NMST 434, Exercise session II: Method of Moments

February 27, 2020

Example 8: MoM in beta distribution

```
$Assumptions = α > 0 && β > 0;
M[m_] = Moment[BetaDistribution[α, β], m];
τ[α_, β_] = {M[1], M[2]}
MoM = Solve[τ[α, β] == {m1, m2}, {α, β}] (* MoM estimators based on X, X^2*)
J = Inverse[D[τ[α, β], {{α, β}}]] // Simplify (* Jacobian of the inverse transform *)
Σ = Table[M[i + j] - M[i] M[j], {i, 1, 2}, {j, 1, 2}] // Simplify
(* Variance matrix of (X, X^2) *)
J.Σ.Transpose[J] // MatrixForm // Simplify
(* asymptotic variance matrix n * AVar(α̂, β̂) *)
$Assumptions = True;
```

$$\left\{\frac{\alpha}{\alpha+\beta}, \frac{\alpha(1+\alpha)}{(\alpha+\beta)(1+\alpha+\beta)}\right\}$$

$$\left\{\left\{\alpha \to \frac{-m1^2 + m1\, m2}{m1^2 - m2}, \beta \to \frac{(-1+m1)(m1-m2)}{m1^2-m2}\right\}\right\}$$

$$\left\{\left\{\frac{(1+\alpha)(\alpha+\beta)(1+2\alpha+2\beta)}{\beta}, -\frac{(\alpha+\beta)(1+\alpha+\beta)^2}{\beta}\right\}, \left\{\frac{(\alpha+\beta)(1+2\alpha^2+\beta+2\alpha(1+\beta))}{\alpha}, -\frac{(\alpha+\beta)(1+\alpha+\beta)^2}{\alpha}\right\}\right\}$$

$$\left\{\left\{\frac{\alpha\beta}{(\alpha+\beta)^2(1+\alpha+\beta)}, \frac{2\alpha(1+\alpha)\beta}{(\alpha+\beta)^2(1+\alpha+\beta)(2+\alpha+\beta)}\right\}, \left\{\frac{2\alpha(1+\alpha)\beta}{(\alpha+\beta)^2(1+\alpha+\beta)(2+\alpha+\beta)}, \frac{2\alpha(1+\alpha)\beta(2\alpha^2+3(1+\beta)+2\alpha(3+\beta))}{(\alpha+\beta)^2(1+\alpha+\beta)^2(2+\alpha+\beta)(3+\alpha+\beta)}\right\}\right\}$$

$$\begin{pmatrix} \frac{\alpha(1+\alpha)(\alpha+\beta)(1+5\beta+4\beta^2+2\beta^3+\alpha^2(3+2\beta)+\alpha(4+7\beta+4\beta^2))}{\beta(1+\alpha+\beta)(2+\alpha+\beta)(3+\alpha+\beta)} & \frac{(1+\alpha)(1+\beta)(\alpha+\beta)(1+\alpha+2\alpha^2+\beta+4\alpha\beta+2\beta^2)}{(1+\alpha+\beta)(2+\alpha+\beta)(3+\alpha+\beta)} \\ \frac{(1+\alpha)(1+\beta)(\alpha+\beta)(1+\alpha+2\alpha^2+\beta+4\alpha\beta+2\beta^2)}{(1+\alpha+\beta)(2+\alpha+\beta)(3+\alpha+\beta)} & \frac{\beta(1+\beta)(\alpha+\beta)(1+2\alpha^3+4\beta+3\beta^2+4\alpha^2(1+\beta)+\alpha(5+7\beta+2\beta^2))}{\alpha(1+\alpha+\beta)(2+\alpha+\beta)(3+\alpha+\beta)} \end{pmatrix}$$

For comparison, without the inverse function theorem.

```
J2 = D[{α, β} /. MoM, {{m1, m2}}] (* Jacobian of the transform *)
J2 /. {m1 → M[1], m2 → M[2]} // Simplify
 (* Jacobian of the transform at the first two moments *)
```

$$\left\{\left\{\left\{\frac{-2\,m1+m2}{m1^2-m2}-\frac{2\,m1\,\left(-m1^2+m1\,m2\right)}{\left(m1^2-m2\right)^2},\ \frac{m1}{m1^2-m2}+\frac{-m1^2+m1\,m2}{\left(m1^2-m2\right)^2}\right\},\ \left\{-\frac{2\,(-1+m1)\,m1\,(m1-m2)}{\left(m1^2-m2\right)^2}+\frac{-1+m1}{m1^2-m2}+\frac{m1-m2}{m1^2-m2},\ \frac{(-1+m1)\,(m1-m2)}{\left(m1^2-m2\right)^2}-\frac{-1+m1}{m1^2-m2}\right\}\right\}\right\}$$

$$\left\{\left\{\left\{\frac{(1+\alpha)\,(\alpha+\beta)\,(1+2\,\alpha+2\,\beta)}{\beta},\ -\frac{(\alpha+\beta)\,(1+\alpha+\beta)^2}{\beta}\right\},\ \left\{\frac{(\alpha+\beta)\,\left(1+2\,\alpha^2+\beta+2\,\alpha\,(1+\beta)\right)}{\alpha},\ -\frac{(\alpha+\beta)\,(1+\alpha+\beta)^2}{\alpha}\right\}\right\}\right\}$$

Example : MoM in Gamma distribution

```
$Assumptions = k > 0 && θ > 0;
M[m_] = Moment[GammaDistribution[k, θ], m];
τ[k_, θ_] = {M[1], M[2]}
MoM = Solve[τ[k, θ] == {m1, m2}, {k, θ}] (* MoM estimators based on X, X^2*)
J = Inverse[D[τ[k, θ], {{k, θ}}]] // Simplify (* Jacobian of the inverse transform *)
Σ = Table[M[i + j] - M[i] M[j], {i, 1, 2}, {j, 1, 2}] // Simplify
(* Variance matrix of (X, X^2) *)
J.Σ.Transpose[J] // MatrixForm // Simplify
 (* asymptotic variance matrix n * AVar(k̂, θ̂) *)
```

$$\left\{k\,\theta,\ k\,(1+k)\,\theta^2\right\}$$

$$\left\{\left\{k\to-\frac{m1^2}{m1^2-m2},\ \theta\to\frac{-m1^2+m2}{m1}\right\}\right\}$$

$$\left\{\left\{\frac{2\,(1+k)}{\theta},\ -\frac{1}{\theta^2}\right\},\ \left\{-2-\frac{1}{k},\ \frac{1}{k\,\theta}\right\}\right\}$$

$$\left\{\left\{k\,\theta^2,\ 2\,k\,(1+k)\,\theta^3\right\},\ \left\{2\,k\,(1+k)\,\theta^3,\ 2\,k\,\left(3+5\,k+2\,k^2\right)\,\theta^4\right\}\right\}$$

$$\begin{pmatrix} 2\,k\,(1+k) & -2\,(1+k)\,\theta \\ -2\,(1+k)\,\theta & \frac{(3+2\,k)\,\theta^2}{k} \end{pmatrix}$$

For comparison, computation of the asymptotic variance matrix without the inverse function theorem.

```
J2 = D[{k, θ} /. MoM[[1]], {{m1, m2}}] /. {m1 → M[1], m2 → M[2]};
J2.Σ.Transpose[J2] // Simplify // MatrixForm
$Assumptions = True;
```

$$\begin{pmatrix} 2\,k\,(1+k) & -2\,(1+k)\,\theta \\ -2\,(1+k)\,\theta & \frac{(3+2\,k)\,\theta^2}{k} \end{pmatrix}$$

Example 15 : MoM estimation in Pareto distribution

```
$Assumptions = α > 0 && β > 4;
M[m_] = Moment[ParetoDistribution[α, β], m];
τ[α_, β_] = {M[1], M[2]}
Solve[{α β/(-1 + β), α^2 β/(-2 + β)} == {m1, m2}, {α, β}] // Simplify
(* MoM estimators based on X, X^2*)
J = Inverse[D[τ[α, β], {{α, β}}]] // Simplify (* Jacobian of the inverse transform *)
Σ = Table[M[i + j] - M[i] M[j], {i, 1, 2}, {j, 1, 2}] // Simplify
(* Variance matrix of (X, X^2) *)
J.Σ.Transpose[J] // MatrixForm // Simplify
 (* asymptotic variance matrix n * AVar(α̂, β̂) *)
```

$$\left\{\begin{cases}\frac{\alpha\beta}{-1+\beta} & 1<\beta\\ \infty & \text{True}\end{cases},\ \begin{cases}\frac{\alpha^2\beta}{-2+\beta} & 2<\beta\\ \infty & \text{True}\end{cases}\right\}$$

$$\left\{\left\{\alpha\to\frac{m2-\sqrt{m2\left(-m1^2+m2\right)}}{m1},\ \beta\to-\frac{-m1^2+m2+\sqrt{m2\left(-m1^2+m2\right)}}{m1^2-m2}\right\},\ \left\{\alpha\to\frac{m2+\sqrt{m2\left(-m1^2+m2\right)}}{m1},\ \beta\to\frac{m1^2-m2+\sqrt{m2\left(-m1^2+m2\right)}}{m1^2-m2}\right\}\right\}$$

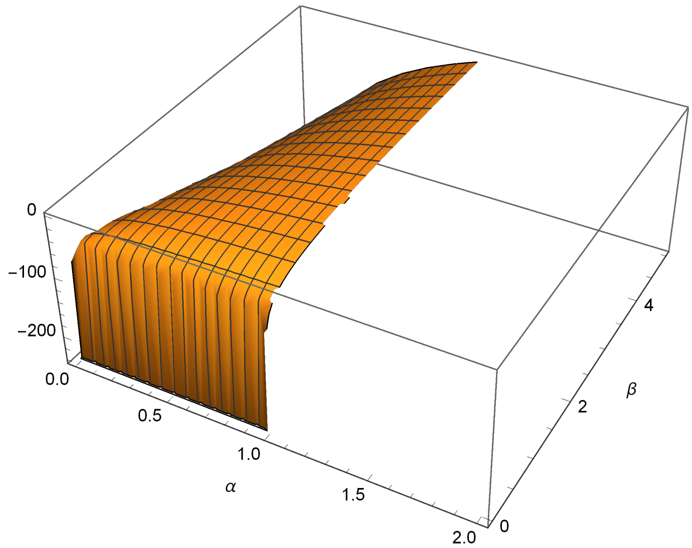
$$\left\{\left\{-2+\frac{1}{\beta}+\beta,\ -\frac{(-2+\beta)^2}{2\alpha\beta}\right\},\ \left\{\frac{(-2+\beta)(-1+\beta)^2}{\alpha},\ -\frac{(-2+\beta)^2(-1+\beta)}{2\alpha^2}\right\}\right\}$$

$$\left\{\left\{\frac{\alpha^2\beta}{(-2+\beta)(-1+\beta)^2},\ \frac{2\alpha^3\beta}{(-3+\beta)\left(2-3\beta+\beta^2\right)}\right\},\ \left\{\frac{2\alpha^3\beta}{(-3+\beta)\left(2-3\beta+\beta^2\right)},\ \frac{4\alpha^4\beta}{(-4+\beta)(-2+\beta)^2}\right\}\right\}$$

$$\begin{pmatrix}\frac{\alpha^2\left(4-3\beta+\beta^2\right)}{(-4+\beta)(-3+\beta)(-2+\beta)\beta} & \frac{\alpha(-1+\beta)\beta}{(-4+\beta)(-3+\beta)}\\ \frac{\alpha(-1+\beta)\beta}{(-4+\beta)(-3+\beta)} & \frac{2(-2+\beta)(-1+\beta)^2\beta}{(-4+\beta)(-3+\beta)}\end{pmatrix}$$

Maximum likelihood estimation

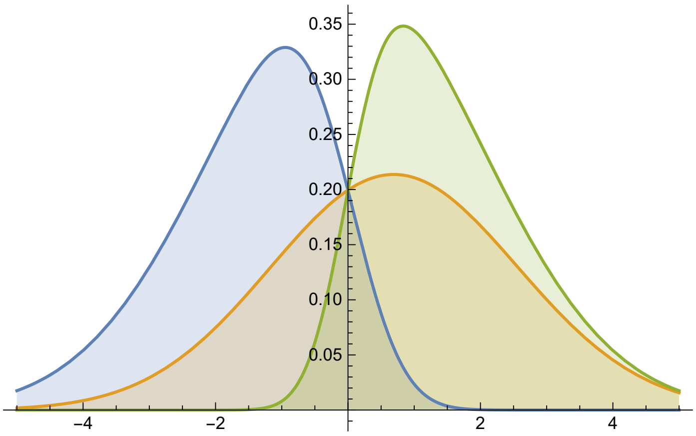
```
data = RandomVariate[ParetoDistribution[1, 4], 20];
logL[α_, β_] = Total[Log[PDF[ParetoDistribution[α, β], data]]];
Plot3D[logL[α, β], {α, 0, Min[data] + 1}, {β, 0, 5}, AxesLabel → Automatic]
$Assumptions = True;
```

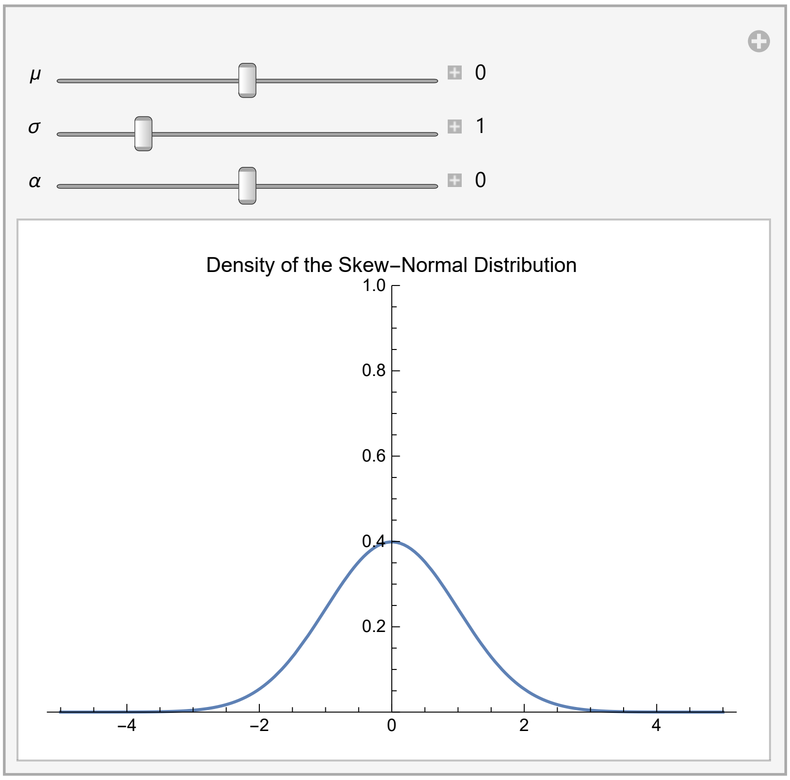


Example: MoM in skew normal distribution

Skew normal distribution SN(μ, σ, α), where $\mu \in$R is the location parameter, $\sigma \in (0,\infty)$ is the scale parameter, and $\alpha \in$R is the asymmetry parameter.

```
$Assumptions = σ > 0;
Plot[Table[PDF[SkewNormalDistribution[0, 2, α], x], {α, {-3, 0.5, 4}}] // Evaluate,
 {x, -5, 5}, Filling → Axis]
Manipulate[Plot[PDF[SkewNormalDistribution[μ, σ, α], x], {x, -5, 5},
  PlotRange → {0, 1}, PlotLabel → "Density of the Skew-Normal Distribution"],
 {{μ, 0}, -5, 5, Appearance → "Labeled"}, {{σ, 1}, 0.01, 5, Appearance → "Labeled"},
 {{α, 0}, -10, 10, Appearance → "Labeled"}]
M[k_] = Moment[SkewNormalDistribution[μ, σ, α], k];
CM[k_] = CentralMoment[SkewNormalDistribution[μ, σ, α], k];
Table[M[k], {k, 1, 3}]
Table[CM[k], {k, 1, 3}]
```





$$\left\{\mu + \frac{\sqrt{\frac{2}{\pi}}\ \alpha\ \sigma}{\sqrt{1+\alpha^2}},\ \mu^2 + \frac{2\sqrt{\frac{2}{\pi}}\ \alpha\ \mu\ \sigma}{\sqrt{1+\alpha^2}} + \sigma^2,\ \mu^3 + \frac{3\sqrt{\frac{2}{\pi}}\ \alpha\ \mu^2\ \sigma}{\sqrt{1+\alpha^2}} + 3\ \mu\ \sigma^2 + \frac{\sqrt{\frac{2}{\pi}}\ \alpha\ \left(3+2\ \alpha^2\right)\ \sigma^3}{\left(1+\alpha^2\right)^{3/2}}\right\}$$

$$\left\{0,\ \frac{\left(\pi - 2\ \alpha^2 + \pi\ \alpha^2\right)\ \sigma^2}{\pi\ \left(1+\alpha^2\right)},\ -\frac{\sqrt{2}\ \left(-4\ \alpha^3 + \pi\ \alpha^3\right)\ \sigma^3}{\pi^{3/2}\ \left(1+\alpha^2\right)^{3/2}}\right\}$$

Direct method of moments estimators, where $E\,X$, $E\,X^2$, and $E\,X^3$ are used.

```
Solve[{M[1] == m1, M[2] == m2, M[3] == m3}, {μ, σ, α}]
```

```
$Aborted
```

Method of moments with $E\,X$, $\text{Var}\,X$, and the skewness $\text{Skew}\,X$.

```
CM[3]/CM[2]^(3/2) // Simplify
Solve[{M[1] == m1, CM[2] == m2, CM[3]/CM[2]^(3/2) == m3}, {μ, σ, α}]
```

$$-\frac{\sqrt{2}\,(-4+\pi)\,\alpha^3}{\left(1+\alpha^2\right)^{3/2}\left(\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}\right)^{3/2}}$$

```
$Aborted
```

Semi-Manual computation based on $E\,X$, Var X, and Skew X. The skewness depends only on α.

```
CM[3]/CM[2]^(3/2) // Simplify
Solve[% == m3, α]
```

$$-\frac{\sqrt{2}\,(-4+\pi)\,\alpha^3}{\left(1+\alpha^2\right)^{3/2}\left(\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}\right)^{3/2}}$$

$$\Big\{\Big\{\alpha\to-\sqrt{\Big(-\frac{4\,\mathrm{m3}^2\,\pi}{-32-8\,\mathrm{m3}^2+16\,\pi+12\,\mathrm{m3}^2\,\pi-2\,\pi^2-6\,\mathrm{m3}^2\,\pi^2+\mathrm{m3}^2\,\pi^3}+\frac{4\,\mathrm{m3}^2\,\pi^2}{-32-8\,\mathrm{m3}^2+16\,\pi+12\,\mathrm{m3}^2\,\pi-2\,\pi^2-6\,\mathrm{m3}^2\,\pi^2+\mathrm{m3}^2\,\pi^3}-\frac{\mathrm{m3}^2\,\pi^3}{-32-8\,\mathrm{m3}^2+16\,\pi+12\,\mathrm{m3}^2\,\pi-2\,\pi^2-6\,\mathrm{m3}^2\,\pi^2+\mathrm{m3}^2\,\pi^3}-}$$

$$\left(192\times2^{1/3}\,\mathrm{m3}^2\,\pi^2\right)\Big/\Big(\left(-32-8\,\mathrm{m3}^2+16\,\pi+12\,\mathrm{m3}^2\,\pi-2\,\pi^2-6\,\mathrm{m3}^2\,\pi^2+\mathrm{m3}^2\,\pi^3\right)\Big(-27\,648\,\mathrm{m3}^2\,\pi^3+6912\,\mathrm{m3}^4\,\pi^3+27\,648\,\mathrm{m3}^2\,\pi^4-13\,824\,\mathrm{m3}^4\,\pi^4-10\,368\,\mathrm{m3}^2\,\pi^5+10\,800\,\mathrm{m3}^4\,\pi^5+1728\,\mathrm{m3}^2\,\pi^6-4104\,\mathrm{m3}^4\,\pi^6-108\,\mathrm{m3}^2\,\pi^7+756\,\mathrm{m3}^4\,\pi^7-54\,\mathrm{m3}^4\,\pi^8+\sqrt{\left(4\left(576\,\mathrm{m3}^2\,\pi^2-576\,\mathrm{m3}^2\,\pi^3+180\,\mathrm{m3}^2\,\pi^4-18\,\mathrm{m3}^2\,\pi^5\right)^3+\left(-27\,648\,\mathrm{m3}^2\,\pi^3+6912\,\mathrm{m3}^4\,\pi^3+27\,648\,\mathrm{m3}^2\,\pi^4-13\,824\,\mathrm{m3}^4\,\pi^4-10\,368\,\mathrm{m3}^2\,\pi^5+10\,800\,\mathrm{m3}^4\,\pi^5+1728\,\mathrm{m3}^2\,\pi^6-4104\,\mathrm{m3}^4\,\pi^6-108\,\mathrm{m3}^2\,\pi^7+756\,\mathrm{m3}^4\,\pi^7-54\,\mathrm{m3}^4\,\pi^8\right)^2\right)}\Big)^{1/3}\Big)+$$

$$\left(192\times2^{1/3}\,\mathrm{m3}^2\,\pi^3\right)\Big/\Big(\left(-32-8\,\mathrm{m3}^2+16\,\pi+12\,\mathrm{m3}^2\,\pi-2\,\pi^2-6\,\mathrm{m3}^2\,\pi^2+\mathrm{m3}^2\,\pi^3\right)\Big(-27\,648\,\mathrm{m3}^2\,\pi^3+6912\,\mathrm{m3}^4\,\pi^3+27\,648\,\mathrm{m3}^2\,\pi^4-13\,824\,\mathrm{m3}^4\,\pi^4-10\,368\,\mathrm{m3}^2\,\pi^5+10\,800\,\mathrm{m3}^4\,\pi^5+1728\,\mathrm{m3}^2\,\pi^6-4104\,\mathrm{m3}^4\,\pi^6-108\,\mathrm{m3}^2\,\pi^7+756\,\mathrm{m3}^4\,\pi^7-54\,\mathrm{m3}^4\,\pi^8+\sqrt{\left(4\left(576\,\mathrm{m3}^2\,\pi^2-576\,\mathrm{m3}^2\,\pi^3+180\,\mathrm{m3}^2\,\pi^4-18\,\mathrm{m3}^2\,\pi^5\right)^3+\left(-27\,648\,\mathrm{m3}^2\,\pi^3+6912\,\mathrm{m3}^4\,\pi^3+27\,648\,\mathrm{m3}^2\,\pi^4-13\,824\,\mathrm{m3}^4\,\pi^4-10\,368\,\mathrm{m3}^2\,\pi^5+10\,800\,\mathrm{m3}^4\,\pi^5+1728\,\mathrm{m3}^2\,\pi^6-4104\,\mathrm{m3}^4\,\pi^6-108\,\mathrm{m3}^2\,\pi^7+756\,\mathrm{m3}^4\,\pi^7-54\,\mathrm{m3}^4\,\pi^8\right)^2\right)}\Big)^{1/3}\Big)-$$

$$\left(60\times2^{1/3}\,\mathrm{m3}^2\,\pi^4\right)\Big/\Big(\left(-32-8\,\mathrm{m3}^2+16\,\pi+12\,\mathrm{m3}^2\,\pi-2\,\pi^2-6\,\mathrm{m3}^2\,\pi^2+\mathrm{m3}^2\,\pi^3\right)\Big(-27\,648\,\mathrm{m3}^2\,\pi^3+6912\,\mathrm{m3}^4\,\pi^3+27\,648\,\mathrm{m3}^2\,\pi^4-13\,824\,\mathrm{m3}^4\,\pi^4-10\,368\,\mathrm{m3}^2\,\pi^5+10\,800\,\mathrm{m3}^4\,\pi^5+1728\,\mathrm{m3}^2\,\pi^6-4104\,\mathrm{m3}^4\,\pi^6-108\,\mathrm{m3}^2\,\pi^7+756\,\mathrm{m3}^4\,\pi^7-54\,\mathrm{m3}^4\,\pi^8+\sqrt{\left(4\left(576\,\mathrm{m3}^2\,\pi^2-576\,\mathrm{m3}^2\,\pi^3+180\,\mathrm{m3}^2\,\pi^4-18\,\mathrm{m3}^2\,\pi^5\right)^3+\left(-27\,648\,\mathrm{m3}^2\,\pi^3+6912\,\mathrm{m3}^4\,\pi^3+27\,648\,\mathrm{m3}^2\,\pi^4-13\,824\,\mathrm{m3}^4\,\pi^4-10\,368\,\mathrm{m3}^2\,\pi^5+10\,800\,\mathrm{m3}^4\,\pi^5+1728\,\mathrm{m3}^2\,\pi^6-4104\,\mathrm{m3}^4\,\pi^6-108\,\mathrm{m3}^2\,\pi^7+756\,\mathrm{m3}^4\,\pi^7-54\,\mathrm{m3}^4\,\pi^8\right)^2\right)}\Big)^{1/3}\Big)+$$

$$\left(6\times2^{1/3}\,\mathrm{m3}^2\,\pi^5\right)\Big/\Big(\left(-32-8\,\mathrm{m3}^2+16\,\pi+12\,\mathrm{m3}^2\,\pi-2\,\pi^2-6\,\mathrm{m3}^2\,\pi^2+\mathrm{m3}^2\,\pi^3\right)\Big(-27\,648\,\mathrm{m3}^2\,\pi^3+6912\,\mathrm{m3}^4\,\pi^3+27\,648\,\mathrm{m3}^2\,\pi^4-13\,824\,\mathrm{m3}^4\,\pi^4-10\,368\,\mathrm{m3}^2\,\pi^5+10\,800\,\mathrm{m3}^4\,\pi^5+1728\,\mathrm{m3}^2\,\pi^6-4104\,\mathrm{m3}^4\,\pi^6-108\,\mathrm{m3}^2\,\pi^7+756\,\mathrm{m3}^4\,\pi^7-54\,\mathrm{m3}^4\,\pi^8+\sqrt{}\Big(4\left(576\,\mathrm{m3}^2\,\pi^2-576\,\mathrm{m3}^2\,\pi^3+180\,\mathrm{m3}^2\,\pi^4-18\,\mathrm{m3}^2\,\pi^5\right)^3+\Big(-27\,648\,\mathrm{m3}^2\,\pi^3+$$

$$6912\,\mathrm{m3}^4\,\pi^3 + 27\,648\,\mathrm{m3}^2\,\pi^4 - 13\,824\,\mathrm{m3}^4\,\pi^4 - 10\,368\,\mathrm{m3}^2\,\pi^5 + 10\,800\,\mathrm{m3}^4\,\pi^5 +$$
$$1728\,\mathrm{m3}^2\,\pi^6 - 4104\,\mathrm{m3}^4\,\pi^6 - 108\,\mathrm{m3}^2\,\pi^7 + 756\,\mathrm{m3}^4\,\pi^7 - 54\,\mathrm{m3}^4\,\pi^8\Big)^2\Big)\Big)^{1/3}\Big) +$$
$$\Big(-27\,648\,\mathrm{m3}^2\,\pi^3 + 6912\,\mathrm{m3}^4\,\pi^3 + 27\,648\,\mathrm{m3}^2\,\pi^4 - 13\,824\,\mathrm{m3}^4\,\pi^4 - 10\,368\,\mathrm{m3}^2\,\pi^5 +$$
$$10\,800\,\mathrm{m3}^4\,\pi^5 + 1728\,\mathrm{m3}^2\,\pi^6 - 4104\,\mathrm{m3}^4\,\pi^6 - 108\,\mathrm{m3}^2\,\pi^7 + 756\,\mathrm{m3}^4\,\pi^7 - 54\,\mathrm{m3}^4\,\pi^8 +$$
$$\sqrt{\Big(4\,\big(576\,\mathrm{m3}^2\,\pi^2 - 576\,\mathrm{m3}^2\,\pi^3 + 180\,\mathrm{m3}^2\,\pi^4 - 18\,\mathrm{m3}^2\,\pi^5\big)^3 + \Big(-27\,648\,\mathrm{m3}^2\,\pi^3 +}$$
$$6912\,\mathrm{m3}^4\,\pi^3 + 27\,648\,\mathrm{m3}^2\,\pi^4 - 13\,824\,\mathrm{m3}^4\,\pi^4 - 10\,368\,\mathrm{m3}^2\,\pi^5 + 10\,800\,\mathrm{m3}^4\,\pi^5 +$$
$$1728\,\mathrm{m3}^2\,\pi^6 - 4104\,\mathrm{m3}^4\,\pi^6 - 108\,\mathrm{m3}^2\,\pi^7 + 756\,\mathrm{m3}^4\,\pi^7 - 54\,\mathrm{m3}^4\,\pi^8\Big)^2\Big)\Big)^{1/3} \Big/$$
$$\Big(3\times 2^{1/3}\,\big(-32 - 8\,\mathrm{m3}^2 + 16\,\pi + 12\,\mathrm{m3}^2\,\pi - 2\,\pi^2 - 6\,\mathrm{m3}^2\,\pi^2 + \mathrm{m3}^2\,\pi^3\big)\Big)\Big)\Big\},$$

$$\Big\{\alpha \to \sqrt{\Big(-\frac{4\,\mathrm{m3}^2\,\pi}{-32 - 8\,\mathrm{m3}^2 + 16\,\pi + 12\,\mathrm{m3}^2\,\pi - 2\,\pi^2 - 6\,\mathrm{m3}^2\,\pi^2 + \mathrm{m3}^2\,\pi^3} +}$$
$$\frac{4\,\mathrm{m3}^2\,\pi^2}{-32 - 8\,\mathrm{m3}^2 + 16\,\pi + 12\,\mathrm{m3}^2\,\pi - 2\,\pi^2 - 6\,\mathrm{m3}^2\,\pi^2 + \mathrm{m3}^2\,\pi^3} -$$
$$\frac{\mathrm{m3}^2\,\pi^3}{-32 - 8\,\mathrm{m3}^2 + 16\,\pi + 12\,\mathrm{m3}^2\,\pi - 2\,\pi^2 - 6\,\mathrm{m3}^2\,\pi^2 + \mathrm{m3}^2\,\pi^3} -$$
$$\big(192\times 2^{1/3}\,\mathrm{m3}^2\,\pi^2\big) \Big/$$
$$\Big(\big(-32 - 8\,\mathrm{m3}^2 + 16\,\pi + 12\,\mathrm{m3}^2\,\pi - 2\,\pi^2 - 6\,\mathrm{m3}^2\,\pi^2 + \mathrm{m3}^2\,\pi^3\big)$$
$$\Big(-27\,648\,\mathrm{m3}^2\,\pi^3 + 6912\,\mathrm{m3}^4\,\pi^3 + 27\,648\,\mathrm{m3}^2\,\pi^4 - 13\,824\,\mathrm{m3}^4\,\pi^4 - 10\,368\,\mathrm{m3}^2\,\pi^5 +$$
$$10\,800\,\mathrm{m3}^4\,\pi^5 + 1728\,\mathrm{m3}^2\,\pi^6 - 4104\,\mathrm{m3}^4\,\pi^6 - 108\,\mathrm{m3}^2\,\pi^7 + 756\,\mathrm{m3}^4\,\pi^7 - 54\,\mathrm{m3}^4\,\pi^8 +$$
$$\sqrt{\Big(4\,\big(576\,\mathrm{m3}^2\,\pi^2 - 576\,\mathrm{m3}^2\,\pi^3 + 180\,\mathrm{m3}^2\,\pi^4 - 18\,\mathrm{m3}^2\,\pi^5\big)^3 + \Big(-27\,648\,\mathrm{m3}^2\,\pi^3 +}$$
$$6912\,\mathrm{m3}^4\,\pi^3 + 27\,648\,\mathrm{m3}^2\,\pi^4 - 13\,824\,\mathrm{m3}^4\,\pi^4 - 10\,368\,\mathrm{m3}^2\,\pi^5 + 10\,800\,\mathrm{m3}^4\,\pi^5 +$$
$$1728\,\mathrm{m3}^2\,\pi^6 - 4104\,\mathrm{m3}^4\,\pi^6 - 108\,\mathrm{m3}^2\,\pi^7 + 756\,\mathrm{m3}^4\,\pi^7 - 54\,\mathrm{m3}^4\,\pi^8\Big)^2\Big)\Big)^{1/3}\Big) +$$
$$\big(192\times 2^{1/3}\,\mathrm{m3}^2\,\pi^3\big) \Big/ \Big(\big(-32 - 8\,\mathrm{m3}^2 + 16\,\pi + 12\,\mathrm{m3}^2\,\pi - 2\,\pi^2 - 6\,\mathrm{m3}^2\,\pi^2 + \mathrm{m3}^2\,\pi^3\big)$$
$$\Big(-27\,648\,\mathrm{m3}^2\,\pi^3 + 6912\,\mathrm{m3}^4\,\pi^3 + 27\,648\,\mathrm{m3}^2\,\pi^4 - 13\,824\,\mathrm{m3}^4\,\pi^4 - 10\,368\,\mathrm{m3}^2\,\pi^5 +$$
$$10\,800\,\mathrm{m3}^4\,\pi^5 + 1728\,\mathrm{m3}^2\,\pi^6 - 4104\,\mathrm{m3}^4\,\pi^6 - 108\,\mathrm{m3}^2\,\pi^7 + 756\,\mathrm{m3}^4\,\pi^7 - 54\,\mathrm{m3}^4\,\pi^8 +$$
$$\sqrt{\Big(4\,\big(576\,\mathrm{m3}^2\,\pi^2 - 576\,\mathrm{m3}^2\,\pi^3 + 180\,\mathrm{m3}^2\,\pi^4 - 18\,\mathrm{m3}^2\,\pi^5\big)^3 + \Big(-27\,648\,\mathrm{m3}^2\,\pi^3 +}$$
$$6912\,\mathrm{m3}^4\,\pi^3 + 27\,648\,\mathrm{m3}^2\,\pi^4 - 13\,824\,\mathrm{m3}^4\,\pi^4 - 10\,368\,\mathrm{m3}^2\,\pi^5 + 10\,800\,\mathrm{m3}^4\,\pi^5 +$$
$$1728\,\mathrm{m3}^2\,\pi^6 - 4104\,\mathrm{m3}^4\,\pi^6 - 108\,\mathrm{m3}^2\,\pi^7 + 756\,\mathrm{m3}^4\,\pi^7 - 54\,\mathrm{m3}^4\,\pi^8\Big)^2\Big)\Big)^{1/3}\Big) -$$
$$\big(60\times 2^{1/3}\,\mathrm{m3}^2\,\pi^4\big) \Big/ \Big(\big(-32 - 8\,\mathrm{m3}^2 + 16\,\pi + 12\,\mathrm{m3}^2\,\pi - 2\,\pi^2 - 6\,\mathrm{m3}^2\,\pi^2 + \mathrm{m3}^2\,\pi^3\big)$$
$$\Big(-27\,648\,\mathrm{m3}^2\,\pi^3 + 6912\,\mathrm{m3}^4\,\pi^3 + 27\,648\,\mathrm{m3}^2\,\pi^4 - 13\,824\,\mathrm{m3}^4\,\pi^4 - 10\,368\,\mathrm{m3}^2\,\pi^5 +$$
$$10\,800\,\mathrm{m3}^4\,\pi^5 + 1728\,\mathrm{m3}^2\,\pi^6 - 4104\,\mathrm{m3}^4\,\pi^6 - 108\,\mathrm{m3}^2\,\pi^7 + 756\,\mathrm{m3}^4\,\pi^7 - 54\,\mathrm{m3}^4\,\pi^8 +$$
$$\sqrt{\Big(4\,\big(576\,\mathrm{m3}^2\,\pi^2 - 576\,\mathrm{m3}^2\,\pi^3 + 180\,\mathrm{m3}^2\,\pi^4 - 18\,\mathrm{m3}^2\,\pi^5\big)^3 + \Big(-27\,648\,\mathrm{m3}^2\,\pi^3 +}$$
$$6912\,\mathrm{m3}^4\,\pi^3 + 27\,648\,\mathrm{m3}^2\,\pi^4 - 13\,824\,\mathrm{m3}^4\,\pi^4 - 10\,368\,\mathrm{m3}^2\,\pi^5 + 10\,800\,\mathrm{m3}^4\,\pi^5 +$$
$$1728\,\mathrm{m3}^2\,\pi^6 - 4104\,\mathrm{m3}^4\,\pi^6 - 108\,\mathrm{m3}^2\,\pi^7 + 756\,\mathrm{m3}^4\,\pi^7 - 54\,\mathrm{m3}^4\,\pi^8\Big)^2\Big)\Big)^{1/3}\Big) +$$
$$\big(6\times 2^{1/3}\,\mathrm{m3}^2\,\pi^5\big) \Big/ \Big(\big(-32 - 8\,\mathrm{m3}^2 + 16\,\pi + 12\,\mathrm{m3}^2\,\pi - 2\,\pi^2 - 6\,\mathrm{m3}^2\,\pi^2 + \mathrm{m3}^2\,\pi^3\big)$$
$$\Big(-27\,648\,\mathrm{m3}^2\,\pi^3 + 6912\,\mathrm{m3}^4\,\pi^3 + 27\,648\,\mathrm{m3}^2\,\pi^4 - 13\,824\,\mathrm{m3}^4\,\pi^4 - 10\,368\,\mathrm{m3}^2\,\pi^5 +$$
$$10\,800\,\mathrm{m3}^4\,\pi^5 + 1728\,\mathrm{m3}^2\,\pi^6 - 4104\,\mathrm{m3}^4\,\pi^6 - 108\,\mathrm{m3}^2\,\pi^7 + 756\,\mathrm{m3}^4\,\pi^7 - 54\,\mathrm{m3}^4\,\pi^8 +$$
$$\sqrt{\Big(4\,\big(576\,\mathrm{m3}^2\,\pi^2 - 576\,\mathrm{m3}^2\,\pi^3 + 180\,\mathrm{m3}^2\,\pi^4 - 18\,\mathrm{m3}^2\,\pi^5\big)^3 + \Big(-27\,648\,\mathrm{m3}^2\,\pi^3 +}$$
$$6912\,\mathrm{m3}^4\,\pi^3 + 27\,648\,\mathrm{m3}^2\,\pi^4 - 13\,824\,\mathrm{m3}^4\,\pi^4 - 10\,368\,\mathrm{m3}^2\,\pi^5 + 10\,800\,\mathrm{m3}^4\,\pi^5 +$$
$$1728\,\mathrm{m3}^2\,\pi^6 - 4104\,\mathrm{m3}^4\,\pi^6 - 108\,\mathrm{m3}^2\,\pi^7 + 756\,\mathrm{m3}^4\,\pi^7 - 54\,\mathrm{m3}^4\,\pi^8\Big)^2\Big)\Big)^{1/3}\Big) +$$
$$\Big(-27\,648\,\mathrm{m3}^2\,\pi^3 + 6912\,\mathrm{m3}^4\,\pi^3 + 27\,648\,\mathrm{m3}^2\,\pi^4 - 13\,824\,\mathrm{m3}^4\,\pi^4 - 10\,368\,\mathrm{m3}^2\,\pi^5 +$$
$$10\,800\,\mathrm{m3}^4\,\pi^5 + 1728\,\mathrm{m3}^2\,\pi^6 - 4104\,\mathrm{m3}^4\,\pi^6 - 108\,\mathrm{m3}^2\,\pi^7 + 756\,\mathrm{m3}^4\,\pi^7 - 54\,\mathrm{m3}^4\,\pi^8 +$$
$$\sqrt{\Big(4\,\big(576\,\mathrm{m3}^2\,\pi^2 - 576\,\mathrm{m3}^2\,\pi^3 + 180\,\mathrm{m3}^2\,\pi^4 - 18\,\mathrm{m3}^2\,\pi^5\big)^3 + \Big(-27\,648\,\mathrm{m3}^2\,\pi^3 +}$$

$$6912\,\text{m3}^4\,\pi^3 + 27\,648\,\text{m3}^2\,\pi^4 - 13\,824\,\text{m3}^4\,\pi^4 - 10\,368\,\text{m3}^2\,\pi^5 + 10\,800\,\text{m3}^4\,\pi^5 +$$
$$1728\,\text{m3}^2\,\pi^6 - 4104\,\text{m3}^4\,\pi^6 - 108\,\text{m3}^2\,\pi^7 + 756\,\text{m3}^4\,\pi^7 - 54\,\text{m3}^4\,\pi^8\Big)^2\Big)\Big)^{1/3} \Big/$$
$$\left(3 \times 2^{1/3}\left(-32 - 8\,\text{m3}^2 + 16\,\pi + 12\,\text{m3}^2\,\pi - 2\,\pi^2 - 6\,\text{m3}^2\,\pi^2 + \text{m3}^2\,\pi^3\right)\right)\Bigg)\Big\},$$

$$\Big\{\alpha \to -\sqrt{\Bigg(-\frac{4\,\text{m3}^2\,\pi}{-32 - 8\,\text{m3}^2 + 16\,\pi + 12\,\text{m3}^2\,\pi - 2\,\pi^2 - 6\,\text{m3}^2\,\pi^2 + \text{m3}^2\,\pi^3} +}$$
$$\frac{4\,\text{m3}^2\,\pi^2}{-32 - 8\,\text{m3}^2 + 16\,\pi + 12\,\text{m3}^2\,\pi - 2\,\pi^2 - 6\,\text{m3}^2\,\pi^2 + \text{m3}^2\,\pi^3} -$$
$$\frac{\text{m3}^2\,\pi^3}{-32 - 8\,\text{m3}^2 + 16\,\pi + 12\,\text{m3}^2\,\pi - 2\,\pi^2 - 6\,\text{m3}^2\,\pi^2 + \text{m3}^2\,\pi^3} +$$
$$\left(96 \times 2^{1/3}\,\text{m3}^2\,\pi^2\right) \Big/$$
$$\Big(\left(-32 - 8\,\text{m3}^2 + 16\,\pi + 12\,\text{m3}^2\,\pi - 2\,\pi^2 - 6\,\text{m3}^2\,\pi^2 + \text{m3}^2\,\pi^3\right)$$
$$\Big(-27\,648\,\text{m3}^2\,\pi^3 + 6912\,\text{m3}^4\,\pi^3 + 27\,648\,\text{m3}^2\,\pi^4 - 13\,824\,\text{m3}^4\,\pi^4 - 10\,368\,\text{m3}^2\,\pi^5 +$$
$$10\,800\,\text{m3}^4\,\pi^5 + 1728\,\text{m3}^2\,\pi^6 - 4104\,\text{m3}^4\,\pi^6 - 108\,\text{m3}^2\,\pi^7 + 756\,\text{m3}^4\,\pi^7 - 54\,\text{m3}^4\,\pi^8 +$$
$$\sqrt{\Big(4\left(576\,\text{m3}^2\,\pi^2 - 576\,\text{m3}^2\,\pi^3 + 180\,\text{m3}^2\,\pi^4 - 18\,\text{m3}^2\,\pi^5\right)^3 + \Big(-27\,648\,\text{m3}^2\,\pi^3 +}$$
$$6912\,\text{m3}^4\,\pi^3 + 27\,648\,\text{m3}^2\,\pi^4 - 13\,824\,\text{m3}^4\,\pi^4 - 10\,368\,\text{m3}^2\,\pi^5 + 10\,800\,\text{m3}^4\,\pi^5 +$$
$$1728\,\text{m3}^2\,\pi^6 - 4104\,\text{m3}^4\,\pi^6 - 108\,\text{m3}^2\,\pi^7 + 756\,\text{m3}^4\,\pi^7 - 54\,\text{m3}^4\,\pi^8\Big)^2\Big)\Big)^{1/3}\Big) -$$
$$\left(96\,i\,2^{1/3}\sqrt{3}\,\text{m3}^2\,\pi^2\right) \Big/ \Big(\left(-32 - 8\,\text{m3}^2 + 16\,\pi + 12\,\text{m3}^2\,\pi - 2\,\pi^2 - 6\,\text{m3}^2\,\pi^2 + \text{m3}^2\,\pi^3\right)$$
$$\Big(-27\,648\,\text{m3}^2\,\pi^3 + 6912\,\text{m3}^4\,\pi^3 + 27\,648\,\text{m3}^2\,\pi^4 - 13\,824\,\text{m3}^4\,\pi^4 - 10\,368\,\text{m3}^2\,\pi^5 +$$
$$10\,800\,\text{m3}^4\,\pi^5 + 1728\,\text{m3}^2\,\pi^6 - 4104\,\text{m3}^4\,\pi^6 - 108\,\text{m3}^2\,\pi^7 + 756\,\text{m3}^4\,\pi^7 - 54\,\text{m3}^4\,\pi^8 +$$
$$\sqrt{\Big(4\left(576\,\text{m3}^2\,\pi^2 - 576\,\text{m3}^2\,\pi^3 + 180\,\text{m3}^2\,\pi^4 - 18\,\text{m3}^2\,\pi^5\right)^3 + \Big(-27\,648\,\text{m3}^2\,\pi^3 +}$$
$$6912\,\text{m3}^4\,\pi^3 + 27\,648\,\text{m3}^2\,\pi^4 - 13\,824\,\text{m3}^4\,\pi^4 - 10\,368\,\text{m3}^2\,\pi^5 + 10\,800\,\text{m3}^4\,\pi^5 +$$
$$1728\,\text{m3}^2\,\pi^6 - 4104\,\text{m3}^4\,\pi^6 - 108\,\text{m3}^2\,\pi^7 + 756\,\text{m3}^4\,\pi^7 - 54\,\text{m3}^4\,\pi^8\Big)^2\Big)\Big)^{1/3}\Big) -$$
$$\left(96 \times 2^{1/3}\,\text{m3}^2\,\pi^3\right) \Big/ \Big(\left(-32 - 8\,\text{m3}^2 + 16\,\pi + 12\,\text{m3}^2\,\pi - 2\,\pi^2 - 6\,\text{m3}^2\,\pi^2 + \text{m3}^2\,\pi^3\right)$$
$$\Big(-27\,648\,\text{m3}^2\,\pi^3 + 6912\,\text{m3}^4\,\pi^3 + 27\,648\,\text{m3}^2\,\pi^4 - 13\,824\,\text{m3}^4\,\pi^4 - 10\,368\,\text{m3}^2\,\pi^5 +$$
$$10\,800\,\text{m3}^4\,\pi^5 + 1728\,\text{m3}^2\,\pi^6 - 4104\,\text{m3}^4\,\pi^6 - 108\,\text{m3}^2\,\pi^7 + 756\,\text{m3}^4\,\pi^7 - 54\,\text{m3}^4\,\pi^8 +$$
$$\sqrt{\Big(4\left(576\,\text{m3}^2\,\pi^2 - 576\,\text{m3}^2\,\pi^3 + 180\,\text{m3}^2\,\pi^4 - 18\,\text{m3}^2\,\pi^5\right)^3 + \Big(-27\,648\,\text{m3}^2\,\pi^3 +}$$
$$6912\,\text{m3}^4\,\pi^3 + 27\,648\,\text{m3}^2\,\pi^4 - 13\,824\,\text{m3}^4\,\pi^4 - 10\,368\,\text{m3}^2\,\pi^5 + 10\,800\,\text{m3}^4\,\pi^5 +$$
$$1728\,\text{m3}^2\,\pi^6 - 4104\,\text{m3}^4\,\pi^6 - 108\,\text{m3}^2\,\pi^7 + 756\,\text{m3}^4\,\pi^7 - 54\,\text{m3}^4\,\pi^8\Big)^2\Big)\Big)^{1/3}\Big) +$$
$$\left(96\,i\,2^{1/3}\sqrt{3}\,\text{m3}^2\,\pi^3\right) \Big/ \Big(\left(-32 - 8\,\text{m3}^2 + 16\,\pi + 12\,\text{m3}^2\,\pi - 2\,\pi^2 - 6\,\text{m3}^2\,\pi^2 + \text{m3}^2\,\pi^3\right)$$
$$\Big(-27\,648\,\text{m3}^2\,\pi^3 + 6912\,\text{m3}^4\,\pi^3 + 27\,648\,\text{m3}^2\,\pi^4 - 13\,824\,\text{m3}^4\,\pi^4 - 10\,368\,\text{m3}^2\,\pi^5 +$$
$$10\,800\,\text{m3}^4\,\pi^5 + 1728\,\text{m3}^2\,\pi^6 - 4104\,\text{m3}^4\,\pi^6 - 108\,\text{m3}^2\,\pi^7 + 756\,\text{m3}^4\,\pi^7 - 54\,\text{m3}^4\,\pi^8 +$$
$$\sqrt{\Big(4\left(576\,\text{m3}^2\,\pi^2 - 576\,\text{m3}^2\,\pi^3 + 180\,\text{m3}^2\,\pi^4 - 18\,\text{m3}^2\,\pi^5\right)^3 + \Big(-27\,648\,\text{m3}^2\,\pi^3 +}$$
$$6912\,\text{m3}^4\,\pi^3 + 27\,648\,\text{m3}^2\,\pi^4 - 13\,824\,\text{m3}^4\,\pi^4 - 10\,368\,\text{m3}^2\,\pi^5 + 10\,800\,\text{m3}^4\,\pi^5 +$$
$$1728\,\text{m3}^2\,\pi^6 - 4104\,\text{m3}^4\,\pi^6 - 108\,\text{m3}^2\,\pi^7 + 756\,\text{m3}^4\,\pi^7 - 54\,\text{m3}^4\,\pi^8\Big)^2\Big)\Big)^{1/3}\Big) +$$
$$\left(30 \times 2^{1/3}\,\text{m3}^2\,\pi^4\right) \Big/ \Big(\left(-32 - 8\,\text{m3}^2 + 16\,\pi + 12\,\text{m3}^2\,\pi - 2\,\pi^2 - 6\,\text{m3}^2\,\pi^2 + \text{m3}^2\,\pi^3\right)$$
$$\Big(-27\,648\,\text{m3}^2\,\pi^3 + 6912\,\text{m3}^4\,\pi^3 + 27\,648\,\text{m3}^2\,\pi^4 - 13\,824\,\text{m3}^4\,\pi^4 - 10\,368\,\text{m3}^2\,\pi^5 +$$
$$10\,800\,\text{m3}^4\,\pi^5 + 1728\,\text{m3}^2\,\pi^6 - 4104\,\text{m3}^4\,\pi^6 - 108\,\text{m3}^2\,\pi^7 + 756\,\text{m3}^4\,\pi^7 - 54\,\text{m3}^4\,\pi^8 +$$
$$\sqrt{\Big(4\left(576\,\text{m3}^2\,\pi^2 - 576\,\text{m3}^2\,\pi^3 + 180\,\text{m3}^2\,\pi^4 - 18\,\text{m3}^2\,\pi^5\right)^3 + \Big(-27\,648\,\text{m3}^2\,\pi^3 +}$$
$$6912\,\text{m3}^4\,\pi^3 + 27\,648\,\text{m3}^2\,\pi^4 - 13\,824\,\text{m3}^4\,\pi^4 - 10\,368\,\text{m3}^2\,\pi^5 + 10\,800\,\text{m3}^4\,\pi^5 +$$
$$1728\,\text{m3}^2\,\pi^6 - 4104\,\text{m3}^4\,\pi^6 - 108\,\text{m3}^2\,\pi^7 + 756\,\text{m3}^4\,\pi^7 - 54\,\text{m3}^4\,\pi^8\Big)^2\Big)\Big)^{1/3}\Big) -$$
$$\left(30\,i\,2^{1/3}\sqrt{3}\,\text{m3}^2\,\pi^4\right) \Big/ \Big(\left(-32 - 8\,\text{m3}^2 + 16\,\pi + 12\,\text{m3}^2\,\pi - 2\,\pi^2 - 6\,\text{m3}^2\,\pi^2 + \text{m3}^2\,\pi^3\right)$$
$$\Big(-27\,648\,\text{m3}^2\,\pi^3 + 6912\,\text{m3}^4\,\pi^3 + 27\,648\,\text{m3}^2\,\pi^4 - 13\,824\,\text{m3}^4\,\pi^4 - 10\,368\,\text{m3}^2\,\pi^5 +$$

$$10800\,m3^4\pi^5 + 1728\,m3^2\pi^6 - 4104\,m3^4\pi^6 - 108\,m3^2\pi^7 + 756\,m3^4\pi^7 - 54\,m3^4\pi^8 +$$
$$\sqrt{}\,(4\,(576\,m3^2\pi^2 - 576\,m3^2\pi^3 + 180\,m3^2\pi^4 - 18\,m3^2\pi^5)^3 + (-27648\,m3^2\pi^3 +$$
$$6912\,m3^4\pi^3 + 27648\,m3^2\pi^4 - 13824\,m3^4\pi^4 - 10368\,m3^2\pi^5 + 10800\,m3^4\pi^5 +$$
$$1728\,m3^2\pi^6 - 4104\,m3^4\pi^6 - 108\,m3^2\pi^7 + 756\,m3^4\pi^7 - 54\,m3^4\pi^8)^2))^{1/3}) -$$
$$(3\times 2^{1/3}\,m3^2\pi^5) \Big/ ((-32 - 8\,m3^2 + 16\pi + 12\,m3^2\pi - 2\pi^2 - 6\,m3^2\pi^2 + m3^2\pi^3)$$
$$(-27648\,m3^2\pi^3 + 6912\,m3^4\pi^3 + 27648\,m3^2\pi^4 - 13824\,m3^4\pi^4 - 10368\,m3^2\pi^5 +$$
$$10800\,m3^4\pi^5 + 1728\,m3^2\pi^6 - 4104\,m3^4\pi^6 - 108\,m3^2\pi^7 + 756\,m3^4\pi^7 - 54\,m3^4\pi^8 +$$
$$\sqrt{}\,(4\,(576\,m3^2\pi^2 - 576\,m3^2\pi^3 + 180\,m3^2\pi^4 - 18\,m3^2\pi^5)^3 + (-27648\,m3^2\pi^3 +$$
$$6912\,m3^4\pi^3 + 27648\,m3^2\pi^4 - 13824\,m3^4\pi^4 - 10368\,m3^2\pi^5 + 10800\,m3^4\pi^5 +$$
$$1728\,m3^2\pi^6 - 4104\,m3^4\pi^6 - 108\,m3^2\pi^7 + 756\,m3^4\pi^7 - 54\,m3^4\pi^8)^2))^{1/3}) +$$
$$(3\,i\,2^{1/3}\sqrt{3}\,m3^2\pi^5) \Big/ ((-32 - 8\,m3^2 + 16\pi + 12\,m3^2\pi - 2\pi^2 - 6\,m3^2\pi^2 + m3^2\pi^3)$$
$$(-27648\,m3^2\pi^3 + 6912\,m3^4\pi^3 + 27648\,m3^2\pi^4 - 13824\,m3^4\pi^4 - 10368\,m3^2\pi^5 +$$
$$10800\,m3^4\pi^5 + 1728\,m3^2\pi^6 - 4104\,m3^4\pi^6 - 108\,m3^2\pi^7 + 756\,m3^4\pi^7 - 54\,m3^4\pi^8 +$$
$$\sqrt{}\,(4\,(576\,m3^2\pi^2 - 576\,m3^2\pi^3 + 180\,m3^2\pi^4 - 18\,m3^2\pi^5)^3 + (-27648\,m3^2\pi^3 +$$
$$6912\,m3^4\pi^3 + 27648\,m3^2\pi^4 - 13824\,m3^4\pi^4 - 10368\,m3^2\pi^5 + 10800\,m3^4\pi^5 +$$
$$1728\,m3^2\pi^6 - 4104\,m3^4\pi^6 - 108\,m3^2\pi^7 + 756\,m3^4\pi^7 - 54\,m3^4\pi^8)^2))^{1/3}) -$$
$$(-27648\,m3^2\pi^3 + 6912\,m3^4\pi^3 + 27648\,m3^2\pi^4 - 13824\,m3^4\pi^4 - 10368\,m3^2\pi^5 +$$
$$10800\,m3^4\pi^5 + 1728\,m3^2\pi^6 - 4104\,m3^4\pi^6 - 108\,m3^2\pi^7 + 756\,m3^4\pi^7 - 54\,m3^4\pi^8 +$$
$$\sqrt{}\,(4\,(576\,m3^2\pi^2 - 576\,m3^2\pi^3 + 180\,m3^2\pi^4 - 18\,m3^2\pi^5)^3 + (-27648\,m3^2\pi^3 +$$
$$6912\,m3^4\pi^3 + 27648\,m3^2\pi^4 - 13824\,m3^4\pi^4 - 10368\,m3^2\pi^5 + 10800\,m3^4\pi^5 +$$
$$1728\,m3^2\pi^6 - 4104\,m3^4\pi^6 - 108\,m3^2\pi^7 + 756\,m3^4\pi^7 - 54\,m3^4\pi^8)^2))^{1/3} \Big/$$
$$(6\times 2^{1/3}\,(-32 - 8\,m3^2 + 16\pi + 12\,m3^2\pi - 2\pi^2 - 6\,m3^2\pi^2 + m3^2\pi^3)) -$$
$$(i\,(-27648\,m3^2\pi^3 + 6912\,m3^4\pi^3 + 27648\,m3^2\pi^4 - 13824\,m3^4\pi^4 - 10368\,m3^2\pi^5 +$$
$$10800\,m3^4\pi^5 + 1728\,m3^2\pi^6 - 4104\,m3^4\pi^6 - 108\,m3^2\pi^7 + 756\,m3^4\pi^7 - 54\,m3^4\pi^8 +$$
$$\sqrt{}\,(4\,(576\,m3^2\pi^2 - 576\,m3^2\pi^3 + 180\,m3^2\pi^4 - 18\,m3^2\pi^5)^3 + (-27648\,m3^2\pi^3 +$$
$$6912\,m3^4\pi^3 + 27648\,m3^2\pi^4 - 13824\,m3^4\pi^4 - 10368\,m3^2\pi^5 + 10800\,m3^4\pi^5 +$$
$$1728\,m3^2\pi^6 - 4104\,m3^4\pi^6 - 108\,m3^2\pi^7 + 756\,m3^4\pi^7 - 54\,m3^4\pi^8)^2))^{1/3}) \Big/$$
$$(2\times 2^{1/3}\sqrt{3}\,(-32 - 8\,m3^2 + 16\pi + 12\,m3^2\pi - 2\pi^2 - 6\,m3^2\pi^2 + m3^2\pi^3))))\},$$

$$\{\alpha \to \sqrt{}\,(-\frac{4\,m3^2\pi}{-32 - 8\,m3^2 + 16\pi + 12\,m3^2\pi - 2\pi^2 - 6\,m3^2\pi^2 + m3^2\pi^3} +$$
$$\frac{4\,m3^2\pi^2}{-32 - 8\,m3^2 + 16\pi + 12\,m3^2\pi - 2\pi^2 - 6\,m3^2\pi^2 + m3^2\pi^3} -$$
$$\frac{m3^2\pi^3}{-32 - 8\,m3^2 + 16\pi + 12\,m3^2\pi - 2\pi^2 - 6\,m3^2\pi^2 + m3^2\pi^3} +$$
$$(96\times 2^{1/3}\,m3^2\pi^2) \Big/$$
$$((-32 - 8\,m3^2 + 16\pi + 12\,m3^2\pi - 2\pi^2 - 6\,m3^2\pi^2 + m3^2\pi^3)$$
$$(-27648\,m3^2\pi^3 + 6912\,m3^4\pi^3 + 27648\,m3^2\pi^4 - 13824\,m3^4\pi^4 - 10368\,m3^2\pi^5 +$$
$$10800\,m3^4\pi^5 + 1728\,m3^2\pi^6 - 4104\,m3^4\pi^6 - 108\,m3^2\pi^7 + 756\,m3^4\pi^7 - 54\,m3^4\pi^8 +$$
$$\sqrt{}\,(4\,(576\,m3^2\pi^2 - 576\,m3^2\pi^3 + 180\,m3^2\pi^4 - 18\,m3^2\pi^5)^3 + (-27648\,m3^2\pi^3 +$$
$$6912\,m3^4\pi^3 + 27648\,m3^2\pi^4 - 13824\,m3^4\pi^4 - 10368\,m3^2\pi^5 + 10800\,m3^4\pi^5 +$$
$$1728\,m3^2\pi^6 - 4104\,m3^4\pi^6 - 108\,m3^2\pi^7 + 756\,m3^4\pi^7 - 54\,m3^4\pi^8)^2))^{1/3}) -$$
$$(96\,i\,2^{1/3}\sqrt{3}\,m3^2\pi^2) \Big/ ((-32 - 8\,m3^2 + 16\pi + 12\,m3^2\pi - 2\pi^2 - 6\,m3^2\pi^2 + m3^2\pi^3)$$

$$\left(-27\,648\,\text{m3}^2\,\pi^3+6912\,\text{m3}^4\,\pi^3+27\,648\,\text{m3}^2\,\pi^4-13\,824\,\text{m3}^4\,\pi^4-10\,368\,\text{m3}^2\,\pi^5+\right.$$
$$10\,800\,\text{m3}^4\,\pi^5+1728\,\text{m3}^2\,\pi^6-4104\,\text{m3}^4\,\pi^6-108\,\text{m3}^2\,\pi^7+756\,\text{m3}^4\,\pi^7-54\,\text{m3}^4\,\pi^8+$$
$$\sqrt{\left(4\left(576\,\text{m3}^2\,\pi^2-576\,\text{m3}^2\,\pi^3+180\,\text{m3}^2\,\pi^4-18\,\text{m3}^2\,\pi^5\right)^3+\left(-27\,648\,\text{m3}^2\,\pi^3+\right.\right.}$$
$$6912\,\text{m3}^4\,\pi^3+27\,648\,\text{m3}^2\,\pi^4-13\,824\,\text{m3}^4\,\pi^4-10\,368\,\text{m3}^2\,\pi^5+10\,800\,\text{m3}^4\,\pi^5+$$
$$\left.\left.\left.1728\,\text{m3}^2\,\pi^6-4104\,\text{m3}^4\,\pi^6-108\,\text{m3}^2\,\pi^7+756\,\text{m3}^4\,\pi^7-54\,\text{m3}^4\,\pi^8\right)^2\right)\right)^{1/3}\Bigg)-$$

$$\left(96\times 2^{1/3}\,\text{m3}^2\,\pi^3\right)\Big/\left(\left(-32-8\,\text{m3}^2+16\,\pi+12\,\text{m3}^2\,\pi-2\,\pi^2-6\,\text{m3}^2\,\pi^2+\text{m3}^2\,\pi^3\right)\right.$$
$$\left(-27\,648\,\text{m3}^2\,\pi^3+6912\,\text{m3}^4\,\pi^3+27\,648\,\text{m3}^2\,\pi^4-13\,824\,\text{m3}^4\,\pi^4-10\,368\,\text{m3}^2\,\pi^5+\right.$$
$$10\,800\,\text{m3}^4\,\pi^5+1728\,\text{m3}^2\,\pi^6-4104\,\text{m3}^4\,\pi^6-108\,\text{m3}^2\,\pi^7+756\,\text{m3}^4\,\pi^7-54\,\text{m3}^4\,\pi^8+$$
$$\sqrt{\left(4\left(576\,\text{m3}^2\,\pi^2-576\,\text{m3}^2\,\pi^3+180\,\text{m3}^2\,\pi^4-18\,\text{m3}^2\,\pi^5\right)^3+\left(-27\,648\,\text{m3}^2\,\pi^3+\right.\right.}$$
$$6912\,\text{m3}^4\,\pi^3+27\,648\,\text{m3}^2\,\pi^4-13\,824\,\text{m3}^4\,\pi^4-10\,368\,\text{m3}^2\,\pi^5+10\,800\,\text{m3}^4\,\pi^5+$$
$$\left.\left.\left.1728\,\text{m3}^2\,\pi^6-4104\,\text{m3}^4\,\pi^6-108\,\text{m3}^2\,\pi^7+756\,\text{m3}^4\,\pi^7-54\,\text{m3}^4\,\pi^8\right)^2\right)\right)^{1/3}\Bigg)+$$

$$\left(96\,i\,2^{1/3}\,\sqrt{3}\,\text{m3}^2\,\pi^3\right)\Big/\left(\left(-32-8\,\text{m3}^2+16\,\pi+12\,\text{m3}^2\,\pi-2\,\pi^2-6\,\text{m3}^2\,\pi^2+\text{m3}^2\,\pi^3\right)\right.$$
$$\left(-27\,648\,\text{m3}^2\,\pi^3+6912\,\text{m3}^4\,\pi^3+27\,648\,\text{m3}^2\,\pi^4-13\,824\,\text{m3}^4\,\pi^4-10\,368\,\text{m3}^2\,\pi^5+\right.$$
$$10\,800\,\text{m3}^4\,\pi^5+1728\,\text{m3}^2\,\pi^6-4104\,\text{m3}^4\,\pi^6-108\,\text{m3}^2\,\pi^7+756\,\text{m3}^4\,\pi^7-54\,\text{m3}^4\,\pi^8+$$
$$\sqrt{\left(4\left(576\,\text{m3}^2\,\pi^2-576\,\text{m3}^2\,\pi^3+180\,\text{m3}^2\,\pi^4-18\,\text{m3}^2\,\pi^5\right)^3+\left(-27\,648\,\text{m3}^2\,\pi^3+\right.\right.}$$
$$6912\,\text{m3}^4\,\pi^3+27\,648\,\text{m3}^2\,\pi^4-13\,824\,\text{m3}^4\,\pi^4-10\,368\,\text{m3}^2\,\pi^5+10\,800\,\text{m3}^4\,\pi^5+$$
$$\left.\left.\left.1728\,\text{m3}^2\,\pi^6-4104\,\text{m3}^4\,\pi^6-108\,\text{m3}^2\,\pi^7+756\,\text{m3}^4\,\pi^7-54\,\text{m3}^4\,\pi^8\right)^2\right)\right)^{1/3}\Bigg)+$$

$$\left(30\times 2^{1/3}\,\text{m3}^2\,\pi^4\right)\Big/\left(\left(-32-8\,\text{m3}^2+16\,\pi+12\,\text{m3}^2\,\pi-2\,\pi^2-6\,\text{m3}^2\,\pi^2+\text{m3}^2\,\pi^3\right)\right.$$
$$\left(-27\,648\,\text{m3}^2\,\pi^3+6912\,\text{m3}^4\,\pi^3+27\,648\,\text{m3}^2\,\pi^4-13\,824\,\text{m3}^4\,\pi^4-10\,368\,\text{m3}^2\,\pi^5+\right.$$
$$10\,800\,\text{m3}^4\,\pi^5+1728\,\text{m3}^2\,\pi^6-4104\,\text{m3}^4\,\pi^6-108\,\text{m3}^2\,\pi^7+756\,\text{m3}^4\,\pi^7-54\,\text{m3}^4\,\pi^8+$$
$$\sqrt{\left(4\left(576\,\text{m3}^2\,\pi^2-576\,\text{m3}^2\,\pi^3+180\,\text{m3}^2\,\pi^4-18\,\text{m3}^2\,\pi^5\right)^3+\left(-27\,648\,\text{m3}^2\,\pi^3+\right.\right.}$$
$$6912\,\text{m3}^4\,\pi^3+27\,648\,\text{m3}^2\,\pi^4-13\,824\,\text{m3}^4\,\pi^4-10\,368\,\text{m3}^2\,\pi^5+10\,800\,\text{m3}^4\,\pi^5+$$
$$\left.\left.\left.1728\,\text{m3}^2\,\pi^6-4104\,\text{m3}^4\,\pi^6-108\,\text{m3}^2\,\pi^7+756\,\text{m3}^4\,\pi^7-54\,\text{m3}^4\,\pi^8\right)^2\right)\right)^{1/3}\Bigg)-$$

$$\left(30\,i\,2^{1/3}\,\sqrt{3}\,\text{m3}^2\,\pi^4\right)\Big/\left(\left(-32-8\,\text{m3}^2+16\,\pi+12\,\text{m3}^2\,\pi-2\,\pi^2-6\,\text{m3}^2\,\pi^2+\text{m3}^2\,\pi^3\right)\right.$$
$$\left(-27\,648\,\text{m3}^2\,\pi^3+6912\,\text{m3}^4\,\pi^3+27\,648\,\text{m3}^2\,\pi^4-13\,824\,\text{m3}^4\,\pi^4-10\,368\,\text{m3}^2\,\pi^5+\right.$$
$$10\,800\,\text{m3}^4\,\pi^5+1728\,\text{m3}^2\,\pi^6-4104\,\text{m3}^4\,\pi^6-108\,\text{m3}^2\,\pi^7+756\,\text{m3}^4\,\pi^7-54\,\text{m3}^4\,\pi^8+$$
$$\sqrt{\left(4\left(576\,\text{m3}^2\,\pi^2-576\,\text{m3}^2\,\pi^3+180\,\text{m3}^2\,\pi^4-18\,\text{m3}^2\,\pi^5\right)^3+\left(-27\,648\,\text{m3}^2\,\pi^3+\right.\right.}$$
$$6912\,\text{m3}^4\,\pi^3+27\,648\,\text{m3}^2\,\pi^4-13\,824\,\text{m3}^4\,\pi^4-10\,368\,\text{m3}^2\,\pi^5+10\,800\,\text{m3}^4\,\pi^5+$$
$$\left.\left.\left.1728\,\text{m3}^2\,\pi^6-4104\,\text{m3}^4\,\pi^6-108\,\text{m3}^2\,\pi^7+756\,\text{m3}^4\,\pi^7-54\,\text{m3}^4\,\pi^8\right)^2\right)\right)^{1/3}\Bigg)-$$

$$\left(3\times 2^{1/3}\,\text{m3}^2\,\pi^5\right)\Big/\left(\left(-32-8\,\text{m3}^2+16\,\pi+12\,\text{m3}^2\,\pi-2\,\pi^2-6\,\text{m3}^2\,\pi^2+\text{m3}^2\,\pi^3\right)\right.$$
$$\left(-27\,648\,\text{m3}^2\,\pi^3+6912\,\text{m3}^4\,\pi^3+27\,648\,\text{m3}^2\,\pi^4-13\,824\,\text{m3}^4\,\pi^4-10\,368\,\text{m3}^2\,\pi^5+\right.$$
$$10\,800\,\text{m3}^4\,\pi^5+1728\,\text{m3}^2\,\pi^6-4104\,\text{m3}^4\,\pi^6-108\,\text{m3}^2\,\pi^7+756\,\text{m3}^4\,\pi^7-54\,\text{m3}^4\,\pi^8+$$
$$\sqrt{\left(4\left(576\,\text{m3}^2\,\pi^2-576\,\text{m3}^2\,\pi^3+180\,\text{m3}^2\,\pi^4-18\,\text{m3}^2\,\pi^5\right)^3+\left(-27\,648\,\text{m3}^2\,\pi^3+\right.\right.}$$
$$6912\,\text{m3}^4\,\pi^3+27\,648\,\text{m3}^2\,\pi^4-13\,824\,\text{m3}^4\,\pi^4-10\,368\,\text{m3}^2\,\pi^5+10\,800\,\text{m3}^4\,\pi^5+$$
$$\left.\left.\left.1728\,\text{m3}^2\,\pi^6-4104\,\text{m3}^4\,\pi^6-108\,\text{m3}^2\,\pi^7+756\,\text{m3}^4\,\pi^7-54\,\text{m3}^4\,\pi^8\right)^2\right)\right)^{1/3}\Bigg)+$$

$$\left(3\,i\,2^{1/3}\,\sqrt{3}\,\text{m3}^2\,\pi^5\right)\Big/\left(\left(-32-8\,\text{m3}^2+16\,\pi+12\,\text{m3}^2\,\pi-2\,\pi^2-6\,\text{m3}^2\,\pi^2+\text{m3}^2\,\pi^3\right)\right.$$
$$\left(-27\,648\,\text{m3}^2\,\pi^3+6912\,\text{m3}^4\,\pi^3+27\,648\,\text{m3}^2\,\pi^4-13\,824\,\text{m3}^4\,\pi^4-10\,368\,\text{m3}^2\,\pi^5+\right.$$
$$10\,800\,\text{m3}^4\,\pi^5+1728\,\text{m3}^2\,\pi^6-4104\,\text{m3}^4\,\pi^6-108\,\text{m3}^2\,\pi^7+756\,\text{m3}^4\,\pi^7-54\,\text{m3}^4\,\pi^8+$$
$$\sqrt{\left(4\left(576\,\text{m3}^2\,\pi^2-576\,\text{m3}^2\,\pi^3+180\,\text{m3}^2\,\pi^4-18\,\text{m3}^2\,\pi^5\right)^3+\left(-27\,648\,\text{m3}^2\,\pi^3+\right.\right.}$$
$$6912\,\text{m3}^4\,\pi^3+27\,648\,\text{m3}^2\,\pi^4-13\,824\,\text{m3}^4\,\pi^4-10\,368\,\text{m3}^2\,\pi^5+10\,800\,\text{m3}^4\,\pi^5+$$
$$\left.\left.\left.1728\,\text{m3}^2\,\pi^6-4104\,\text{m3}^4\,\pi^6-108\,\text{m3}^2\,\pi^7+756\,\text{m3}^4\,\pi^7-54\,\text{m3}^4\,\pi^8\right)^2\right)\right)^{1/3}\Bigg)-$$

$$\left(-27\,648\,\text{m3}^2\,\pi^3+6912\,\text{m3}^4\,\pi^3+27\,648\,\text{m3}^2\,\pi^4-13\,824\,\text{m3}^4\,\pi^4-10\,368\,\text{m3}^2\,\pi^5+\right.$$
$$10\,800\,\text{m3}^4\,\pi^5+1728\,\text{m3}^2\,\pi^6-4104\,\text{m3}^4\,\pi^6-108\,\text{m3}^2\,\pi^7+756\,\text{m3}^4\,\pi^7-54\,\text{m3}^4\,\pi^8+$$

$$\surd\left(4\,\left(576\,m3^2\,\pi^2-576\,m3^2\,\pi^3+180\,m3^2\,\pi^4-18\,m3^2\,\pi^5\right)^3+\left(-27\,648\,m3^2\,\pi^3+\right.\right.$$
$$6912\,m3^4\,\pi^3+27\,648\,m3^2\,\pi^4-13\,824\,m3^4\,\pi^4-10\,368\,m3^2\,\pi^5+10\,800\,m3^4\,\pi^5+$$
$$\left.\left.\left.1728\,m3^2\,\pi^6-4104\,m3^4\,\pi^6-108\,m3^2\,\pi^7+756\,m3^4\,\pi^7-54\,m3^4\,\pi^8\right)^2\right)\right)^{1/3}\Big/$$
$$\left(6\times 2^{1/3}\left(-32-8\,m3^2+16\,\pi+12\,m3^2\,\pi-2\,\pi^2-6\,m3^2\,\pi^2+m3^2\,\pi^3\right)\right)-$$
$$\Big(\mathrm{i}\,\Big(-27\,648\,m3^2\,\pi^3+6912\,m3^4\,\pi^3+27\,648\,m3^2\,\pi^4-13\,824\,m3^4\,\pi^4-10\,368\,m3^2\,\pi^5+$$
$$10\,800\,m3^4\,\pi^5+1728\,m3^2\,\pi^6-4104\,m3^4\,\pi^6-108\,m3^2\,\pi^7+756\,m3^4\,\pi^7-54\,m3^4\,\pi^8+$$
$$\surd\left(4\,\left(576\,m3^2\,\pi^2-576\,m3^2\,\pi^3+180\,m3^2\,\pi^4-18\,m3^2\,\pi^5\right)^3+\left(-27\,648\,m3^2\,\pi^3+\right.\right.$$
$$6912\,m3^4\,\pi^3+27\,648\,m3^2\,\pi^4-13\,824\,m3^4\,\pi^4-10\,368\,m3^2\,\pi^5+10\,800\,m3^4\,\pi^5+$$
$$\left.\left.\left.1728\,m3^2\,\pi^6-4104\,m3^4\,\pi^6-108\,m3^2\,\pi^7+756\,m3^4\,\pi^7-54\,m3^4\,\pi^8\right)^2\right)\right)^{1/3}\Big)\Big/$$
$$\left(2\times 2^{1/3}\sqrt{3}\,\left(-32-8\,m3^2+16\,\pi+12\,m3^2\,\pi-2\,\pi^2-6\,m3^2\,\pi^2+m3^2\,\pi^3\right)\right)\Big)\Big\},$$

$$\Big\{\alpha\to-\surd\Big(-\frac{4\,m3^2\,\pi}{-32-8\,m3^2+16\,\pi+12\,m3^2\,\pi-2\,\pi^2-6\,m3^2\,\pi^2+m3^2\,\pi^3}+$$
$$\frac{4\,m3^2\,\pi^2}{-32-8\,m3^2+16\,\pi+12\,m3^2\,\pi-2\,\pi^2-6\,m3^2\,\pi^2+m3^2\,\pi^3}-$$
$$\frac{m3^2\,\pi^3}{-32-8\,m3^2+16\,\pi+12\,m3^2\,\pi-2\,\pi^2-6\,m3^2\,\pi^2+m3^2\,\pi^3}+$$
$$\left(96\times 2^{1/3}\,m3^2\,\pi^2\right)\Big/$$
$$\Big(\left(-32-8\,m3^2+16\,\pi+12\,m3^2\,\pi-2\,\pi^2-6\,m3^2\,\pi^2+m3^2\,\pi^3\right)$$
$$\Big(-27\,648\,m3^2\,\pi^3+6912\,m3^4\,\pi^3+27\,648\,m3^2\,\pi^4-13\,824\,m3^4\,\pi^4-10\,368\,m3^2\,\pi^5+$$
$$10\,800\,m3^4\,\pi^5+1728\,m3^2\,\pi^6-4104\,m3^4\,\pi^6-108\,m3^2\,\pi^7+756\,m3^4\,\pi^7-54\,m3^4\,\pi^8+$$
$$\surd\left(4\,\left(576\,m3^2\,\pi^2-576\,m3^2\,\pi^3+180\,m3^2\,\pi^4-18\,m3^2\,\pi^5\right)^3+\left(-27\,648\,m3^2\,\pi^3+\right.\right.$$
$$6912\,m3^4\,\pi^3+27\,648\,m3^2\,\pi^4-13\,824\,m3^4\,\pi^4-10\,368\,m3^2\,\pi^5+10\,800\,m3^4\,\pi^5+$$
$$\left.\left.\left.1728\,m3^2\,\pi^6-4104\,m3^4\,\pi^6-108\,m3^2\,\pi^7+756\,m3^4\,\pi^7-54\,m3^4\,\pi^8\right)^2\right)\right)^{1/3}\Big)+$$
$$\left(96\,\mathrm{i}\,2^{1/3}\sqrt{3}\,m3^2\,\pi^2\right)\Big/\Big(\left(-32-8\,m3^2+16\,\pi+12\,m3^2\,\pi-2\,\pi^2-6\,m3^2\,\pi^2+m3^2\,\pi^3\right)$$
$$\Big(-27\,648\,m3^2\,\pi^3+6912\,m3^4\,\pi^3+27\,648\,m3^2\,\pi^4-13\,824\,m3^4\,\pi^4-10\,368\,m3^2\,\pi^5+$$
$$10\,800\,m3^4\,\pi^5+1728\,m3^2\,\pi^6-4104\,m3^4\,\pi^6-108\,m3^2\,\pi^7+756\,m3^4\,\pi^7-54\,m3^4\,\pi^8+$$
$$\surd\left(4\,\left(576\,m3^2\,\pi^2-576\,m3^2\,\pi^3+180\,m3^2\,\pi^4-18\,m3^2\,\pi^5\right)^3+\left(-27\,648\,m3^2\,\pi^3+\right.\right.$$
$$6912\,m3^4\,\pi^3+27\,648\,m3^2\,\pi^4-13\,824\,m3^4\,\pi^4-10\,368\,m3^2\,\pi^5+10\,800\,m3^4\,\pi^5+$$
$$\left.\left.\left.1728\,m3^2\,\pi^6-4104\,m3^4\,\pi^6-108\,m3^2\,\pi^7+756\,m3^4\,\pi^7-54\,m3^4\,\pi^8\right)^2\right)\right)^{1/3}\Big)-$$
$$\left(96\times 2^{1/3}\,m3^2\,\pi^3\right)\Big/\Big(\left(-32-8\,m3^2+16\,\pi+12\,m3^2\,\pi-2\,\pi^2-6\,m3^2\,\pi^2+m3^2\,\pi^3\right)$$
$$\Big(-27\,648\,m3^2\,\pi^3+6912\,m3^4\,\pi^3+27\,648\,m3^2\,\pi^4-13\,824\,m3^4\,\pi^4-10\,368\,m3^2\,\pi^5+$$
$$10\,800\,m3^4\,\pi^5+1728\,m3^2\,\pi^6-4104\,m3^4\,\pi^6-108\,m3^2\,\pi^7+756\,m3^4\,\pi^7-54\,m3^4\,\pi^8+$$
$$\surd\left(4\,\left(576\,m3^2\,\pi^2-576\,m3^2\,\pi^3+180\,m3^2\,\pi^4-18\,m3^2\,\pi^5\right)^3+\left(-27\,648\,m3^2\,\pi^3+\right.\right.$$
$$6912\,m3^4\,\pi^3+27\,648\,m3^2\,\pi^4-13\,824\,m3^4\,\pi^4-10\,368\,m3^2\,\pi^5+10\,800\,m3^4\,\pi^5+$$
$$\left.\left.\left.1728\,m3^2\,\pi^6-4104\,m3^4\,\pi^6-108\,m3^2\,\pi^7+756\,m3^4\,\pi^7-54\,m3^4\,\pi^8\right)^2\right)\right)^{1/3}\Big)-$$
$$\left(96\,\mathrm{i}\,2^{1/3}\sqrt{3}\,m3^2\,\pi^3\right)\Big/\Big(\left(-32-8\,m3^2+16\,\pi+12\,m3^2\,\pi-2\,\pi^2-6\,m3^2\,\pi^2+m3^2\,\pi^3\right)$$
$$\Big(-27\,648\,m3^2\,\pi^3+6912\,m3^4\,\pi^3+27\,648\,m3^2\,\pi^4-13\,824\,m3^4\,\pi^4-10\,368\,m3^2\,\pi^5+$$
$$10\,800\,m3^4\,\pi^5+1728\,m3^2\,\pi^6-4104\,m3^4\,\pi^6-108\,m3^2\,\pi^7+756\,m3^4\,\pi^7-54\,m3^4\,\pi^8+$$
$$\surd\left(4\,\left(576\,m3^2\,\pi^2-576\,m3^2\,\pi^3+180\,m3^2\,\pi^4-18\,m3^2\,\pi^5\right)^3+\left(-27\,648\,m3^2\,\pi^3+\right.\right.$$
$$6912\,m3^4\,\pi^3+27\,648\,m3^2\,\pi^4-13\,824\,m3^4\,\pi^4-10\,368\,m3^2\,\pi^5+10\,800\,m3^4\,\pi^5+$$
$$\left.\left.\left.1728\,m3^2\,\pi^6-4104\,m3^4\,\pi^6-108\,m3^2\,\pi^7+756\,m3^4\,\pi^7-54\,m3^4\,\pi^8\right)^2\right)\right)^{1/3}\Big)+$$
$$\left(30\times 2^{1/3}\,m3^2\,\pi^4\right)\Big/\Big(\left(-32-8\,m3^2+16\,\pi+12\,m3^2\,\pi-2\,\pi^2-6\,m3^2\,\pi^2+m3^2\,\pi^3\right)$$

$$(-27648\,\mathrm{m3}^2\pi^3+6912\,\mathrm{m3}^4\pi^3+27648\,\mathrm{m3}^2\pi^4-13824\,\mathrm{m3}^4\pi^4-10368\,\mathrm{m3}^2\pi^5+$$
$$10800\,\mathrm{m3}^4\pi^5+1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8+$$
$$\sqrt{}(4\,(576\,\mathrm{m3}^2\pi^2-576\,\mathrm{m3}^2\pi^3+180\,\mathrm{m3}^2\pi^4-18\,\mathrm{m3}^2\pi^5)^3+(-27648\,\mathrm{m3}^2\pi^3+$$
$$6912\,\mathrm{m3}^4\pi^3+27648\,\mathrm{m3}^2\pi^4-13824\,\mathrm{m3}^4\pi^4-10368\,\mathrm{m3}^2\pi^5+10800\,\mathrm{m3}^4\pi^5+$$
$$1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8)^2))^{1/3})+$$

$$(30\,i\,2^{1/3}\sqrt{3}\,\mathrm{m3}^2\pi^4)\,/\,((-32-8\,\mathrm{m3}^2+16\pi+12\,\mathrm{m3}^2\pi-2\pi^2-6\,\mathrm{m3}^2\pi^2+\mathrm{m3}^2\pi^3)$$
$$(-27648\,\mathrm{m3}^2\pi^3+6912\,\mathrm{m3}^4\pi^3+27648\,\mathrm{m3}^2\pi^4-13824\,\mathrm{m3}^4\pi^4-10368\,\mathrm{m3}^2\pi^5+$$
$$10800\,\mathrm{m3}^4\pi^5+1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8+$$
$$\sqrt{}(4\,(576\,\mathrm{m3}^2\pi^2-576\,\mathrm{m3}^2\pi^3+180\,\mathrm{m3}^2\pi^4-18\,\mathrm{m3}^2\pi^5)^3+(-27648\,\mathrm{m3}^2\pi^3+$$
$$6912\,\mathrm{m3}^4\pi^3+27648\,\mathrm{m3}^2\pi^4-13824\,\mathrm{m3}^4\pi^4-10368\,\mathrm{m3}^2\pi^5+10800\,\mathrm{m3}^4\pi^5+$$
$$1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8)^2))^{1/3})-$$

$$(3\times 2^{1/3}\,\mathrm{m3}^2\pi^5)\,/\,((-32-8\,\mathrm{m3}^2+16\pi+12\,\mathrm{m3}^2\pi-2\pi^2-6\,\mathrm{m3}^2\pi^2+\mathrm{m3}^2\pi^3)$$
$$(-27648\,\mathrm{m3}^2\pi^3+6912\,\mathrm{m3}^4\pi^3+27648\,\mathrm{m3}^2\pi^4-13824\,\mathrm{m3}^4\pi^4-10368\,\mathrm{m3}^2\pi^5+$$
$$10800\,\mathrm{m3}^4\pi^5+1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8+$$
$$\sqrt{}(4\,(576\,\mathrm{m3}^2\pi^2-576\,\mathrm{m3}^2\pi^3+180\,\mathrm{m3}^2\pi^4-18\,\mathrm{m3}^2\pi^5)^3+(-27648\,\mathrm{m3}^2\pi^3+$$
$$6912\,\mathrm{m3}^4\pi^3+27648\,\mathrm{m3}^2\pi^4-13824\,\mathrm{m3}^4\pi^4-10368\,\mathrm{m3}^2\pi^5+10800\,\mathrm{m3}^4\pi^5+$$
$$1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8)^2))^{1/3})-$$

$$(3\,i\,2^{1/3}\sqrt{3}\,\mathrm{m3}^2\pi^5)\,/\,((-32-8\,\mathrm{m3}^2+16\pi+12\,\mathrm{m3}^2\pi-2\pi^2-6\,\mathrm{m3}^2\pi^2+\mathrm{m3}^2\pi^3)$$
$$(-27648\,\mathrm{m3}^2\pi^3+6912\,\mathrm{m3}^4\pi^3+27648\,\mathrm{m3}^2\pi^4-13824\,\mathrm{m3}^4\pi^4-10368\,\mathrm{m3}^2\pi^5+$$
$$10800\,\mathrm{m3}^4\pi^5+1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8+$$
$$\sqrt{}(4\,(576\,\mathrm{m3}^2\pi^2-576\,\mathrm{m3}^2\pi^3+180\,\mathrm{m3}^2\pi^4-18\,\mathrm{m3}^2\pi^5)^3+(-27648\,\mathrm{m3}^2\pi^3+$$
$$6912\,\mathrm{m3}^4\pi^3+27648\,\mathrm{m3}^2\pi^4-13824\,\mathrm{m3}^4\pi^4-10368\,\mathrm{m3}^2\pi^5+10800\,\mathrm{m3}^4\pi^5+$$
$$1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8)^2))^{1/3})-$$

$$(-27648\,\mathrm{m3}^2\pi^3+6912\,\mathrm{m3}^4\pi^3+27648\,\mathrm{m3}^2\pi^4-13824\,\mathrm{m3}^4\pi^4-10368\,\mathrm{m3}^2\pi^5+$$
$$10800\,\mathrm{m3}^4\pi^5+1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8+$$
$$\sqrt{}(4\,(576\,\mathrm{m3}^2\pi^2-576\,\mathrm{m3}^2\pi^3+180\,\mathrm{m3}^2\pi^4-18\,\mathrm{m3}^2\pi^5)^3+(-27648\,\mathrm{m3}^2\pi^3+$$
$$6912\,\mathrm{m3}^4\pi^3+27648\,\mathrm{m3}^2\pi^4-13824\,\mathrm{m3}^4\pi^4-10368\,\mathrm{m3}^2\pi^5+10800\,\mathrm{m3}^4\pi^5+$$
$$1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8)^2))^{1/3}\,/$$
$$(6\times 2^{1/3}\,(-32-8\,\mathrm{m3}^2+16\pi+12\,\mathrm{m3}^2\pi-2\pi^2-6\,\mathrm{m3}^2\pi^2+\mathrm{m3}^2\pi^3))+$$
$$(i\,(-27648\,\mathrm{m3}^2\pi^3+6912\,\mathrm{m3}^4\pi^3+27648\,\mathrm{m3}^2\pi^4-13824\,\mathrm{m3}^4\pi^4-10368\,\mathrm{m3}^2\pi^5+$$
$$10800\,\mathrm{m3}^4\pi^5+1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8+$$
$$\sqrt{}(4\,(576\,\mathrm{m3}^2\pi^2-576\,\mathrm{m3}^2\pi^3+180\,\mathrm{m3}^2\pi^4-18\,\mathrm{m3}^2\pi^5)^3+(-27648\,\mathrm{m3}^2\pi^3+$$
$$6912\,\mathrm{m3}^4\pi^3+27648\,\mathrm{m3}^2\pi^4-13824\,\mathrm{m3}^4\pi^4-10368\,\mathrm{m3}^2\pi^5+10800\,\mathrm{m3}^4\pi^5+$$
$$1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8)^2))^{1/3})\,/$$
$$(2\times 2^{1/3}\sqrt{3}\,(-32-8\,\mathrm{m3}^2+16\pi+12\,\mathrm{m3}^2\pi-2\pi^2-6\,\mathrm{m3}^2\pi^2+\mathrm{m3}^2\pi^3))))\},$$

$$\{\alpha\to\sqrt{}(-\frac{4\,\mathrm{m3}^2\pi}{-32-8\,\mathrm{m3}^2+16\pi+12\,\mathrm{m3}^2\pi-2\pi^2-6\,\mathrm{m3}^2\pi^2+\mathrm{m3}^2\pi^3}+$$
$$\frac{4\,\mathrm{m3}^2\pi^2}{-32-8\,\mathrm{m3}^2+16\pi+12\,\mathrm{m3}^2\pi-2\pi^2-6\,\mathrm{m3}^2\pi^2+\mathrm{m3}^2\pi^3}-$$
$$\frac{\mathrm{m3}^2\pi^3}{-32-8\,\mathrm{m3}^2+16\pi+12\,\mathrm{m3}^2\pi-2\pi^2-6\,\mathrm{m3}^2\pi^2+\mathrm{m3}^2\pi^3}+$$
$$(96\times 2^{1/3}\,\mathrm{m3}^2\pi^2)\,/$$

$$\Big(\big(-32-8\,\mathtt{m3}^2+16\,\pi+12\,\mathtt{m3}^2\,\pi-2\,\pi^2-6\,\mathtt{m3}^2\,\pi^2+\mathtt{m3}^2\,\pi^3\big)$$
$$\big(-27\,648\,\mathtt{m3}^2\,\pi^3+6912\,\mathtt{m3}^4\,\pi^3+27\,648\,\mathtt{m3}^2\,\pi^4-13\,824\,\mathtt{m3}^4\,\pi^4-10\,368\,\mathtt{m3}^2\,\pi^5+$$
$$10\,800\,\mathtt{m3}^4\,\pi^5+1728\,\mathtt{m3}^2\,\pi^6-4104\,\mathtt{m3}^4\,\pi^6-108\,\mathtt{m3}^2\,\pi^7+756\,\mathtt{m3}^4\,\pi^7-54\,\mathtt{m3}^4\,\pi^8+$$
$$\sqrt{}\big(4\,\big(576\,\mathtt{m3}^2\,\pi^2-576\,\mathtt{m3}^2\,\pi^3+180\,\mathtt{m3}^2\,\pi^4-18\,\mathtt{m3}^2\,\pi^5\big)^3+\big(-27\,648\,\mathtt{m3}^2\,\pi^3+$$
$$6912\,\mathtt{m3}^4\,\pi^3+27\,648\,\mathtt{m3}^2\,\pi^4-13\,824\,\mathtt{m3}^4\,\pi^4-10\,368\,\mathtt{m3}^2\,\pi^5+10\,800\,\mathtt{m3}^4\,\pi^5+$$
$$1728\,\mathtt{m3}^2\,\pi^6-4104\,\mathtt{m3}^4\,\pi^6-108\,\mathtt{m3}^2\,\pi^7+756\,\mathtt{m3}^4\,\pi^7-54\,\mathtt{m3}^4\,\pi^8\big)^2\big)\big)^{1/3}\Big)+$$

$$\big(96\,i\,2^{1/3}\,\sqrt{3}\,\mathtt{m3}^2\,\pi^2\big)\Big/\Big(\big(-32-8\,\mathtt{m3}^2+16\,\pi+12\,\mathtt{m3}^2\,\pi-2\,\pi^2-6\,\mathtt{m3}^2\,\pi^2+\mathtt{m3}^2\,\pi^3\big)$$
$$\big(-27\,648\,\mathtt{m3}^2\,\pi^3+6912\,\mathtt{m3}^4\,\pi^3+27\,648\,\mathtt{m3}^2\,\pi^4-13\,824\,\mathtt{m3}^4\,\pi^4-10\,368\,\mathtt{m3}^2\,\pi^5+$$
$$10\,800\,\mathtt{m3}^4\,\pi^5+1728\,\mathtt{m3}^2\,\pi^6-4104\,\mathtt{m3}^4\,\pi^6-108\,\mathtt{m3}^2\,\pi^7+756\,\mathtt{m3}^4\,\pi^7-54\,\mathtt{m3}^4\,\pi^8+$$
$$\sqrt{}\big(4\,\big(576\,\mathtt{m3}^2\,\pi^2-576\,\mathtt{m3}^2\,\pi^3+180\,\mathtt{m3}^2\,\pi^4-18\,\mathtt{m3}^2\,\pi^5\big)^3+\big(-27\,648\,\mathtt{m3}^2\,\pi^3+$$
$$6912\,\mathtt{m3}^4\,\pi^3+27\,648\,\mathtt{m3}^2\,\pi^4-13\,824\,\mathtt{m3}^4\,\pi^4-10\,368\,\mathtt{m3}^2\,\pi^5+10\,800\,\mathtt{m3}^4\,\pi^5+$$
$$1728\,\mathtt{m3}^2\,\pi^6-4104\,\mathtt{m3}^4\,\pi^6-108\,\mathtt{m3}^2\,\pi^7+756\,\mathtt{m3}^4\,\pi^7-54\,\mathtt{m3}^4\,\pi^8\big)^2\big)\big)^{1/3}\Big)-$$

$$\big(96\times 2^{1/3}\,\mathtt{m3}^2\,\pi^3\big)\Big/\Big(\big(-32-8\,\mathtt{m3}^2+16\,\pi+12\,\mathtt{m3}^2\,\pi-2\,\pi^2-6\,\mathtt{m3}^2\,\pi^2+\mathtt{m3}^2\,\pi^3\big)$$
$$\big(-27\,648\,\mathtt{m3}^2\,\pi^3+6912\,\mathtt{m3}^4\,\pi^3+27\,648\,\mathtt{m3}^2\,\pi^4-13\,824\,\mathtt{m3}^4\,\pi^4-10\,368\,\mathtt{m3}^2\,\pi^5+$$
$$10\,800\,\mathtt{m3}^4\,\pi^5+1728\,\mathtt{m3}^2\,\pi^6-4104\,\mathtt{m3}^4\,\pi^6-108\,\mathtt{m3}^2\,\pi^7+756\,\mathtt{m3}^4\,\pi^7-54\,\mathtt{m3}^4\,\pi^8+$$
$$\sqrt{}\big(4\,\big(576\,\mathtt{m3}^2\,\pi^2-576\,\mathtt{m3}^2\,\pi^3+180\,\mathtt{m3}^2\,\pi^4-18\,\mathtt{m3}^2\,\pi^5\big)^3+\big(-27\,648\,\mathtt{m3}^2\,\pi^3+$$
$$6912\,\mathtt{m3}^4\,\pi^3+27\,648\,\mathtt{m3}^2\,\pi^4-13\,824\,\mathtt{m3}^4\,\pi^4-10\,368\,\mathtt{m3}^2\,\pi^5+10\,800\,\mathtt{m3}^4\,\pi^5+$$
$$1728\,\mathtt{m3}^2\,\pi^6-4104\,\mathtt{m3}^4\,\pi^6-108\,\mathtt{m3}^2\,\pi^7+756\,\mathtt{m3}^4\,\pi^7-54\,\mathtt{m3}^4\,\pi^8\big)^2\big)\big)^{1/3}\Big)-$$

$$\big(96\,i\,2^{1/3}\,\sqrt{3}\,\mathtt{m3}^2\,\pi^3\big)\Big/\Big(\big(-32-8\,\mathtt{m3}^2+16\,\pi+12\,\mathtt{m3}^2\,\pi-2\,\pi^2-6\,\mathtt{m3}^2\,\pi^2+\mathtt{m3}^2\,\pi^3\big)$$
$$\big(-27\,648\,\mathtt{m3}^2\,\pi^3+6912\,\mathtt{m3}^4\,\pi^3+27\,648\,\mathtt{m3}^2\,\pi^4-13\,824\,\mathtt{m3}^4\,\pi^4-10\,368\,\mathtt{m3}^2\,\pi^5+$$
$$10\,800\,\mathtt{m3}^4\,\pi^5+1728\,\mathtt{m3}^2\,\pi^6-4104\,\mathtt{m3}^4\,\pi^6-108\,\mathtt{m3}^2\,\pi^7+756\,\mathtt{m3}^4\,\pi^7-54\,\mathtt{m3}^4\,\pi^8+$$
$$\sqrt{}\big(4\,\big(576\,\mathtt{m3}^2\,\pi^2-576\,\mathtt{m3}^2\,\pi^3+180\,\mathtt{m3}^2\,\pi^4-18\,\mathtt{m3}^2\,\pi^5\big)^3+\big(-27\,648\,\mathtt{m3}^2\,\pi^3+$$
$$6912\,\mathtt{m3}^4\,\pi^3+27\,648\,\mathtt{m3}^2\,\pi^4-13\,824\,\mathtt{m3}^4\,\pi^4-10\,368\,\mathtt{m3}^2\,\pi^5+10\,800\,\mathtt{m3}^4\,\pi^5+$$
$$1728\,\mathtt{m3}^2\,\pi^6-4104\,\mathtt{m3}^4\,\pi^6-108\,\mathtt{m3}^2\,\pi^7+756\,\mathtt{m3}^4\,\pi^7-54\,\mathtt{m3}^4\,\pi^8\big)^2\big)\big)^{1/3}\Big)+$$

$$\big(30\times 2^{1/3}\,\mathtt{m3}^2\,\pi^4\big)\Big/\Big(\big(-32-8\,\mathtt{m3}^2+16\,\pi+12\,\mathtt{m3}^2\,\pi-2\,\pi^2-6\,\mathtt{m3}^2\,\pi^2+\mathtt{m3}^2\,\pi^3\big)$$
$$\big(-27\,648\,\mathtt{m3}^2\,\pi^3+6912\,\mathtt{m3}^4\,\pi^3+27\,648\,\mathtt{m3}^2\,\pi^4-13\,824\,\mathtt{m3}^4\,\pi^4-10\,368\,\mathtt{m3}^2\,\pi^5+$$
$$10\,800\,\mathtt{m3}^4\,\pi^5+1728\,\mathtt{m3}^2\,\pi^6-4104\,\mathtt{m3}^4\,\pi^6-108\,\mathtt{m3}^2\,\pi^7+756\,\mathtt{m3}^4\,\pi^7-54\,\mathtt{m3}^4\,\pi^8+$$
$$\sqrt{}\big(4\,\big(576\,\mathtt{m3}^2\,\pi^2-576\,\mathtt{m3}^2\,\pi^3+180\,\mathtt{m3}^2\,\pi^4-18\,\mathtt{m3}^2\,\pi^5\big)^3+\big(-27\,648\,\mathtt{m3}^2\,\pi^3+$$
$$6912\,\mathtt{m3}^4\,\pi^3+27\,648\,\mathtt{m3}^2\,\pi^4-13\,824\,\mathtt{m3}^4\,\pi^4-10\,368\,\mathtt{m3}^2\,\pi^5+10\,800\,\mathtt{m3}^4\,\pi^5+$$
$$1728\,\mathtt{m3}^2\,\pi^6-4104\,\mathtt{m3}^4\,\pi^6-108\,\mathtt{m3}^2\,\pi^7+756\,\mathtt{m3}^4\,\pi^7-54\,\mathtt{m3}^4\,\pi^8\big)^2\big)\big)^{1/3}\Big)+$$

$$\big(30\,i\,2^{1/3}\,\sqrt{3}\,\mathtt{m3}^2\,\pi^4\big)\Big/\Big(\big(-32-8\,\mathtt{m3}^2+16\,\pi+12\,\mathtt{m3}^2\,\pi-2\,\pi^2-6\,\mathtt{m3}^2\,\pi^2+\mathtt{m3}^2\,\pi^3\big)$$
$$\big(-27\,648\,\mathtt{m3}^2\,\pi^3+6912\,\mathtt{m3}^4\,\pi^3+27\,648\,\mathtt{m3}^2\,\pi^4-13\,824\,\mathtt{m3}^4\,\pi^4-10\,368\,\mathtt{m3}^2\,\pi^5+$$
$$10\,800\,\mathtt{m3}^4\,\pi^5+1728\,\mathtt{m3}^2\,\pi^6-4104\,\mathtt{m3}^4\,\pi^6-108\,\mathtt{m3}^2\,\pi^7+756\,\mathtt{m3}^4\,\pi^7-54\,\mathtt{m3}^4\,\pi^8+$$
$$\sqrt{}\big(4\,\big(576\,\mathtt{m3}^2\,\pi^2-576\,\mathtt{m3}^2\,\pi^3+180\,\mathtt{m3}^2\,\pi^4-18\,\mathtt{m3}^2\,\pi^5\big)^3+\big(-27\,648\,\mathtt{m3}^2\,\pi^3+$$
$$6912\,\mathtt{m3}^4\,\pi^3+27\,648\,\mathtt{m3}^2\,\pi^4-13\,824\,\mathtt{m3}^4\,\pi^4-10\,368\,\mathtt{m3}^2\,\pi^5+10\,800\,\mathtt{m3}^4\,\pi^5+$$
$$1728\,\mathtt{m3}^2\,\pi^6-4104\,\mathtt{m3}^4\,\pi^6-108\,\mathtt{m3}^2\,\pi^7+756\,\mathtt{m3}^4\,\pi^7-54\,\mathtt{m3}^4\,\pi^8\big)^2\big)\big)^{1/3}\Big)-$$

$$\big(3\times 2^{1/3}\,\mathtt{m3}^2\,\pi^5\big)\Big/\Big(\big(-32-8\,\mathtt{m3}^2+16\,\pi+12\,\mathtt{m3}^2\,\pi-2\,\pi^2-6\,\mathtt{m3}^2\,\pi^2+\mathtt{m3}^2\,\pi^3\big)$$
$$\big(-27\,648\,\mathtt{m3}^2\,\pi^3+6912\,\mathtt{m3}^4\,\pi^3+27\,648\,\mathtt{m3}^2\,\pi^4-13\,824\,\mathtt{m3}^4\,\pi^4-10\,368\,\mathtt{m3}^2\,\pi^5+$$
$$10\,800\,\mathtt{m3}^4\,\pi^5+1728\,\mathtt{m3}^2\,\pi^6-4104\,\mathtt{m3}^4\,\pi^6-108\,\mathtt{m3}^2\,\pi^7+756\,\mathtt{m3}^4\,\pi^7-54\,\mathtt{m3}^4\,\pi^8+$$
$$\sqrt{}\big(4\,\big(576\,\mathtt{m3}^2\,\pi^2-576\,\mathtt{m3}^2\,\pi^3+180\,\mathtt{m3}^2\,\pi^4-18\,\mathtt{m3}^2\,\pi^5\big)^3+\big(-27\,648\,\mathtt{m3}^2\,\pi^3+$$
$$6912\,\mathtt{m3}^4\,\pi^3+27\,648\,\mathtt{m3}^2\,\pi^4-13\,824\,\mathtt{m3}^4\,\pi^4-10\,368\,\mathtt{m3}^2\,\pi^5+10\,800\,\mathtt{m3}^4\,\pi^5+$$
$$1728\,\mathtt{m3}^2\,\pi^6-4104\,\mathtt{m3}^4\,\pi^6-108\,\mathtt{m3}^2\,\pi^7+756\,\mathtt{m3}^4\,\pi^7-54\,\mathtt{m3}^4\,\pi^8\big)^2\big)\big)^{1/3}\Big)-$$

$$
\begin{aligned}
&\left(3\,\mathrm{i}\,2^{1/3}\sqrt{3}\,\mathrm{m3}^2\pi^5\right)\Big/\Bigg(\left(-32-8\,\mathrm{m3}^2+16\,\pi+12\,\mathrm{m3}^2\pi-2\,\pi^2-6\,\mathrm{m3}^2\pi^2+\mathrm{m3}^2\pi^3\right)\\
&\quad\Big(-27\,648\,\mathrm{m3}^2\pi^3+6912\,\mathrm{m3}^4\pi^3+27\,648\,\mathrm{m3}^2\pi^4-13\,824\,\mathrm{m3}^4\pi^4-10\,368\,\mathrm{m3}^2\pi^5+\\
&\qquad 10\,800\,\mathrm{m3}^4\pi^5+1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8+\\
&\qquad\sqrt{\Big(4\left(576\,\mathrm{m3}^2\pi^2-576\,\mathrm{m3}^2\pi^3+180\,\mathrm{m3}^2\pi^4-18\,\mathrm{m3}^2\pi^5\right)^3+\left(-27\,648\,\mathrm{m3}^2\pi^3+\right.}\\
&\qquad\qquad 6912\,\mathrm{m3}^4\pi^3+27\,648\,\mathrm{m3}^2\pi^4-13\,824\,\mathrm{m3}^4\pi^4-10\,368\,\mathrm{m3}^2\pi^5+10\,800\,\mathrm{m3}^4\pi^5+\\
&\qquad\qquad \left.1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8\right)^2\Big)\Big)^{1/3}\Bigg)-\\
&\Big(-27\,648\,\mathrm{m3}^2\pi^3+6912\,\mathrm{m3}^4\pi^3+27\,648\,\mathrm{m3}^2\pi^4-13\,824\,\mathrm{m3}^4\pi^4-10\,368\,\mathrm{m3}^2\pi^5+\\
&\quad 10\,800\,\mathrm{m3}^4\pi^5+1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8+\\
&\quad\sqrt{\Big(4\left(576\,\mathrm{m3}^2\pi^2-576\,\mathrm{m3}^2\pi^3+180\,\mathrm{m3}^2\pi^4-18\,\mathrm{m3}^2\pi^5\right)^3+\left(-27\,648\,\mathrm{m3}^2\pi^3+\right.}\\
&\qquad 6912\,\mathrm{m3}^4\pi^3+27\,648\,\mathrm{m3}^2\pi^4-13\,824\,\mathrm{m3}^4\pi^4-10\,368\,\mathrm{m3}^2\pi^5+10\,800\,\mathrm{m3}^4\pi^5+\\
&\qquad \left.1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8\right)^2\Big)\Big)^{1/3}\Big/\\
&\left(6\times 2^{1/3}\left(-32-8\,\mathrm{m3}^2+16\,\pi+12\,\mathrm{m3}^2\pi-2\,\pi^2-6\,\mathrm{m3}^2\pi^2+\mathrm{m3}^2\pi^3\right)\right)+\\
&\Bigg(\mathrm{i}\,\Big(-27\,648\,\mathrm{m3}^2\pi^3+6912\,\mathrm{m3}^4\pi^3+27\,648\,\mathrm{m3}^2\pi^4-13\,824\,\mathrm{m3}^4\pi^4-10\,368\,\mathrm{m3}^2\pi^5+\\
&\qquad 10\,800\,\mathrm{m3}^4\pi^5+1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8+\\
&\qquad\sqrt{\Big(4\left(576\,\mathrm{m3}^2\pi^2-576\,\mathrm{m3}^2\pi^3+180\,\mathrm{m3}^2\pi^4-18\,\mathrm{m3}^2\pi^5\right)^3+\left(-27\,648\,\mathrm{m3}^2\pi^3+\right.}\\
&\qquad\qquad 6912\,\mathrm{m3}^4\pi^3+27\,648\,\mathrm{m3}^2\pi^4-13\,824\,\mathrm{m3}^4\pi^4-10\,368\,\mathrm{m3}^2\pi^5+10\,800\,\mathrm{m3}^4\pi^5+\\
&\qquad\qquad \left.1728\,\mathrm{m3}^2\pi^6-4104\,\mathrm{m3}^4\pi^6-108\,\mathrm{m3}^2\pi^7+756\,\mathrm{m3}^4\pi^7-54\,\mathrm{m3}^4\pi^8\right)^2\Big)\Big)^{1/3}\Big/\\
&\left(2\times 2^{1/3}\sqrt{3}\left(-32-8\,\mathrm{m3}^2+16\,\pi+12\,\mathrm{m3}^2\pi-2\,\pi^2-6\,\mathrm{m3}^2\pi^2+\mathrm{m3}^2\pi^3\right)\right)\Bigg)\}\}
\end{aligned}
$$

Manual computation gives closed form expressions for the estimators of the parameters.

Asymptotic distribution of $\hat{\mu}$, $(\hat{\sigma})^2$, $\hat{\alpha}$.

Direct approach. A two-step procedure.

1. Find the joint asymptotic distribution of the sample mean, sample variance, and the sample skewness. (3-dimensional Δ-theorem).

2. Use the 3-dimensional Δ-theorem again to find the asymptotic distribution of the estimators.

Step 1.

```
Σ = Table[M[i + j] - M[i] M[j], {i, 1, 3}, {j, 1, 3}];
(* Variance matrix of (EX, EX², EX³) *)
g[a_, b_, c_] = {a, b - a², (c - 3 a b + 2 a³)/(b - a²)^(3/2)};
(* Function that takes the sample averages of X, X²,
X³ to the sample mean, sample variance, and sample skewness *)
J1 = D[g[a, b, c], {{a, b, c}}] /. {a → M[1], b → M[2], c → M[3]} // Simplify;
V0 = J1.Σ.Transpose[J1] // Simplify;
MatrixForm[V0] (* n AVar(sample mean, sample variance, sample skewness) *)
```

$$
\begin{pmatrix}
\frac{\left(\pi-2\alpha^2+\pi\alpha^2\right)\sigma^2}{\pi\left(1+\alpha^2\right)} & -\frac{\sqrt{2}\left(-4+\pi\right)\alpha^3\sigma^3}{\pi^{3/2}\left(1+\alpha^2\right)^{3/2}} & \cdots\\
-\frac{\sqrt{2}\left(-4+\pi\right)\alpha^3\sigma^3}{\pi^{3/2}\left(1+\alpha^2\right)^{3/2}} & \frac{2\left(-4\pi\alpha^2-8\alpha^4+\pi^2\left(1+\alpha^2\right)^2\right)\sigma^4}{\pi^2\left(1+\alpha^2\right)^2} & \cdots\\
\frac{\sqrt{\pi}\,\alpha^4\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}\left(-8\left(3+2\alpha^2\right)+\pi\left(8+5\alpha^2\right)\right)\sigma}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^3} & \frac{\sqrt{2}\,\alpha^3\sqrt{1+\alpha^2}\left(\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}\right)^{3/2}\left(-3\pi^2\left(1+\alpha^2\right)+16\alpha^2\left(3+2\alpha^2\right)-2\pi\left(-6+2\alpha^2+5\alpha^4\right)\right)\sigma^2}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^4} & \cdots
\end{pmatrix}
$$

Check: what does the computation give for α=0? (This is the normal distribution)

```
V0 /. α → 0 // MatrixForm
```

$$\begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & 2\sigma^4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Step 2.

$$\delta[a\_, b\_, c\_] = \frac{c^{1/3}}{\sqrt{\left(\frac{2}{\pi}\left(c^{2/3} + \left(\frac{4-\pi}{2}\right)^{2/3}\right)\right)}};$$

$$h[a\_, b\_, c\_] = \left\{a - \sqrt{\left(\frac{2}{\pi}\right)}\, \delta[a, b, c] \sqrt{\left(\frac{b}{1 - \frac{2}{\pi}\delta[a, b, c]^2}\right)}, \frac{b}{1 - \frac{2}{\pi}\delta[a, b, c]^2}, \frac{\delta[a, b, c]}{\sqrt{\left(1 - \delta[a, b, c]^2\right)}}\right\};$$

```
 (* Function that takes the sample mean, variance,
and skewness to the estimators *)
```

$$J2 = D[h[a, b, c], \{\{a, b, c\}\}] \,/.\, \left\{a \to M[1], b \to CM[2], c \to \frac{CM[3]}{CM[2]^{3/2}}\right\} \,//\, Simplify;$$

```
(* The Jacobian of function h at the population mean, variance, and skewness *)
V = J2.V0.Transpose[J2] // Simplify;
V // MatrixForm (* Finally, n * the asymptotic variance matrix of our estimators *)
V
```

$$\left(\frac{9\left(\pi-2\alpha^2+\pi\alpha^2\right)}{1+\alpha^2}+\frac{9\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\left(-4\pi\alpha^2-8\alpha^4+\pi^2\left(1+\alpha^2\right)^2\right)}{\left(1+\alpha^2\right)\left(\pi-2\alpha^2+\pi\alpha^2\right)}+\frac{18\left(-4+\pi\right)\alpha^3\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{1/3}\sqrt{1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}}}{\left(1+\alpha^2\right)^{3/2}\sqrt{\frac{\left(\pi-2\alpha^2+\pi\alpha^2\right)\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\right)}{1+\alpha^2}}}-\frac{24\pi\alpha\left(3+2\alpha^2\right)\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-\cdots}{\cdots}}}{\left(\pi-2\alpha^2+\pi\cdots\right.}\cdots\right.}{\left(-4+\pi\right)\left(1+\alpha^2\right)^{3/2}\sqrt{\frac{\left(\pi-\cdots\right.}{\cdots}}}\cdots$$

$$\sqrt{2}\left(-\frac{6\pi\alpha\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\left(-8\left(3+2\alpha^2\right)+\pi\left(8+5\alpha^2\right)\right)}{\left(-4+\pi\right)\left(1+\alpha^2\right)^{3/2}}-\frac{9\left(-4+\pi\right)\alpha^3\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\right)}{\left(1+\alpha^2\right)^{3/2}}-\frac{3\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{1/3}\sqrt{1+2\left(\frac{\alpha\cdots}{\cdots}\right.\cdots}}{\cdots}\cdots\right.$$

$$\left\{\left\{\frac{1}{9\pi}\left(\frac{9\left(\pi-2\alpha^2+\pi\alpha^2\right)}{1+\alpha^2}+\frac{9\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\left(-4\pi\alpha^2-8\alpha^4+\pi^2\left(1+\alpha^2\right)^2\right)}{\left(1+\alpha^2\right)\left(\pi-2\alpha^2+\pi\alpha^2\right)}+\right.\right.\right.$$

$$\left(18\left(-4+\pi\right)\alpha^3\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{1/3}\sqrt{1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}}\right)\Big/$$

$$\left(\left(1+\alpha^2\right)^{3/2}\sqrt{\frac{\left(\pi-2\alpha^2+\pi\alpha^2\right)\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\right)}{1+\alpha^2}}\right)-$$

$$\left(24\pi\alpha\left(3+2\alpha^2\right)\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{1/3}\sqrt{1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}}\right)\Big/$$

$$\left(\left(-4+\pi\right)\left(1+\alpha^2\right)^{3/2}\sqrt{\frac{\left(\pi-2\alpha^2+\pi\alpha^2\right)\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\right)}{1+\alpha^2}}\right)+$$

$$\left(3\pi^2\alpha\left(8+5\alpha^2\right)\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{1/3}\sqrt{1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}}\right)\Big/$$

$$\left((-4+\pi)\left(1+\alpha^2\right)^{3/2}\sqrt{\frac{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\right)}{1+\alpha^2}}\right.+$$

$$\left(3\,\pi\,\alpha\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{1/3}\left(-8\left(3+2\,\alpha^2\right)+\pi\left(8+5\,\alpha^2\right)\right)\right.$$

$$\left.\left.\sqrt{1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}}\right)\right/$$

$$\left((-4+\pi)\left(1+\alpha^2\right)^{3/2}\sqrt{\frac{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\right)}{1+\alpha^2}}\right.+$$

$$\left(6\,\pi\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\left(3\,\pi^2\left(1+\alpha^2\right)-16\,\alpha^2\left(3+2\,\alpha^2\right)+2\,\pi\left(-6+2\,\alpha^2+5\,\alpha^4\right)\right)\right)\Big/$$

$$\left((-4+\pi)\left(1+\alpha^2\right)\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)\right)+$$

$$\left(\pi^2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\left(6\,\pi^3\left(1+\alpha^2\right)^5-32\,\alpha^6\left(3+2\,\alpha^2\right)^2+\right.\right.$$

$$\left.\left.4\,\pi\,\alpha^4\left(6+48\,\alpha^2+65\,\alpha^4+23\,\alpha^6\right)-\pi^2\,\alpha^2\left(60+168\,\alpha^2+209\,\alpha^4+136\,\alpha^6+35\,\alpha^8\right)\right)\right)\Big/$$

$$\left.\left((-4+\pi)^2\,\alpha^6\left(1+\alpha^2\right)\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)\right)\right)\sigma^2,\ \frac{1}{9\,(-4+\pi)\,\pi^{3/2}\left(1+\alpha^2\right)\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}$$

$$\sqrt{2}\left(-3\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)\sqrt{\frac{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\right)}{1+\alpha^2}}\right.$$

$$\left(3\,(4-\pi)\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{1/3}\right.$$

$$\left(4\,\pi\,\alpha^2+8\,\alpha^4-\pi^2\left(1+\alpha^2\right)^2\right)\sqrt{1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}}\ +$$

$$3\,(-4+\pi)^2\,\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\right)}{1+\alpha^2}}\ +$$

$$\pi\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{1/3}\sqrt{1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}}$$

$$\left(3\pi^2\left(1+\alpha^2\right)-16\alpha^2\left(3+2\alpha^2\right)+2\pi\left(-6+2\alpha^2+5\alpha^4\right)\right)\Bigg) +$$

$$\frac{1}{\alpha^3\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}2\pi\left(-\frac{3\alpha^7\left(\pi-2\alpha^2+\pi\alpha^2\right)\left(-8\left(3+2\alpha^2\right)+\pi\left(8+5\alpha^2\right)\right)}{\left(1+\alpha^2\right)\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{1/3}}-\right.$$

$$\left(3\alpha^6\sqrt{\left(\frac{1}{1+\alpha^2}\left(\pi-2\alpha^2+\pi\alpha^2\right)\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\right)\right)}\right)$$

$$\left(3\pi^2\left(1+\alpha^2\right)-16\alpha^2\left(3+2\alpha^2\right)+2\pi\left(-6+2\alpha^2+5\alpha^4\right)\right)\Bigg)\Bigg/$$

$$\left(\sqrt{1+\alpha^2}\sqrt{1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}}\right)-$$

$$\left(\pi\sqrt{\left(\frac{1}{1+\alpha^2}\left(\pi-2\alpha^2+\pi\alpha^2\right)\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\right)\right)}\right)$$

$$\left(6\pi^3\left(1+\alpha^2\right)^5-32\alpha^6\left(3+2\alpha^2\right)^2+4\pi\alpha^4\left(6+48\alpha^2+65\alpha^4+23\alpha^6\right)-\right.$$

$$\left.\pi^2\alpha^2\left(60+168\alpha^2+209\alpha^4+136\alpha^6+35\alpha^8\right)\right)\Bigg/$$

$$\left.\left.\left.\left((-4+\pi)\sqrt{1+\alpha^2}\sqrt{1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}}\right)\right)\right)\sigma^3,$$

$$-\left(\left(\pi\left(6\alpha^4\left(\pi-2\alpha^2+\pi\alpha^2\right)^2\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}\left(-8\left(3+2\alpha^2\right)+\pi\left(8+5\alpha^2\right)\right)+\right.\right.\right.$$

$$\left(6\alpha^6\sqrt{\left(\frac{1}{1+\alpha^2}\left(\pi-2\alpha^2+\pi\alpha^2\right)\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\right)\right)}\right.$$

$$\left.\left(3\pi^2\left(1+\alpha^2\right)-16\alpha^2\left(3+2\alpha^2\right)+2\pi\left(-6+2\alpha^2+5\alpha^4\right)\right)\right)\Big/$$

$$\left(\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\sqrt{1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}}\right)+$$

$$\left(2\pi\sqrt{\left(\frac{1}{1+\alpha^2}\left(\pi-2\alpha^2+\pi\alpha^2\right)\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\right)\right)}\right.$$

$$\left(6\pi^3\left(1+\alpha^2\right)^5-32\alpha^6\left(3+2\alpha^2\right)^2+4\pi\alpha^4\left(6+48\alpha^2+65\alpha^4+23\alpha^6\right)-\right.$$

$$\left.\left.\pi^2\alpha^2\left(60+168\alpha^2+209\alpha^4+136\alpha^6+35\alpha^8\right)\right)\right)\Big/$$

$$\Bigg((-4+\pi)\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{(\pi-2\alpha^2+\pi\alpha^2)^2}\right)^{2/3}\sqrt{1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{(\pi-2\alpha^2+\pi\alpha^2)^2}\right)^{2/3}}\Bigg)\Bigg)\,\sigma\Bigg)\Big/$$

$$\Bigg(18\,(4-\pi)^{1/3}\,(\pi-2\alpha^2+\pi\alpha^2)^5\left(-\frac{(-4+\pi)\,\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{(\pi-2\alpha^2+\pi\alpha^2)^2}\right)^{2/3}$$

$$\sqrt{2+4\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{(\pi-2\alpha^2+\pi\alpha^2)^2}\right)^{2/3}}$$

$$\left(-1-2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{(\pi-2\alpha^2+\pi\alpha^2)^2}\right)^{2/3}+\pi\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{(\pi-2\alpha^2+\pi\alpha^2)^2}\right)^{2/3}\right)$$

$$\sqrt{1-\frac{\pi\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{(\pi-2\alpha^2+\pi\alpha^2)^2}\right)^{2/3}}{1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{(\pi-2\alpha^2+\pi\alpha^2)^2}\right)^{2/3}}}\Bigg)\Bigg)\Big\},$$

$$\Big\{\frac{1}{9\,\pi^{3/2}}\sqrt{2}\;\Bigg(-\Bigg(\Bigg(6\,\pi\,\alpha\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{(\pi-2\alpha^2+\pi\alpha^2)^2}\right)^{2/3}\left(-8\,(3+2\alpha^2)+\pi\,(8+5\alpha^2)\right)\Bigg)\Big/$$

$$\left(\left(-4+\pi\right)\left(1+\alpha^2\right)^{3/2}\right)\Bigg) - \frac{9\left(-4+\pi\right)\alpha^3\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\right)}{\left(1+\alpha^2\right)^{3/2}} -$$

$$\left(3\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{1/3}\sqrt{1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}}\right.$$

$$\left(3\left(-4\pi\alpha^2-8\alpha^4+\pi^2\left(1+\alpha^2\right)^2\right)\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\right)+\right.$$

$$\frac{1}{-4+\pi}2\pi\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}$$

$$\left.\left.\left(3\pi^2\left(1+\alpha^2\right)-16\alpha^2\left(3+2\alpha^2\right)+2\pi\left(-6+2\alpha^2+5\alpha^4\right)\right)\right)\right)\Big/$$

$$\left(\left(1+\alpha^2\right)^2\sqrt{\frac{\left(\pi-2\alpha^2+\pi\alpha^2\right)\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\right)}{1+\alpha^2}}\right)+$$

$$\left(\pi\sqrt{\frac{\left(\pi-2\alpha^2+\pi\alpha^2\right)\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\right)}{1+\alpha^2}}\right.$$

$$\left(3\,\alpha^3\sqrt{1+\alpha^2}\left(\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}\right)^{3/2}\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\right)\left(-3\,\pi^2\left(1+\alpha^2\right)+16\,\alpha^2\left(3+2\,\alpha^2\right)-2\,\pi\left(-6+2\,\alpha^2+5\,\alpha^4\right)\right)-\left(2\,\pi\left(6\,\pi^3\left(1+\alpha^2\right)^5-32\,\alpha^6\left(3+2\,\alpha^2\right)^2+4\,\pi\,\alpha^4\left(6+48\,\alpha^2+65\,\alpha^4+23\,\alpha^6\right)-\pi^2\,\alpha^2\left(60+168\,\alpha^2+209\,\alpha^4+136\,\alpha^6+35\,\alpha^8\right)\right)\right)\Big/\left((-4+\pi)\left(1+\alpha^2\right)\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{1/3}\right)\right)\Big/\left((-4+\pi)\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^4\right.$$

$$\left.\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\sqrt{1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}}\right)\sigma^3,$$

$$\frac{1}{9\,\pi^2\left(1+\alpha^2\right)^2}2\left(3\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\right)\right.$$

$$\left(3\left(-4\,\pi\,\alpha^2-8\,\alpha^4+\pi^2\left(1+\alpha^2\right)^2\right)\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\right)+\frac{1}{-4+\pi}2\,\pi\right.$$

$$\left.\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\left(3\,\pi^2\left(1+\alpha^2\right)-16\,\alpha^2\left(3+2\,\alpha^2\right)+2\,\pi\left(-6+2\,\alpha^2+5\,\alpha^4\right)\right)\right)-$$

$$\left(2\,\pi\left(1+\alpha^2\right)\left(3\,\alpha^3\sqrt{1+\alpha^2}\left(\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}\right)^{3/2}\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\right)\right.\right.$$

$$\left(-3\,\pi^2\left(1+\alpha^2\right)+16\,\alpha^2\left(3+2\,\alpha^2\right)-2\,\pi\left(-6+2\,\alpha^2+5\,\alpha^4\right)\right)-$$

$$\left(2\,\pi\left(6\,\pi^3\left(1+\alpha^2\right)^5-32\,\alpha^6\left(3+2\,\alpha^2\right)^2+4\,\pi\,\alpha^4\left(6+48\,\alpha^2+65\,\alpha^4+23\,\alpha^6\right)-\pi^2\right.\right.$$

$$\alpha^2 \left(60 + 168\,\alpha^2 + 209\,\alpha^4 + 136\,\alpha^6 + 35\,\alpha^8\right)\Big)\Big) \Big/$$

$$\left((-4+\pi)\,\left(1+\alpha^2\right)\left(\frac{\alpha^3\,\sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{1/3}\right)\Bigg)\Bigg)\Bigg) \Big/$$

$$\left((-4+\pi)\,\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^3\left(\frac{\alpha^3\,\sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{1/3}\right)\Bigg)\Bigg)\,\sigma^4,$$

$$-\Bigg(\Bigg(\sqrt{\pi}\,\Bigg(3\,\alpha^3\,\sqrt{1+\alpha^2}\,\left(\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}\right)^{3/2}\left(1+2\left(\frac{\alpha^3\,\sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\right)$$

$$\left(-3\,\pi^2\left(1+\alpha^2\right)+16\,\alpha^2\left(3+2\,\alpha^2\right)-2\,\pi\left(-6+2\,\alpha^2+5\,\alpha^4\right)\right) -$$
$$\Big(2\,\pi\,\Big(6\,\pi^3\left(1+\alpha^2\right)^5-32\,\alpha^6\left(3+2\,\alpha^2\right)^2+4\,\pi\,\alpha^4\left(6+48\,\alpha^2+65\,\alpha^4+23\,\alpha^6\right)-$$
$$\pi^2\,\alpha^2\left(60+168\,\alpha^2+209\,\alpha^4+136\,\alpha^6+35\,\alpha^8\right)\Big)\Big) \Big/$$

$$\left((-4+\pi)\,\left(1+\alpha^2\right)\left(\frac{\alpha^3\,\sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{1/3}\right)\Bigg)\Bigg)\,\sigma^2\Bigg) \Big/$$

$$\left(9\,(4-\pi)^{1/3}\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^4\left(-\frac{(-4+\pi)\,\alpha^3\,\sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\right.$$

$$\sqrt{1+2\left(\frac{\alpha^3\,\sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}}$$

$$\left(-1-2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}+\pi\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\right)$$

$$\left.\left.\sqrt{1-\frac{\pi\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}}{1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}}}\right)\right)\Bigg\},$$

$$\Bigg\{\pi-\left(\left(3\alpha^4\left(1+\alpha^2\right)^2\left(\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}\right)^{3/2}\left(-8\left(3+2\alpha^2\right)+\pi\left(8+5\alpha^2\right)\right)\right)\middle/\right.$$

$$\left.\left(\sqrt{1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}}\right)\right)-$$

$$\left(3\alpha^6\left(3\pi^2\left(1+\alpha^2\right)-16\alpha^2\left(3+2\alpha^2\right)+2\pi\left(-6+2\alpha^2+5\alpha^4\right)\right)\right)\Big/$$

$$\left(\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\sqrt{\frac{\left(\pi-2\alpha^2+\pi\alpha^2\right)\left(1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\right)}{1+\alpha^2}}\right)+$$

$$\left(\pi\left(-6\pi^3\left(1+\alpha^2\right)^5+32\alpha^6\left(3+2\alpha^2\right)^2-4\pi\alpha^4\left(6+48\alpha^2+65\alpha^4+23\alpha^6\right)+\pi^2\alpha^2\right.\right.$$

$$\left.\left.\left(60+168\alpha^2+209\alpha^4+136\alpha^6+35\alpha^8\right)\right)\right)\Big/\left((-4+\pi)\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\right.$$

$$\left.\left.\left.\sqrt{\frac{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)\left(1+2\left(\frac{\alpha^3\,\sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\right)}{1+\alpha^2}}\right)\right)\sigma\right)\Big/$$

$$\left(9\,(4-\pi)^{1/3}\left(1+\alpha^2\right)\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^4\left(-\frac{(-4+\pi)\,\alpha^3\,\sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\right.$$

$$\left(-1-2\left(\frac{\alpha^3\,\sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}+\pi\left(\frac{\alpha^3\,\sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\right)$$

$$\left.\sqrt{2-\frac{2\,\pi\left(\frac{\alpha^3\,\sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}}{1+2\left(\frac{\alpha^3\,\sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}}}\right),$$

$$-\left(\left(\sqrt{\pi}\left(192\,\alpha^5\left(3+2\,\alpha^2\right)\left(-\alpha+\frac{\alpha^3}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)\left(\frac{\alpha^3\,\sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}}\right)+\right.\right.\right.$$

$$3\,\pi^3\left(1+\alpha^2\right)\left(4+12\,\alpha^2+12\,\alpha^4+4\,\alpha^6+\frac{3\,\alpha^6}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)\left(\frac{\alpha^3\,\sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}}\right)-$$

$$8\,\pi\,\alpha^3 \left(18\,\alpha - 12\,\alpha^3 - 19\,\alpha^5 - \frac{18\,\alpha^3}{\left(\pi - 2\,\alpha^2 + \pi\,\alpha^2\right) \left(\frac{\alpha^3 \sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}} + \right.$$

$$\left. \frac{24\,\alpha^5}{\left(\pi - 2\,\alpha^2 + \pi\,\alpha^2\right) \left(\frac{\alpha^3 \sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}} + \frac{27\,\alpha^7}{\left(\pi - 2\,\alpha^2 + \pi\,\alpha^2\right) \left(\frac{\alpha^3 \sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}}\right) -$$

$$2\,\pi^2\,\alpha^2 \left(48 + 72\,\alpha^2 + 56\,\alpha^4 + 23\,\alpha^6 + \frac{36\,\alpha^4}{\left(\pi - 2\,\alpha^2 + \pi\,\alpha^2\right) \left(\frac{\alpha^3 \sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}} + \right.$$

$$\left.\left. \frac{12\,\alpha^6}{\left(\pi - 2\,\alpha^2 + \pi\,\alpha^2\right) \left(\frac{\alpha^3 \sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}} - \frac{15\,\alpha^8}{\left(\pi - 2\,\alpha^2 + \pi\,\alpha^2\right) \left(\frac{\alpha^3 \sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}}\right)\right)$$

$$\left.\sigma^2\right) \Big/ \left(9\,(-4+\pi)^2\,\alpha^3\,\left(1+\alpha^2\right)^{5/2} \left(\frac{\pi - 2\,\alpha^2 + \pi\,\alpha^2}{1+\alpha^2}\right)^{3/2} \sqrt{1 + 2\left(\frac{\alpha^3 \sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}} \right.$$

$$\left.\left(-1 - 2\left(\frac{\alpha^3 \sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3} + \pi\left(\frac{\alpha^3 \sqrt{1+\alpha^2}\,\sqrt{\frac{\pi-2\,\alpha^2+\pi\,\alpha^2}{1+\alpha^2}}}{\left(\pi-2\,\alpha^2+\pi\,\alpha^2\right)^2}\right)^{2/3}\right)\right)$$

$$\left.\left.\sqrt{1-\frac{\pi\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}}{1+2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}}}\right)\right),$$

$$-\left(\left(\pi^3\left(-\frac{(-4+\pi)\,\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\right.\right.$$

$$\left(6\pi^3\left(1+\alpha^2\right)^5-32\alpha^6\left(3+2\alpha^2\right)^2+4\pi\alpha^4\left(6+48\alpha^2+65\alpha^4+23\alpha^6\right)-\right.$$

$$\left.\left.\pi^2\alpha^2\left(60+168\alpha^2+209\alpha^4+136\alpha^6+35\alpha^8\right)\right)\right)\Big/\left(18\,(4-\pi)^{8/3}\,\alpha^6\left(\pi-2\alpha^2+\pi\alpha^2\right)^2\right.$$

$$\left.\left.\left(-1-2\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}+\pi\left(\frac{\alpha^3\sqrt{1+\alpha^2}\sqrt{\frac{\pi-2\alpha^2+\pi\alpha^2}{1+\alpha^2}}}{\left(\pi-2\alpha^2+\pi\alpha^2\right)^2}\right)^{2/3}\right)^3\right)\right)\}\}$$

Application to the simulated data from the R script.

```
μR = -5.0063024817859105;
σR = 3.0325019891716396;
αR = -1.9521085711315336; (* Point estimators obtained from R *)
V /. {μ → μR, σ → σR, α → αR} // MatrixForm
V /. {μ → μR, σ → σR, α → -αR} // MatrixForm
(* The result appears to work properly only with α positive,
because of c^(1/3) in the definition of δ *)
c^(1/3) /. c → αR
```

$$\begin{pmatrix} -7.38718+24.2238\,i & 99.8318-20.7154\,i & 6.77632-3.61509\,i \\ 99.8318-20.7154\,i & -276.963-270.908\,i & -28.1672-14.5179\,i \\ 6.77632-3.61509\,i & -28.1672-14.5179\,i & -2.50667-0.169835\,i \end{pmatrix}$$

$$\begin{pmatrix} 28.4502 & -122.534 & -46.0704 \\ -122.534 & 696.881 & 216.375 \\ -46.0704 & 216.375 & 90.4974 \end{pmatrix}$$

$0.624892 + 1.08234\,i$

A better approach. Use the inverse function theorem (Theorem 4).

Why do we substitute σ twice in the first line of the following code?

```
J2b =
  Inverse[D[{M[1], CM[2], CM[3]/CM[2]^(3/2)} /. σ → √s2, {{μ, s2, α}}] /. s2 → σ^2] // Simplify;
J2b /. {μ → μR, σ → σR, α → αR} // N
J2 /. {μ → μR, σ → σR, α → αR} // N
Vfinal = J2b.V0.Transpose[J2b] // Simplify;
Vfinal // MatrixForm
Vfinal /. {μ → μR, σ → σR, α → αR} // MatrixForm // N
```

{{1., 0.236198, -1.62999}, {0., 2.01729, -7.02029}, {0., 0., 3.52364}}

{{1., -0.118099 - 0.204554 ⅈ, 0.814996 + 1.41161 ⅈ},
 {0., 0.491353 + 0.881002 ⅈ, 3.51015 - 6.07975 ⅈ}, {0., 0., 0.0198551 - 0.586774 ⅈ}}

$$\begin{pmatrix} \frac{\pi\,(-4\,\alpha^2\,(9+18\,\alpha^2+11\,\alpha^4)+3\,\pi\,(2+6\,\alpha^2+9\,\alpha^4+5\,\alpha^6))\,\sigma^2}{9\,(-4+\pi)^2\,\alpha^4\,(1+\alpha^2)} & -\frac{2\,\sqrt{2\,\pi}\,(-4\,\alpha^2\,(9+18\,\alpha^2+11\,\alpha^4)+3\,\pi\,(2+6\,\alpha^2+9\,\alpha^4+5\,\alpha}{9\,(-4+\pi)^2\,\alpha^3\,(1+\alpha^2)^{3/2}} \\ -\frac{2\,\sqrt{2\,\pi}\,(-4\,\alpha^2\,(9+18\,\alpha^2+11\,\alpha^4)+3\,\pi\,(2+6\,\alpha^2+9\,\alpha^4+5\,\alpha^6))\,\sigma^3}{9\,(-4+\pi)^2\,\alpha^3\,(1+\alpha^2)^{3/2}} & \frac{2\,(-32\,\alpha^6+9\,\pi^2\,(\alpha+\alpha^3)^2-12\,\pi\,(-2+3\,\alpha^4+\alpha^6))\,\sigma^4}{9\,(-4+\pi)^2\,\alpha^2\,(1+\alpha^2)^2} \\ -\frac{\sqrt{2\,\pi}\,(3\,\pi^2\,(1+\alpha^2)^4-4\,\alpha^4\,(9+15\,\alpha^2+4\,\alpha^4)-4\,\pi\,\alpha^2\,(6+9\,\alpha^2+4\,\alpha^4+\alpha^6))\,\sigma}{9\,(-4+\pi)^2\,\alpha^4\,\sqrt{1+\alpha^2}} & \frac{(-32\,\alpha^6\,(3+2\,\alpha^2)-8\,\pi\,\alpha^2\,(12+27\,\alpha^2+17\,\alpha^4+2\,\alpha^6)+3\,\pi^2\,(4+16\,\alpha^2+27\,\alpha}{9\,(-4+\pi)^2\,\alpha^3\,(1+\alpha^2)} \end{pmatrix}$$

$$\begin{pmatrix} 28.4502 & 122.534 & -46.0704 \\ 122.534 & 696.881 & -216.375 \\ -46.0704 & -216.375 & 90.4974 \end{pmatrix}$$

We get (pretty much) the same result in a much more compact form.

Real data analysis - Barolo dataset.

```
data = Import["C:\\Users\\snagy\\Documents\\R\\Barolo.txt", "Data"];
(* Replace the path to where the Barolo.txt file is in your computer *)
data = Flatten[data];
Length[data]
data
```

260

```
{3.52636, 3.91202, 5.19296, 5.03695, 4.09434, 2.80336, 3.63759, 5.24702, 4.06044,
 3.91202, 3.80666, 4.29046, 4.60517, 3.91202, 4.40672, 4.17439, 3.91202, 5.07517,
 4.78749, 4.16821, 3.3322, 3.3322, 3.58352, 3.8712, 3.93183, 4.15888, 3.46574,
 5.37064, 4.49981, 4.17439, 4.74493, 4.00733, 3.98898, 3.91202, 5.273, 3.49651,
 3.82864, 3.43399, 3.31419, 3.43399, 3.94449, 4.31749, 4.31749, 4.34381, 3.93183,
 4.60517, 4.55388, 3.92197, 3.91202, 4.52179, 4.46591, 3.68888, 4.65396, 4.00733,
 4.38203, 3.7612, 4.49981, 4.74493, 5.79909, 3.62434, 4.06044, 3.71357, 3.82864,
 3.55535, 4.00733, 4.02535, 3.91202, 3.93183, 4.2485, 4.09434, 5.21494, 4.47734,
 3.43399, 3.71357, 4.34381, 3.71357, 4.00733, 4.63473, 5.29832, 3.55535, 4.27667,
 4.02535, 4.17439, 3.68888, 3.27714, 4.2485, 3.77276, 3.29584, 3.2581, 3.68888,
 4.17439, 4.58497, 3.65842, 3.94546, 4.00733, 4.47734, 3.63759, 4.17439, 3.38439,
 4.00733, 3.4012, 3.93183, 4.06044, 3.52636, 3.68888, 3.73767, 3.85015, 3.8712,
 4.02535, 4.04305, 4.04305, 4.09434, 4.11087, 4.15888, 4.15888, 4.21951, 4.23411,
 4.27667, 4.27667, 4.27667, 4.45435, 4.46591, 4.48864, 4.49981, 4.49981, 4.55388,
 4.56435, 4.59512, 4.60517, 4.60517, 4.67283, 4.78749, 4.78749, 4.78749, 4.86753,
 4.94164, 4.94164, 4.94164, 4.97673, 4.97673, 4.97673, 4.98361, 5.07517, 5.10595,
 5.10595, 5.10595, 5.39363, 5.54126, 5.70378, 5.73657, 5.76832, 5.76832, 5.79909,
 5.82895, 5.85793, 6.21461, 3.46574, 3.49651, 3.55535, 3.58352, 3.61092, 3.73767,
 3.73767, 3.73767, 3.73767, 3.73767, 3.80666, 3.80666, 3.80666, 3.8712, 3.91202,
 4.00733, 4.00733, 4.00733, 4.00733, 4.00733, 4.00733, 4.00733, 4.00733, 4.09434,
 4.09434, 4.09434, 4.09434, 4.09434, 4.09434, 4.17439, 4.21951, 4.21951, 4.21951,
 4.2485, 4.2485, 4.31749, 4.31749, 4.38203, 4.38203, 4.39445, 4.49981, 4.49981,
 4.49981, 4.49981, 4.49981, 4.60517, 4.60517, 4.74493, 4.78749, 4.78749, 4.78749,
 4.78749, 4.78749, 4.78749, 4.86753, 4.94164, 5.01064, 5.01064, 5.01064, 5.1358,
 5.1358, 5.1358, 5.34711, 5.39363, 5.39363, 3.20275, 3.21084, 3.26194, 3.29213,
 3.47816, 3.65584, 3.66868, 3.73767, 3.73767, 3.74242, 3.77735, 3.85227, 3.86912,
 3.86912, 3.90197, 3.91801, 3.91999, 3.95508, 4.02177, 4.04655, 4.10264, 4.10429,
 4.10429, 4.10429, 4.14313, 4.14789, 4.20618, 4.25703, 4.2683, 4.28414, 4.49536,
 4.51415, 4.55282, 4.61314, 4.73883, 4.93591, 4.93591, 4.93591, 5.14166}
```

```
Moment[data, 1]
CentralMoment[data, 2]
CentralMoment[data, 3] / CentralMoment[data, 2]^(3/2)
MoM = h[Moment[data, 1], CentralMoment[data, 2], CentralMoment[data, 3] / CentralMoment[data, 2]^(3/2)]
(* MoM estimators of the parameters *)
VBarolo = V /. {μ → MoM[[1]], σ → √MoM[[2]], α → MoM[[3]]}
(* n * AVar matrix of the estimators *)
VBarolo2 = Vfinal /. {μ → MoM[[1]], σ → √MoM[[2]], α → MoM[[3]]}
(* n * AVar matrix of the estimators *)
MoM[[3]] +
 {-1, 1} Quantile[NormalDistribution[], .975] √(VBarolo[[3, 3]] / Length[data])
(* Asymptotic confidence interval for α *)
```

```
4.2724

0.357609

0.601578

{3.60317, 0.805488, 2.62678}

{{1.82344, -2.44063, -17.0677},
 {-2.44063, 4.56434, 24.9606}, {-17.0677, 24.9606, 198.367}}

{{1.82344, -2.44063, -17.0677},
 {-2.44063, 4.56434, 24.9606}, {-17.0677, 24.9606, 198.367}}

{0.914808, 4.33874}
```

Numerical solution to the same problem.
The asymptotic variance matrix was in the approach using the inverse function theorem obtained ***without*** any numerical computations - the only numerical computation needed here is to evaluate the point estimates.

```
NSolve[{M[1] == Moment[data, 1], CM[2] == CentralMoment[data, 2],
   CM[3] / CM[2]^(3/2) == CentralMoment[data, 3] / CentralMoment[data, 2]^(3/2)}, {μ, σ, α}, Reals]
σ^2 /.
 %[[
  2]]
```

```
{{σ → -0.89749, α → -2.62678, μ → 3.60317}, {σ → 0.89749, α → 2.62678, μ → 3.60317}}

0.805488
```

Maximum likelihood estimation

Closed form expression for MLE cannot be obtained. Numerical solution.

```
L[μ_, σ_, α_] = Times @@ PDF[SkewNormalDistribution[μ, σ, α], data];
NMaximize[{L[μ, σ, α], σ > 0}, {μ, σ, α}]
logL[μ_, σ_, α_] = Total[Log[PDF[SkewNormalDistribution[μ, σ, α], data]]];
MLE = NMaximize[{logL[μ, σ, α], σ > 0}, {μ, σ, α}]
```

{3.73356 × 10^-150, {μ → 2.76362, σ → 1.70051, α → 1.64774}}

{-225.239, {μ → 3.59032, σ → 0.907113, α → 2.85073}}

Observed Fisher information matrix

```
D2logL = - D[logL[μ, σ, α], {{μ, σ, α}, 2}] / Length[data] /. MLE[[2]]
Inverse[D2logL]
```

{{3.56053, 1.26417, 0.179293},
 {1.26417, 2.75444, -0.103057}, {0.179293, -0.103057, 0.0327933}}

{{0.829172, -0.623483, -6.49276},
 {-0.623483, 0.880244, 6.17509}, {-6.49276, 6.17509, 85.3984}}

```
(α /. MLE[[2]]) + {-1, 1} Quantile[NormalDistribution[], .975]
  √(Inverse[D2logL][[3, 3]] / Length[data])
(* Asymptotic confidence interval for α *)
```

{1.72745, 3.974}