NMSA332

37, 29, $X \sim B(a,b)$ $\quad \dfrac{f(x)}{f(y)} = \dfrac{(\prod x_i)^{a-1} \prod (1-x_i)^{b-1}}{(\prod y_i)^{a-1} \prod (1-y_i)^{b-1}} = \left(\prod \dfrac{x_i}{y_i}\right)^{a-1} \left(\prod \dfrac{1-x_i}{1-y_i}\right)^{b-1}$

$\Rightarrow (\prod X_i , \prod (1-X_i))^T$ alebo $(\sum \log X_i , \sum \log(1-X_i))^T$ sú obe min. suf.

38, 30, $X \sim N(\mu_1, \sigma^2)$ $\quad Y \sim N(\mu_2, \sigma^2)$

i) $f(x,y) = \dfrac{c}{\sigma^{n+m}} \exp\left\{-\dfrac{1}{2\sigma^2}\left[\sum^n x_i^2 - 2\mu_1 \sum x_i + n\mu_1^2 - \sum^m y_i^2 + 2\mu_2 \sum y_i - m\mu_2^2\right]\right\}$

$\Rightarrow (\sum X_i , \sum X_i^2 , \sum Y_i , \sum Y_i^2)^T$ je suf.

ii) $E_{\mu_1,\mu_2,\sigma^2} \; S^2_{n,X} - S^2_{m,Y} = 0 \quad \forall \mu_1, \mu_2, \sigma^2$

39, 31, $P(\mathbb{X} = x) = \dfrac{e^{-n\lambda} \lambda^{\sum x_i}}{\prod x! \cdot C(k,\lambda)^n} \; I[0 \le x_1 \le k] \dots I[0 \le x_n \le k] \Rightarrow (\sum X_i , \max X_i)^T$ je suf.

Využitie suf. štatistík $\quad$ 39, 40 urobiť $\quad$ 45 urobiť

39 +1 32, $X \sim Ge(p)$ $\quad P(X=x) = p(1-p)^x \quad x \in \mathbb{N}_0$

a) $ET = E\,I[X=0] = p$

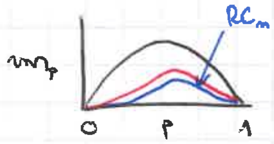
b) (p. 174) $\Rightarrow S = \sum X_i$ postačujúca $\quad E[T \mid S=s] = E[I[X=0] \mid S=s] = P(X=0 \mid S=s) =$

25

$= \dfrac{P(S=s \mid X_1 = 0)\, P(X_1=0)}{P(S=s)} = p \cdot \dfrac{P(\sum_{i=2}^n X_i = s)}{P(\sum_{i=1}^n X_i = s)} = p \dfrac{\binom{n+s-2}{s} p^{n-1} (1-p)^s}{\binom{n+s-1}{s} p^n (1-p)^s} =$

$P(\sum_{i=1}^n X_i = s) = \binom{n+s-1}{s} p^n (1-p)^s \quad s \in \mathbb{N}_0$

$= \dfrac{\frac{(n+s-2)!}{s!\,(n-2)!}}{\frac{(n+s-1)!}{s!\,(n-1)!}} = \dfrac{n-1}{n+s-1}$ $\qquad RC_n = \dfrac{p^2 q}{n}$



$mn\, T = \dfrac{p(1-p)}{n}$

$mn\, u(\bar{x})$ je možné dopočítať.

$E[T \mid S] = \dfrac{n-1}{n+S-1} = \dfrac{1 - 1/n}{1 + \bar{X} - 1/n} = u(\bar{X})$

c) $P(X=x) = p(1-p)^x = \exp\{x \log(1-p)\} \cdot p \Rightarrow \sum X_i = S$ je úplná postač. pre $\theta = \log(1-p)$

$a(\theta) = 1 - e^\theta = 1 - (1-p) = p \qquad E_\theta T^2 = E_p T^2 < \infty \;\forall p \Rightarrow$ Lehmann-Scheffé $u(\bar{X})$ je najlepší nestr. odhad p

d) $E\,I[X_1 = 1] = P(X_1 = 1) = p(1-p) \qquad T = I[X_1 = 1]$ nestranný odhad.

$E[T \mid S=s] = p(1-p) \cdot \dfrac{P(\sum_{i=2}^n X_i = s-1)}{P(\sum X_i = s)} = p(1-p) \dfrac{\binom{n+s-3}{s-1} p^{n-1} (1-p)^{s-1}}{\binom{n+s-1}{s} p^n (1-p)^s} =$

$= \dfrac{\frac{(n+s-3)!}{(s-1)!\,(n-2)!}}{\frac{(n+s-1)!}{s!\,(n-1)!}} = \dfrac{(n-1) \cdot s}{(n+s-1)(n+s-2)}$

$E[T \mid S] = \dfrac{(n-1)\, S}{(n+S-1)(n+S-2)} = u(S)$

40 33, $\mathbb{X} \sim M(1; p, 1-2p, p)$ s hodnotami $\{-1, 0, 1\}$

a) $ET = P(X=1) = p$

b) $P(\mathbb{X} = (x_1, x_2, x_3)^T) = p^{x_1} (1-2p)^{x_2} p^{x_3}$ $\quad$ kde $(x_1, x_2, x_3)^T \in \{0,1\}^3$, $\sum x_i = 1$ $\quad$ počet -1 .. počet $+1$

$= p^{x_1 + x_3} (1-2p)^{x_2}$

$= p^{I[X \neq 0]} (1-2p)^{1 - I[X \neq 0]}$

c) $E[I[X=1] \mid \underbrace{\sum I[X_i \neq 0]}_{S \sim Bi(n, 2p)} = s] = p \cdot \dfrac{P(S=s \mid X_1 = 1)}{P(S=s)} = p \dfrac{\binom{n-1}{s-1} (2p)^{s-1} (1-2p)^{n-s}}{\binom{n}{s} (2p)^s (1-2p)^{n-s}}$

$$= \frac{\frac{(m-1)!}{(s-1)!(m-s)!\cdot 2}}{\frac{m!}{s!(m-s)!}} = \frac{s}{2m} \qquad E[T|S] = u(S) = \frac{S}{2m} = \frac{1}{2}\frac{1}{m}\sum I[X_i=1]$$

d) $P(X=x) = \binom{m}{x_1 \dots x_3} \prod p_i^{x_i} = \exp\{\sum x_i \log p_i\} = \exp\{\log p \cdot (x_1+x_3) + \log(1-2p)\cdot x_2\}$

$= \exp\{(x_1+x_3)\log p + (1-(x_1+x_3))\log(1-2p)\} = \exp\{(x_1+x_3)\log\frac{p}{1-2p}\}\exp\{\log(1-2p)\}$

$\Rightarrow$ exponenciální rodina $\Rightarrow$ úplná suf. stat $\Rightarrow$ nejlepší nestranný odhad

41 34) $X \sim alt(p)$

a) $T = I[X_1 = 1]$ $\quad P(X=x) = p^x(1-p)^{1-x} = \exp\{x\log p + (1-x)\log(1-p)\} =$

$= \exp\{x \log\frac{p}{1-p}\}(1-p) \Rightarrow S = \sum X_i$ je úplná postačující $\quad$ $\approx$ Pr 5 $\quad E[X_1|\sum X_i] \overset{X_i \text{ iid}}{=} \frac{\sum X_i}{m}$

$E[T|S=s] = E[\frac{S}{m}|S=s] = s/m \quad u(S) = \frac{S}{m} = \bar{X}$ je nejlepší nestranný odhad p

b) $T' = \bar{X}(1-\bar{X}) \qquad \bar{X} \sim Bi(m,p)/m \qquad Y \sim Bi(m,p) \qquad \operatorname{var} Y = mp(1-p) = EY^2 - (mp)^2$

$EY^2 = mp(1-p) + m^2p^2$

$ET' = p - \frac{1}{m^2}EY^2 = p - \frac{p(1-p)}{m} - p^2 = p(1-p)(1-\frac{1}{m})$

$\Rightarrow T := \frac{m}{m-1}\bar{X}(1-\bar{X})$ je $ET = p(1-p)$

$E[T|S] = T$ (fce $\bar{X}$) je nejlepší nestr. odhad $p(1-p)$.

42 35) $X \sim Po(\lambda)$

a) $T_1 = \bar{X} \quad ET_1 = \lambda \qquad P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{1}{x!}e^{-\lambda}e^{x\log\lambda} \Rightarrow \sum X_i$ je úplná postačující pro $\log\lambda$

$E[T_1|\sum X_i] = T_1$ je nejlepší nestr. odhad $\lambda = \exp\{\log\lambda\}$

b) $T_2' = \frac{\sum I[X_i = 0]}{m} \qquad ET_2 = e^{-\lambda} \qquad \sum_{i=1}^m X_i \sim Po(m\lambda)$

$E[T_2|\sum X_i = s] = E[X_1 = 0|\sum X_i = s] = e^{-\lambda}\frac{e^{-(m-1)\lambda}[(m-1)\lambda]^s/s!}{e^{-m\lambda}(m\lambda)^s/s!} = \left(\frac{m-1}{m}\right)^s$

$\Rightarrow (1-\frac{1}{m})^{\sum X_i}$ je nejlepší nestr. odhad $e^{-\lambda}$.

c) $\approx$ Pr 9 $\quad T_1$ je eficientní, T_2 není. 10

43 36) $X \sim N(\mu,\sigma^2) \qquad f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\{-\frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2)\} = \frac{1}{\sqrt{2\pi}}\sqrt{\zeta}\exp\{-\frac{\zeta}{2}(x^2) + \tau x\}e^{-\frac{\tau^2}{2\zeta}}$

$(\sum X_i, \sum X_i^2)^T$ jsou $\sim$ úplné postač. pro $(\zeta,\tau)^T \qquad \frac{1}{\sigma^2} =: \zeta \qquad \frac{\mu}{\sigma^2} =: \tau$

$a(\zeta,\tau) = 1/\sqrt{\zeta} = \sigma$ $\qquad$ a) $E[S_m^2|\sum X_i, \sum X_i^2] = S_m^2$

a) $E\tilde{\sigma} = E a_m \frac{S\sqrt{m-1}}{\sigma}\cdot\sigma = \sigma a_m EX_{m-1} = \sigma$ nestranný $\quad$ Podobně jako v Pr 1, 13 $\left(\text{kdy } \mu \neq 0\right)$ ~~ale není v Pr 12 není je eficientní~~

$\frac{S^2(m-1)}{\sigma^2} \sim \chi^2_{m-1} \qquad \frac{S\sqrt{m-1}}{\sigma} \sim \chi_{m-1}$

$E[\tilde{\sigma}|\sum X_i, \sum X_i^2] = \tilde{\sigma}$ nejlepší nestranný

b) medián není je $\approx$ LS ref a jednoznačnost

b) $a(\zeta,\tau) = \tau/\zeta + u_\alpha \cdot 1/\sqrt{\zeta} = \mu + u_\alpha\sigma \qquad E(\bar{X} + u_\alpha\tilde{\sigma}_m) = \mu + u_\alpha\sigma$ nestranný a je fce

$(\sum X_i, \sum X_i^2)^T \Rightarrow$ nejlepší nestranný

c) $\mu^2 = -\sigma^2 + EX^2 \qquad T := \frac{1}{m}\sum X_i^2 - \cancel{\tilde{\sigma}}\, S^2 \qquad ET = EX^2 - \sigma^2 = \mu^2$ a je fce $(\sum X_i, \sum X_i^2)^T$

$\Rightarrow$ nejlepší nestranný $\qquad$ $\approx$ Pr 8iv) nedosahují RC_m

e) $\sigma^2 = EX^2 - \mu^2 \qquad$ odhadneme $\quad \frac{1}{m}\sum X_i^2 - S_m^2$ nestranný, fce úplný postač. stat, $\Rightarrow$ nejlepší nestranný

$\mu^2 = EX^2 - \sigma^2$

NMSA 332

(44) 37) $X \sim N(\mu, \mu^2)$ $\quad f(x) = \left(\frac{1}{\sqrt{2\pi\mu^2}}\right)^n \exp\left\{-\frac{1}{2\mu^2}\left[\sum x_i^2 - 2\mu \sum x_i + n\mu^2\right]\right\} = (2\pi\mu^2)^{-\frac{n}{2}} e^{-\frac{n}{2}}\cdot$

$\exp\left\{-\frac{1}{2\mu^2}\sum x_i^2 + \frac{1}{\mu}\sum x_i\right\} \Rightarrow (\sum x_i, \sum x_i^2)^T$ je sufficientná štatistika

$\frac{f(x)}{f(y)} = \exp\left\{-\frac{1}{2\mu^2}\left(\sum x_i^2 - \sum y_i^2\right) + \frac{1}{\mu}\left(\sum x_i - \sum y_i\right)\right\}$ je minimálna suf. štat.

i) $T_1 = \bar{X}$, $T_2 = a_n \sqrt{(n-1)S_n^2}$ sú oba fce $(\sum x_i, \sum x_i^2)'$

$E T_1 = \mu \qquad E T_2 = a_n E \sqrt{\frac{(n-1)S_n^2}{\sigma^2}} \cdot \sigma = \sigma = \mu$ sú oba nestranné

ii) $\operatorname{var} T_1 = \frac{\mu^2}{n} \qquad \operatorname{var} T_2 = a_n^2 \operatorname{var} \sqrt{\frac{(n-1)S_n^2}{\sigma^2}} \cdot \mu = \mu^2 a_n^2 \operatorname{var} \sqrt{\chi^2_{n-1}} =$

$= \mu^2 a_n^2 \left(n-1 - a_n^{-2}\right) = \mu^2 \left(\frac{n-1}{a_n^2} - 1\right)$

podľa p. 17 ani jeden odhad nedosahuje RC a S_n je lepší

$(\sum x_i, \sum x_i^2)'$ nie je úplná $\quad E_\mu\left[\frac{n}{n+1}(\bar{X})^2 - S^2\right] = 0 \quad \forall \mu$

(45) 38) $X \sim f(x) = \lambda e^{-\lambda(x-\delta)} I[x > \delta]$ $\quad \lambda$ známe.

i) $f(x) = \lambda^n e^{-\lambda \sum x_i} e^{\lambda\delta}\, I[\min x_i > \delta] \Rightarrow \min X_i$ je sufficientná

$X_1' .. X_n' \sim Exp(\lambda) \Rightarrow \min X_i' \sim Exp(n\lambda)$ analogicky $\min X_i \sim (n\lambda) e^{-n\lambda(x-\delta)} I[x>\delta]$

nech $0 = E_\delta w(\min X_i) = \int_\delta^\infty (n\lambda) e^{-n\lambda(x-\delta)} w(x)\, dx = n\lambda e^{n\lambda\delta} \int_\delta^\infty w(x) e^{-n\lambda x} dx$

diaľ $0 = \int_\delta^\infty w(x) e^{-n\lambda x} dx \quad \forall \delta \quad / \frac{\partial}{\partial \delta}$

$0 = 0 - w(\delta) e^{-n\lambda\delta} \quad / \cdot e^{n\lambda\delta}$

$0 = w(\delta) \quad \forall \delta \Rightarrow$ úplná postač. štatistika $\min X_i$

$E \min X_i = \delta + \frac{1}{n\lambda} \Rightarrow T = \min X_i - 1/n\lambda$ je najlepší nestr. odhad

ii) neregulárny systém hustôt $\quad \operatorname{var} \min X_i = \operatorname{var} T = \frac{1}{(n\lambda)^2}$ a RC je iba rádu $\frac{c}{n}$ "preskakuje" vidlo či by mohol byť RC medzou

(46) 39) $X \sim f(x) = \lambda e^{-\lambda x} I[x > 0]$ $\qquad f(x) = \lambda^n e^{-\lambda \sum x_i} I[\min x_i > 0]$

$\sum X_i$ je úplná postačujúca. $\qquad X_i \sim \underset{\Gamma(1, 1/\lambda)}{\underbrace{Exp(\lambda)}} \Rightarrow \sum X_i \sim \Gamma(n, 1/\lambda) \sim \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma(n)} I[x>0]$

i) $E \sum X_i = \frac{n}{\lambda}$ ale $E \frac{1}{\bar{X}} = n \int_0^\infty \frac{\lambda^n x^{n-2} e^{-\lambda x}}{\Gamma(n)} dx \underset{\lambda x = t}{=} \frac{n\lambda^{n-1}}{\Gamma(n)\lambda^{n-2}} \int_0^\infty t^{n-2} e^{-t} dt =$

$= \frac{n\lambda\Gamma(n-1)}{\Gamma(n)} = \frac{n}{n-1}\lambda \Rightarrow \frac{n-1}{\cancel{n}} \cdot \frac{\cancel{n}}{\sum X_i}$ je nestranný odhad λ

$\Rightarrow T = \frac{n-1}{\sum X_i}$ je najlepší nestranný odhad λ.

ii) R (Fn IS): RC: $\frac{\lambda^2}{n} \qquad \operatorname{var} \frac{n-1}{\sum X_i} = (n-1)^2 \cdot \operatorname{var} \underset{\text{inverse gamma distribution}}{\underbrace{\frac{1}{\Gamma(n, 1/\lambda)}}} = \frac{\lambda^2}{n-2}$

nedosahuje RC medzu.

iii) $E\left(\frac{1}{\bar{X}}\right)^k = (n\lambda)^k \frac{\Gamma(n-k)}{\Gamma(n)}$

47 40) $X \sim R(0,\theta)$ $\quad f(x) = \theta^{-m} I[0 < \min x_i]\, I[\max x_i < \theta]$

$\Rightarrow$ $\max X_i$ je suf. štat. podľa pr ~~23~~ 31 je úplná

$$E \max X_i \overset{\text{pr 23 / pr 30}}{=} \int_0^\theta \frac{x\, m\, x^{m-1}}{\theta^m}\,dx = \frac{m}{m+1}\theta \quad \Rightarrow \quad \frac{m+1}{m}\max X_i \text{ je najlepší nestr. odhad.}$$

RC medza neexistuje – neregulárny systém

i) $E\, 2\bar{X} = \theta \quad \operatorname{var} 2\bar{X} = 4\,\theta^2/12m = \theta^2/3m$

$$\operatorname{var}\left(\frac{m+1}{m}\right)\max X_i = \left(\frac{m+1}{m}\right)^2\left[\int_0^\theta \frac{x^2 m\, x^{m-1}}{\theta^m}\,dx\right] - \theta^2 = \left(\frac{m+1}{m}\right)^2 \frac{\theta^2 m}{m+2} - \theta^2$$

$$= \theta^2\left(\frac{(m+1)^2}{m(m+2)} - 1\right) = \theta^2/(m(m+2)) \quad a \quad \frac{1}{3m} \geq \frac{1}{m(m+2)} \text{ pre } m \geq 1$$

pre $m = 1$ majú rovnaké rozptyly (rovnaké odhady)

ii) $\frac{m+1}{m}\max X_i$

iii) RC medza neexistuje

48 41) $\mathbb{X} \sim M(1; p_1 \ldots p_k)$ $\quad P(\mathbb{X} = \mathbb{x}) = \prod p_i^{x_i} = \prod e^{x_i \log p_i} = e^{\sum x_i \log p_i}$

i) $\left(\sum_{i=1}^m X_{i1}, \ldots, \sum_{i=1}^m X_{ik}\right)'$ je úplná suficientná pre $\mathbb{p}$

ii) $E \sum_{i=1}^m X_{i1} \sum_{i=1}^m X_{i2} \underset{\mathbb{Y}}{=} E\, Y_1 Y_2 = \operatorname{cov}(Y_1, Y_2) + EY_1\, EY_2 = -m p_1 p_2 + m^2 p_1 p_2$

$\mathbb{Y} = (Y_1 \ldots Y_k) \sim M(m, p_1 \ldots p_k) \qquad = m p_1 p_2 (m-1)$

$\operatorname{cov}(Y_1, Y_2) = -m p_1 p_2 \quad EY_i = m p_i$

$\Rightarrow \frac{1}{m(m-1)} \sum_{i=1}^m X_{i1} \cdot \sum_{i=1}^m X_{i2}$ je najlepší nestr. odhad $p_1 p_2$.