow 3. \\ Ax}}{S(h)Ax} \Bigg) = 0$$

3.
$$S(h)x^t - x^t = S(h)\int_0^t S(s)x\,ds - \int_0^t S(s)x\,ds \qquad S(h) \in \mathcal{L}(X) + 2.$$
$$= \int_0^t S(s+h)x\,ds - \int_0^t S(s)x\,ds$$
$$= \int_h^{t+h} S(s)x\,ds - \int_0^t S(s)x\,ds$$
$$= \int_t^{t+h} S(s)x\,ds - \int_0^h S(s)x\,ds \quad , \quad L1,2. \ (t \mapsto S(t)x \text{ cont.})$$

$$\lim_{h\to 0+} \frac{1}{h}(S(h)x^t - x^t) = \lim_{h\to 0+} \left(\frac{1}{h}\int_t^{t+h} S(s)x\,ds - \frac{1}{h}\int_0^h S(s)x\,ds \right) = S(t)x - S(0)x \overset{1.}{=} S(t)x - x$$

Remark: $u : t \mapsto u(t) = S(t)u_0$ is a classical solution to $\frac{d}{dt}u(t) = Au(t)$, $u(0) = u_0$, if $u \in D(A)$.

$$\frac{d}{dt}u(t) = \frac{d}{dt}S(t)u_0 \overset{T1,2.}{=} S(t)Au_0 = AS(t)u_0 = Au(t).$$

Definition (closed operator). We say that an unbounded operator $(A, D(A))$ is closed iff $u_n \in D(A)$, $u_n \to u$, $Au_n \to v \Rightarrow u \in D(A)$ and $Au = v$.

Remark: unboundedness & closedness is natural for derivative operators, e.g. $A: u(t) \mapsto \frac{d}{dt}u(t)$, $D(A) = W^{1,1}(I,X)$. $u_n(t) \in W^{1,1}(I,X)$, $u_n(t) \to u(t)$ in $L^1(I,X)$, $\frac{d}{dt}u_n(t) \to g(t)$ in $L^1(I,X) \Rightarrow u(t) \in W^{1,1}(I,X)$ & $\frac{d}{dt}u(t) = g(t)$.

Theorem 2 (density and closedness of a generator): Let $(A, D(A))$ be a generator of a C_0-semigroup $S(t)$ in X. Then $D(A)$ is dense in X and $(A, D(A))$ is closed.

Proof. Density. For any $x \in X$ and $t \geq 0$, $x^t \in D(A)$ (T1, 3.) and

$$x = \lim_{t \to 0+} \frac{x^t}{t} \left(= \lim_{t \to 0+} \frac{1}{t} \int_0^t S(s)x \, ds = S(0)x = x \right)$$

Closedness. Let $x_n \in D(A)$, $x_n \to x$, $Ax_n \to y$.

We want to show that $x \in D(A)$, $Ax = y$.

Claim: $s \mapsto S(s)x$ is C^1. Indeed, $\frac{d}{dt} S(t)x = S(t)Ax$ (and RHS is continuous in t).

Newton-Leibniz: $S(h)x_n - S(0)x_n = \int_0^h \frac{d}{ds} S(s) x_n \, ds = \int_0^h S(s) A x_n \, ds$, $n \to \infty$

$\frac{1}{h}(S(h)x - x) = \frac{1}{h} \int_0^h S(s) y \, ds$, $h \to 0+$

$Ax = S(0)y = y$.

Remark: by closedness of A + T1, 3. : (simple if A continuous)

$$A\left(\int_0^t S(s)x \, ds \right) = S(t)x - x = \int_0^t S(s) A x \, ds = \int_0^t A S(s) x \, ds$$

By Theorem 1, $(A, D(A))$ generator of $S(t)$ ⇒ $\forall u_0 \in D(A)$, $u(t) = S(t)u_0$ a solution to (1).

For $(A, D(A))$ given, can we find a C_0-semigroup $S(t)$ s.t. A generates $S(t)$?

Lemma 2 (uniqueness of a semigroup): Let $S(t)$, $\tilde{S}(t)$ be semigroups with the same generator. Then $S(t) = \tilde{S}(t)$ $\forall t > 0$.

Proof. Define $y(t) = S(T-t)\tilde{S}(t)x$, $x \in D(A)$ and $T > 0$ fixed.

$y \in C([0,T]; X)$ by continuity of $S, \tilde{S}$ & $T - t \geq 0$

$y'(t) = \frac{d}{dt}\left(S(T-t)\tilde{S}(t)x\right) = -S(T-t)A\tilde{S}(t)x + S(T-t)A\tilde{S}(t)x = 0 \quad \forall t \in (0,T)$

$\Rightarrow \forall t \in [0,T]$, $y(t) = y(0) = S(T)x = y(T) = \tilde{S}(T)x \quad \forall x \in D(A)$

$D(A)$ is dense in X (T2) and $T > 0$ arbitrary ⇒ $S(t) = \tilde{S}(t)$ $\forall t > 0$.

Definition (resolvent): Let $(A, D(A))$ be unbounded operator. We define

resolvent set $\rho(A) = \{\lambda \in \mathbb{R};\ \lambda I - A : D(A) \to X$ is one-to-one and onto$\}$

resolvent $R(\lambda, A) = (\lambda I - A)^{-1} : X \to D(A)$, $\lambda \in \rho(A)$.

Remark: $(A, D(A))$ closed ⇒ $\lambda I - A : D(A) \to X$ continuous ⇒ $R(\lambda, A) \in \mathcal{L}(X)$

Lemma 3 (properties of resolvent operators): Let $(A, D(A))$ be a generator of a C_0-semigroup $S(t)$, let $\|S(t)\|_{\mathcal{L}(X)} \le M e^{\omega t}$. It holds:

1. $A R(\lambda,A)x = \lambda R(\lambda,A)x - x \qquad \forall x \in X$
2. $R(\lambda,A)Ax = \lambda R(\lambda,A)x - x \qquad \forall x \in D(A)$
3. $R(\lambda,A)x - R(\mu,A)x = (\mu-\lambda) R(\lambda,A)R(\mu,A)x \qquad \forall x \in X, \quad R(\lambda,A)R(\mu,A) = R(\mu,A)R(\lambda,A)$
4. $\forall \lambda > \omega: \ \lambda \in \rho(A), \quad R(\lambda,A) = {}_0\int^\infty e^{-\lambda t} S(t)x\,dt, \quad \|R(\lambda,A)\|_{\mathcal{L}(X)} \le \frac{M}{\lambda-\omega}$.

Proof: 1. $A R(\lambda,A)x = [(A-\lambda I)+\lambda I]R(\lambda,A)x = \lambda R(\lambda,A)x - x$, 2. - the same

3. LHS: $(\lambda I - A)(\mu I - A)(R(\lambda,A) - R(\mu,A))x$

$$= (\lambda I - A)(\mu R(\lambda,A) - \lambda R(\lambda,A) + I - I)x = (\mu-\lambda)(\lambda I - A)R(\lambda,A)x = (\mu-\lambda)x$$

RHS: $(\lambda I - A)(\mu I - A)(\mu-\lambda)R(\mu,A)R(\lambda,A) = (\mu-\lambda)(\lambda I - A)(\mu I - A)R(\mu,A)R(\lambda,A)x = (\mu-\lambda)x$

LHS: $(\mu I - A)(\lambda I - A)R(\lambda,A)R(\mu,A) = I$

RHS: $(\mu I - A)(\lambda I - A)R(\mu,A)R(\lambda,A) = (\mu I - A)(\lambda R(\mu,A) - \mu R(\mu,A) + I)R(\lambda,A)$

$$= (\mu I - A)[R(\mu,A)(\lambda-\mu) + I]R(\lambda,A)$$

$$= [(\lambda - \mu + \mu)I - A]R(\lambda,A) = I$$

4. $S(t)$ gen. by $(A, D(A)) \iff \tilde{S}(t) = e^{-\omega t}S(t)$ gen. by $(\hat{A}, D(\hat{A}))$, $\hat{A} = A - \omega I$, $D(\hat{A}) = D(A)$,

$R(\lambda,\hat{A}) = R(\lambda+\omega, A) \Rightarrow$ WLOG $\omega = 0$.

Then $\|S(t)\|_{\mathcal{L}(X)} \le M$, $\lambda > 0$, denote $\tilde{R}x = {}_0\int^\infty e^{-\lambda t}S(t)x\,dt$ (well-defined)

$\|\tilde{R}x\| < {}_0\int^\infty e^{-\lambda t} M\|x\|\,dt = \frac{M}{\lambda}\|x\| \Rightarrow \tilde{R} \in \mathcal{L}(X), \ \|\tilde{R}\|_{\mathcal{L}(X)} \le \frac{M}{\lambda}$

$\tilde{R}x \in D(A)$: let $h > 0$ and $x \in X$:

$$S(h)\tilde{R}x - \tilde{R}x = {}_0\int^\infty e^{-\lambda t}[S(t+h)x - S(t)x]\,dt$$

$$= {}_h\int^\infty e^{-\lambda(t-h)} S(t)x\,dt - {}_0\int^\infty e^{-\lambda t}S(t)x\,dt$$

$$= e^{\lambda h}\left({}_0\int^\infty e^{-\lambda t}S(t)x\,dt - {}_0\int^h e^{-\lambda t}S(t)x\,dt\right) - {}_0\int^\infty e^{-\lambda t}S(t)x\,dt$$

$$= (e^{\lambda h} - 1)\,{}_0\int^\infty e^{-\lambda t}S(t)x\,dt - e^{\lambda h}\,{}_0\int^h e^{-\lambda t}S(t)x\,dt$$

$$\lim_{h\to 0+} \frac{1}{h}(S(h)\tilde{R}x - \tilde{R}x) = \lim_{h\to 0+}\left(\frac{e^{\lambda h}-1}{h}\right){}_0\int^\infty e^{-\lambda t}S(t)x\,dt - \lim_{h\to 0+} e^{\lambda h}\frac{1}{h}\,{}_0\int^h e^{-\lambda t}S(t)x\,dt$$

$$= \lambda\hat{R}x - X$$

$$\left.\begin{aligned}
&\Rightarrow \quad x = \lambda\tilde{R}x - A\tilde{R}x = (\lambda I - A)\tilde{R}x, \quad x \in X \\
&\text{For } x \in D(A), \quad A\tilde{R}x = A\left({}_0\int^\infty e^{-\lambda t}S(t)x\,dt\right) = {}_0\int^\infty e^{-\lambda t}S(t)Ax\,dt = \tilde{R}Ax \\
&\Rightarrow A\tilde{R}x \underset{\substack{\uparrow \\ x\in X}}{=} \lambda\tilde{R}x - x \underset{\substack{\uparrow \\ x\in D(A)}}{=} \tilde{R}Ax \Rightarrow x = \tilde{R}(\lambda I - A)x, \quad x \in D(A)
\end{aligned}\right\} \begin{array}{l}\lambda I - A \text{ is} \\ \text{one-to-one} \\ \text{and} \\ \text{onto}\end{array}$$

$\lambda > 0$ was arbitrary $\Rightarrow$ $\rho(A) \supset (0,\infty)$ and $\tilde{R} = (\lambda I - A)^{-1} = R(\lambda, A)$.

Definition (semigroup of contractions): We say that $S(t)$ is contraction semigroup if $\|S(t)\|_{\mathcal{L}(X)} \leq 1 \quad \forall\, t \geq 0$.

Theorem 3 (Hille-Yosida (for contractions)): For $(A, D(A))$ an unbounded operator, the following is equivalent:

1. $\exists$ C_0-semigroup of contractions generated by $(A, D(A))$
2. $(A, D(A))$ is closed, $D(A)$ is dense in X, $(0,\infty) \subset \rho(A)$ and $\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}$.

Proof. „1. $\Rightarrow$ 2." already proven by T2 (closed, dense) and L3 (resolvent)

„2. $\Rightarrow$ 1." Yosida's approximation: $A_n = n A R(n, A)$, $n \in \mathbb{N}$.

Strategy: $A_n x \to Ax$ as $n \to \infty$, $S(t) = \lim\limits_{n\to\infty} e^{tA_n}$ $\quad (A_n \in \mathcal{L}(X))$

Step 1. Properties of A_n.

$$A_n = n A R(n,A) \overset{L3,1.}{=} n^2 R(n,A) - nI \in \mathcal{L}(X), \text{ because } R(n,A) \in \mathcal{L}(X)\ \forall n$$

$$\|nR(n,A)x - x\|_X = \|R(n,A)Ax\|_X \leq \|R(n,A)\|_{\mathcal{L}(X)} \|Ax\|_X \leq \tfrac{1}{n}\|Ax\|_X \to 0 \text{ as } n \to \infty$$

$\Rightarrow nR(n,A)x \to x \quad \forall x \in D(A)$, $D(A)$ dense in X & $\|nR(n,A)\| \leq 1$ $\Rightarrow$ $\forall x \in X$.

$$A_n x = nAR(n,A)x = nR(n,A)Ax \to Ax \quad \forall x \in D(A)$$

Step 2. Approximation of the semigroup $S(t)$ by $S_n(t)$.

$$S_n(t) := e^{tA_n} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A_n^k \in \mathcal{L}(X) \qquad (A_n \in \mathcal{L}(X))$$

$$e^{tA_n} = e^{-ntI + n^2 t R(n,A)} = e^{-nt} \cdot e^{n^2 t R(n,A)}$$

$$\|e^{n^2 t R(n,A)}\|_{\mathcal{L}(X)} = \Big\|\sum_{k=0}^{\infty} \frac{(nt)^k (nR(n,A))^k}{k!}\Big\|_{\mathcal{L}(X)} \leq \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} \underbrace{\|nR(n,A)\|_{\mathcal{L}(X)}^k}_{\leq 1} \leq e^{nt}$$

$\Rightarrow \|S_n(t)\|_{\mathcal{L}(X)} \leq e^{-nt} \cdot e^{nt} = 1 \quad \Rightarrow S_n(t)$ are contractions.

Step 3. Existence of the limit

Let $x \in D(A)$ and $t > 0$.

$$S_n(t)x - S_m(t)x = \int_0^t \frac{d}{ds}[S_m(t-s) S_n(s) x]\, ds = \int_0^t \frac{d}{ds}[e^{(t-s)A_m} e^{sA_n} x]\, ds$$

$$= \int_0^t (-A_m e^{(t-s)A_m} e^{sA_n} x + A_n e^{(t-s)A_m} e^{sA_n} x)\, ds$$

$$\Big(R(n,A)R(m,A) \overset{L3,3.}{=} R(m,A)R(n,A) \Rightarrow A_n A_m = A_m A_n \Rightarrow A_n S_m(t) = S_m(t) A_n \quad \forall t > 0\Big)$$

$$= \int_0^t e^{(t-s)A_m} e^{sA_n} (A_n x - A_m x)\, ds$$

$$\Rightarrow \|S_n(t)x - S_m(t)x\|_X \leq \int_0^t \|S_m(t-s)S_n(s)\|_{\mathcal{L}(X)} \|A_n x - A_m x\|_X\, ds \leq t\, \|A_n x - A_m x\|_X \xrightarrow{n,m\to\infty} 0$$

$\Rightarrow \{S_n(t)x\}$ satisfy BC condition uniformly w.r.t. $t \in [0,T]$ $\Rightarrow$ $\exists$ limit $\forall x \in D(A)$

$D(A)$ dense in X, $\|S_n(t)\|_{\mathcal{L}(X)} \le 1 \quad \Rightarrow \exists$ limit $\forall x \in X$.

Step 4. Check that $S(t)$ is generated by $(A, D(A))$.

Denote the generator of $S(t)$ by $(\tilde{A}, D(\tilde{A}))$ and let $x \in D(A)$. $\quad x \overset{?}{\in} D(\tilde{A})$

$$S_n(t)x - x = \int_0^t \frac{d}{ds} S_n(s)x \, ds = \int_0^t S_n(s) A_n x \, ds$$

$$\|S_n(s)A_n x - S(s)Ax\|_X \le \underbrace{\|S_n(s)\|_{\mathcal{L}(X)}}_{\le 1} \underbrace{\|A_n x - Ax\|_X}_{\to 0} + \underbrace{\|S_n(s) - S(s)\|_{\mathcal{L}(X)}}_{\to 0} \underbrace{\|Ax\|_X}_{\text{fixed}} \xrightarrow{n\to\infty} 0$$

Take $n \to \infty$

$$S(t)x - x = \int_0^t S(s)Ax\,ds \quad \Rightarrow \frac{1}{t}(S(t)x - x) = \frac{1}{t}\int_0^t S(s)Ax\,ds$$

take $t \to 0+$: $\quad \tilde{A}x = Ax \quad \forall x \in D(A) \quad \Rightarrow x \in D(\tilde{A}) \qquad (D(A) \subseteq D(\tilde{A}))$

Now, for any $\lambda > 0$, $\lambda \in \rho(A)$ (by assumption) and $\lambda \in \rho(\tilde{A})$ (by L3,4.)

Therefore, $\lambda I - \tilde{A} = \lambda I - A$ on $D(A)$. Also, $(\lambda I - \tilde{A})|_{D(A)}$ is one-to-one and onto

$\Rightarrow D(\tilde{A}) \subseteq D(A) \quad \Rightarrow \quad D(\tilde{A}) = D(A) \quad \& \quad \tilde{A} = A \quad \Rightarrow A$ is the generator.

Theorem (generalized H-Y): $(A, D(A))$ generates C_0-semigroup, satisfying $\|S(t)\|_{\mathcal{L}(X)} \le Me^{\omega t}$

$\Leftrightarrow$ $(A, D(A))$ is closed, densely defined and $\forall \lambda > \omega$, $\lambda \in \rho(A)$ and $\|R^n(\lambda, A)\|_{\mathcal{L}(X)} \le \frac{M}{(\lambda-\omega)^n}$ $\forall n \in \mathbb{N}$.

Application of the semigroup theory in the PDEs

$\partial_t u - \Delta u = 0$ in $(0,T) \times \Omega$

$u(0) = u_0$ in Ω, $u = 0$ on $(0,T) \times \partial\Omega$

1. set $S(t)u_0 := u(t)$, a solution, then $S(0)u_0 = u_0$, and $S(t+s)u_0 = u(t+s) = S(t)u(s) = S(t)S(s)u_0$ S is a sg

$\lim\limits_{t\to 0} \frac{S(t)u_0 - u_0}{t} = \lim\limits_{t \to 0} \frac{u(t) - u_0}{t} = \lim\limits_{t\to 0} ⨍_0^t \partial_t u \Rightarrow \partial_t u(0) = \Delta u(0)$

2. $\partial_t u = Au$, A generates semigroup S, $u(t) := S(t)u_0$

$\Rightarrow \partial_t u = \partial_t(S(t)u_0) = AS(t)u_0 = Au(t) \Rightarrow u$ is a solution

$\Rightarrow$ A generates a sg S and vice versa, S defines a solution

$Au = \Delta u$, $\quad A: D(A) \to X$

$D(A) = W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$, $\quad X = L^2(\Omega)$

to apply H-Y, we need ① $(A, D(A))$ is closed operator

② $(0,\infty) \subseteq \rho(A)$, $\|R(\lambda, A)\| \le \frac{1}{\lambda}$

① $u^n \to u$ in L^2, $Au^n \to f$ in L^2 $\overset{?}{\Rightarrow}$ $u \in D(A)$, $Au = f$

$u^n \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$, $Au^n = f^n$: u^n a weak sol. of $\Delta u^n = f^n$, $u^n = 0$ on $\partial B_1(0)$

continuous dependence on data: $f^n \to f$ in $L^2 \Rightarrow u^n \to u$ in $W_0^{1,2}(\Omega)$

$u \in W_0^{1,2}(\Omega)$, $\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$ $\Leftrightarrow \int_\Omega \nabla u \cdot \nabla v = -\int_\Omega fv \quad \forall v \in W_0^{1,2}(\Omega) \xRightarrow{\text{regularity}} u \in W^{2,2}(\Omega)$

② $\forall \lambda > 0$, $\quad (\lambda I - A): W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega) \to L^2(\Omega)$

$\lambda u - \Delta u = f$, $\qquad u = R_\lambda f$, dream: $\|u\|_2 \le \frac{1}{\lambda}\|f\|_2$

$\lambda\|u\|_2^2 \le \int_\Omega \lambda|u|^2 + |\nabla u|^2 = \int_\Omega fu \le \|f\|_2\|u\|_2 \Rightarrow \|u\|_2 \le \frac{\|f\|_2}{\lambda}$

$\omega := -\lambda_1$, smallest eigenvalue : $\lambda\|u\|_2^2 + \lambda_1\|u\|_2^2 \le \int_\Omega \lambda|u|^2 + |\nabla u|^2 \Rightarrow \|u\|_2 \le \frac{\|f\|_2}{\lambda + \lambda_1}$

$\Uparrow$

$\|S(t)u_0\| \le e^{-\lambda_1 t}\|u_0\|$

22.5.2019

Semigroup for wave equation

$$\partial_{tt} u - \Delta u = 0 \quad \text{in } (0,T)\times\Omega$$
$$u(0) = u_0 \quad \text{in } \Omega$$
$$\partial_t u(0) = v_0 \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega \times (0,T)$$

$$\partial_t u = v \qquad U = (u, v)$$
$$\partial_t v = \Delta u \qquad A(U): (u,v) \mapsto (v, \Delta u)$$

$$\partial_t U = A(U)$$

$$X := \{ U = (u,v),\ u \in W_0^{1,2}(\Omega),\ v \in L^2(\Omega) \} = W_0^{1,2}(\Omega) \times L^2(\Omega)$$

$$D(A) \subseteq X, \quad D(A) = [W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)] \times L^2(\Omega)$$

Use of Hille-Yossida theorem:

1. A is closed - the same as for parabolic equation
2. estimates for resolvent (different)

show that $\forall \lambda > 0$ $(\lambda I - A)$ is onto and invertible

$$R_\lambda := (\lambda I - A)^{-1}, \quad \| R_\lambda \| \leq \frac{M}{\lambda}.$$

$$(\lambda I - A)(U) = (\lambda u - v, \lambda v - \Delta u)$$

$$\forall (f_1, f_2)\ \exists!\ (u,v) \quad \lambda u - v = f_1 \in W_0^{1,2},\ \lambda v - \Delta u = f_2 \in L^2(\Omega)$$

$$v = \lambda u - f_1 \Rightarrow \lambda^2 u - \Delta u = f_2 + \lambda f_1 \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$
$$\Big\} \Rightarrow \exists!\ u \in W_0^{1,2}(\Omega) \Rightarrow \exists!\ v \in L^2(\Omega)$$

Now, we want

$$\| U \|^2 \leq \frac{\|f\|^2}{\lambda^2} . M \quad \dots ? \quad \|\nabla u\|_2^2 + \|v\|_2^2 \leq \frac{1}{\lambda^2} (\|\nabla f_1\|_2^2 + \|f_2\|_2^2)$$

Equations: $\lambda u - v = f_1$ | multiply by u, v, $\int$

$\lambda v - \Delta u = f_2$ | test by $u\lambda$

$$\int_\Omega \lambda u^2 - u.v = \int_\Omega f_1 u$$
$$\int_\Omega \lambda^2 u.v - \lambda v^2 = \lambda \int_\Omega f_1 v$$
$$\int_\Omega \lambda^2 u.v + \lambda |\nabla u|^2 = \lambda \int_\Omega u f_2$$

$$\Big\} \quad \lambda \int_\Omega v^2 + |\nabla u|^2 = \lambda \int_\Omega f_2 u - \lambda \int_\Omega f_1 v$$
$$= \int_\Omega (f_1 + v) f_2 - (\Delta u + f_2) f_1$$
$$= \int_\Omega f_1 f_2 + f_2 v - f_1 f_2 + \nabla u . \nabla f_1$$

$$= \int_\Omega f_2 v + \nabla u . \nabla f_1 \leq \|f_2\|_2 \|v\|_2 + \|\nabla u\|_2 \|f_1\|_2 \leq \frac{1}{2} \lambda (\|v\|_2^2 + \|\nabla u\|_2^2) + \frac{1}{2\lambda} (\|f_2\|_2^2 + \|\nabla f_1\|_2^2)$$

$$\Rightarrow \quad \|u\|_X^2 = \|v\|_2^2 + \|\nabla u\|_2^2 \leq \frac{1}{\lambda^2}\left(\|f_2\|_2^2 + \|\nabla f_1\|_2^2\right) = \|(f_1, f_2)\|_X^2$$

Semigroup when you cannot use regularity

$$\partial_t u + Lu = 0 \quad \text{in } (0,T)\times\Omega \qquad Lu = -\operatorname{div} A(x)\nabla u + \vec{b}\cdot\nabla u + cu$$

$$u = 0 \quad \text{on } (0,T)\times\partial\Omega$$

$$u(0) = u_0$$

$\partial_t u = A(u) := -Lu \qquad X = L^2(\Omega)$

$$D(A) := \{u \in W_0^{1,2}(\Omega);\ \|Lu\|_2 = \sup_{\substack{\varphi\in C_0^\infty(\Omega)\\ \|\varphi\|_2 \leq 1}} \int_\Omega A\nabla u\cdot\nabla\varphi + \vec{b}\cdot\nabla u\,\varphi + cu\varphi < \infty\}$$

density of $D(A)$ in X is clear

Comment: $D(A)$ can be equivalently defined as

$$D(A) := \{u \in W_0^{1,2}(\Omega);\ \exists f \in L^2(\Omega) \text{ solving } \forall v \in W_0^{1,2}(\Omega) \int_\Omega A\nabla u\cdot\nabla v + \vec{b}\cdot\nabla u\, v + cuv = \int_\Omega fv\}$$

A is closed: $u^n \to u$ in $L^2(\Omega)$, $Lu^n \to f$ in $L^2(\Omega)$ $\overset{?}{\Rightarrow}$ $Lu = f$

it means: $\{u^n\}_{n=1}^\infty, \{f^n\}_{n=1}^\infty$: $\int_\Omega A\nabla u^n\cdot\nabla v + \vec{b}\cdot\nabla u^n v + cu^n v = \int_\Omega f^n v \quad \forall v \in W_0^{1,2}(\Omega)$

$\overset{?}{\Rightarrow} \quad \int_\Omega A\nabla u\cdot\nabla v + \vec{b}\cdot\nabla u v + cuv = \int_\Omega fv \quad \forall v \in W_0^{1,2}(\Omega)$

Proof. 1. show that $u \in W_0^{1,2}(\Omega)$

we know $u^n \to u$ in $L^2(\Omega)$ $\Rightarrow$ $\|u^n\|_2 \leq c$, test by u^n

$$\Rightarrow \int_\Omega A\nabla u^n\cdot\nabla u^n = \int_\Omega f^n u^n - \vec{b}\cdot\nabla u^n u^n - c(u^n)^2$$

$$\leq \|f^n\|_2\|u^n\|_2 + \|\vec{b}\|_\infty\|\nabla u^n\|_2\|u^n\|_2 + \|c\|_\infty\|u^n\|_2^2 \leq c(1 + \|\nabla u^n\|_2)$$

ellipticity: $c_1\|\nabla u^n\|_2^2 \leq c(1 + \|\nabla u^n\|_2) \overset{\text{Young}}{\leq} \frac{c_1}{2}\|\nabla u^n\|_2^2 + \tilde{c} \Rightarrow \|\nabla u^n\|_2^2 \leq c$ constant

for a subsequence $u^n \rightharpoonup u$ in $W_0^{1,2}(\Omega)$

2, $R_\lambda \quad \exists\gamma\ \forall\lambda > \gamma \quad (\lambda I - A)$ is invertible and onto, $\|R_\lambda\| = \|(\lambda I - A)^{-1}\| \leq \frac{M}{\lambda - \gamma}$

$\Leftrightarrow \exists\gamma\ \forall\lambda > \gamma\ \forall f \in L^2(\Omega)\ \exists!\, u \quad \int_\Omega \lambda uv + \int_\Omega A\nabla u\cdot\nabla v + \vec{b}\nabla u v + cuv = \int_\Omega fv \quad \forall v \in W_0^{1,2}(\Omega)$

and $\|u\|_2 \leq \frac{\|f\|_2}{\lambda - \gamma}$

In winter semester, we did: $\exists\gamma^*\ \forall f \in L^2\ \exists!\, u \quad \gamma^* u + Lu = f$

$\lambda I - \gamma^* I + \gamma^* I + Lu = f$

"new" abbreviation.

$\partial_t u - \Delta u = 0$ in $(0,T)\times\Omega$

$u(0) = u_0$ in Ω

$u_0 \in L^2(\Omega) \Rightarrow \exists u \in L^2(0,T; W_0^{1,2}(\Omega)) \cap W^{1,2}(0,T; (W_0^{1,2}(\Omega))^*) \hookrightarrow C(0,T; L^2(\Omega))$

$u_0 \in L^p(\Omega) \Rightarrow u \in C(0,T; L^p(\Omega))$ will be done by semigroup

$2 < p < \infty$: $X = L^p(\Omega)$; $Au = \Delta u$

$D(A) = \{W_0^{1,2}(\Omega) \ni u ; \Delta u \in L^p(\Omega)\}$

estimate for resolvent

$$\left.\begin{aligned} \lambda u - \Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}\right\} \overset{?}{\Rightarrow} \|u\|_p \le \frac{\|f\|_p}{\lambda}$$

$\exists!\, u \in W_0^{1,2}$

I multiply by $|u|^{p-2}u$ (formally: I don't know whether it is in $W^{1,2}$)

$$\lambda \int_\Omega |u|^p + \underbrace{\int_\Omega \nabla u \cdot \nabla(|u|^{p-2}u)}_{\ge 0} = \int_\Omega f |u|^{p-2} u \le \|f\|_p \, \||u|^{p-1}\|_{p'} = \|f\|_p \, \|u\|_p^{p-1}$$

method: $\nabla u \cdot \nabla(|u|^{p-2}u) = |\nabla u|^2 (|u|^{p-2})(p-1)$

correction: test by $\left(\frac{|u|}{1+\varepsilon|u|}\right)^{p-2} u \in W_0^{1,2}(\Omega)$ $\forall \varepsilon > 0$, then let $\varepsilon \to 0+$

Semigroup with right hand side

$$\left.\begin{aligned} \partial_t u - \Delta u &= f && \text{in } (0,T)\times\Omega \\ u &= 0 && \text{in } (0,T)\times\partial\Omega \\ u(0) &= u_0 && \text{in } \Omega \end{aligned}\right\} \begin{aligned} &\partial_t u = Au + f \\ &A \text{ is a generator of } S \text{ (we already have)} \\ &f \in L^1(0,T; L^2(\Omega)) \end{aligned}$$

set: $u(t) := S(t)u + \int_0^t S(t-\tau) f(\tau)\, d\tau$

compute
$$\begin{aligned} \partial_t u(t) &= \partial_t(S(t)u_0) + \partial_t\left(\int_0^t S(t-\tau) f(\tau)\, d\tau\right) \\ &= \partial_t(S(t)u_0) + S(0)f(t) + \int_0^t \partial_t(S(t-\tau)f(\tau))\, d\tau \} \\ &= AS(t)u_0 + \int_0^t AS(t-\tau)f(\tau)\, d\tau + f(t) \\ &= f(t) + A\left(S(t)u_0 + \int_0^t S(t-\tau) f(\tau)\, d\tau\right) = f(t) + Au(t) \end{aligned}$$