Mathematical analysis in thermodynamics of incompressible fluids

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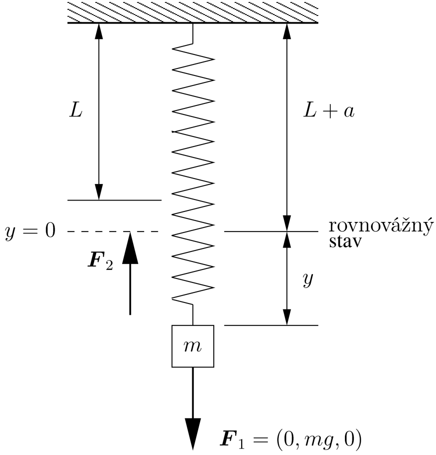
Contents

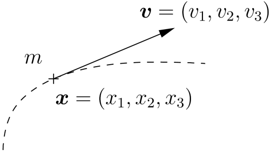
Part #1

Mathematically self-consistent models of classical mechanics - models for the system Spring - Weight

System **Spring - Weight**/Description and assumptions

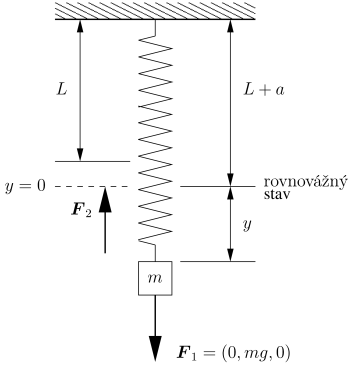


- Bodies (weights) modeled as mass-points
- Three Newton's postulates:
  - $\mathbf{F} = \mathbf{0} \implies$ straight-line motion
  - $\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\frac{d^2\mathbf{x}}{dt^2}$



  - Any $\mathbf{F}$ exerts reaction $-\mathbf{F}$
- Motion allowed only in the vertical direction
- Mass of the spring is neglected

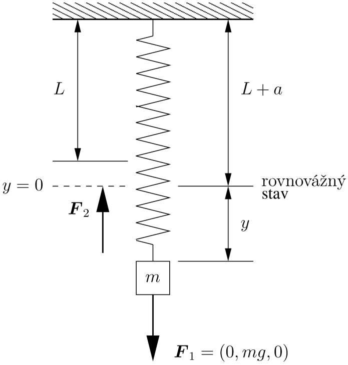
System **Spring** - **Weight**/Assumptions characterizing material properties



- Linear Spring:
  $\mathbf{F}_2 = (0, -k(y+a), 0) \quad (k > 0)$
- Resistance due to environment is neglected

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \qquad y(0) = y_0, \quad \frac{dy}{dt}(0) = y_1$$

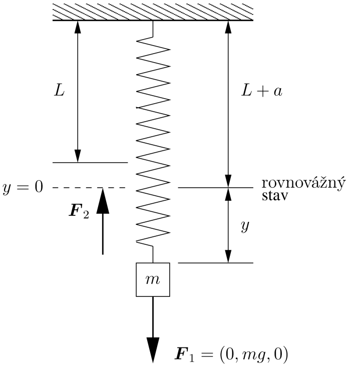
System **Spring** - **Weight**/Assumptions characterizing material properties



- Linear Spring:
  $\mathbf{F}_2 = (0, -k(y + a), 0) \quad (k > 0)$
- Resistance proportional to the velocity:
  $\mathbf{F}_3 = (0, -b\frac{dy}{dt}, 0) \qquad (b > 0)$

$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = 0 \qquad y(0) = y_0, \quad \frac{dy}{dt}(0) = y_1$$

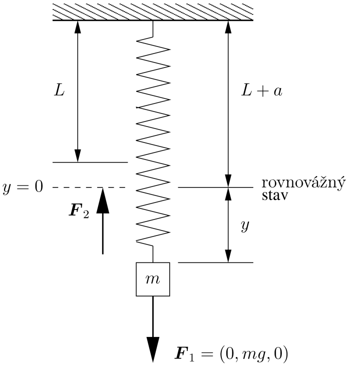
System **Spring** - **Weight**/Assumptions characterizing material properties



- Linear Spring:
  $\mathbf{F}_2 = (0, -k(y+a), 0) \quad (k > 0)$
- Resistance force due to environment depends on the velocity non-linearly:
  $\mathbf{F}_3 = \left(0, h\left(\frac{dy}{dt}\right), 0\right)$

$$m\frac{d^2y}{dt^2} + h\left(\frac{dy}{dt}\right) + ky = 0 \qquad y(0) = y_0, \quad \frac{dy}{dt}(0) = y_1$$

System **Spring** - **Weight**/Assumptions characterizing material properties



- Non-linear Spring: $\mathbf{F}_2 = (0, g(y+a), 0)$
- Environment resistance neglected, linear, or non-linear

$$\boxed{\frac{d^2y}{dt^2} + h(\frac{dy}{dt}) + g(y) = 0}$$

$$\boxed{\frac{d^2y}{dt^2} = f(y, \frac{dy}{dt})}$$

- Free fall due to gravity: $\mathbf{F}_2 = (0, 0, 0)$

$$\boxed{\frac{d^2y}{dt^2} + h(\frac{dy}{dt}) = 0 \quad \Longleftrightarrow \quad \frac{dv}{dt} + h(v) = 0}$$

$$\boxed{\frac{dv}{dt} = f(v) \qquad v(0) = v_0}$$

System **Spring - Weight**/Mathematically self-consistent models

- Simplifying assumptions $\Longrightarrow$ very crude approximation of the reality
- Independently how accurate are models we are interested in **mathematical self-consistency of the models**: notion of solution
  - **existence** for arbitrary set of data (T, v_0 (or y_0 and y_1), m, ...)
  - **uniqueness**
  - **continuous dependence of solution on data**
  - **boundedness** of the velocity
  - **long time behavior of solutions**.
- Mathematical self-consistency of models of incompressible fluid thermodynamics
- Derivation of fluid thermodynamics models stems from the principles of classical mechanics

System **Spring - Weight**/Simple observations

- Free fall due to gravity: first order equation for the velocity
- Mathematical self-consistency of the equation of a "slightly" generalized form $\frac{dv}{dt} = f(v)$, $v(0) = v_0$. Counterexamples:
  - existence/boundedness for any time interval - $f(v) = v^2$
  - uniqueness - $f(v) = v^{2/3}$
- $m\frac{dv}{dt} + bv = f \implies \frac{m}{2}\frac{d}{dt}|v|^2 + \frac{b}{m}|v|^2 = fv \implies$

$$|v(t)|^2 \leq |v_0|^2 e^{-\frac{b}{m}t} + \frac{f^2}{b^2}(1 - e^{-\frac{b}{m}t}) \qquad \text{pro } t > 0$$

- Derived models have a limited region where they can be useful

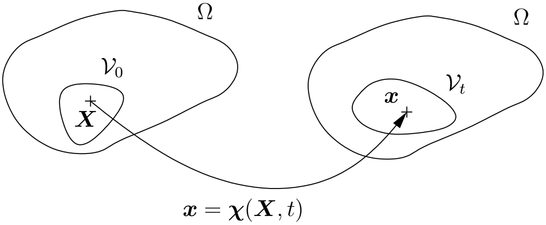
Part #2

Thermodynamics of incompressible fluids

Fluid

Definition

Fluid is a body that, in time scale of observation of interest, undergoes discernible deformation due to the application of a sufficiently small shear stress



$$\mathbf{v} = \frac{\partial \boldsymbol{\chi}}{\partial t} \qquad \mathbf{F}_{\boldsymbol{\chi}} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}$$

Long-lasting physical experiment

In 1927 at University of Queensland: liquid asphalt put inside the closed vessel, after three years the vessel was open and the asphalt has started to drop slowly.

Year	Event
1930	Plug trimmed off
1938 (Dec)	1st drop
1947 (Feb)	2nd drop
1954 (Apr)	3rd drop
1962 (May)	4th drop
1970 (Aug)	5th drop
1979 (Apr)	6th drop
1988 (Jul)	7th drop
2000 (28 Nov)	8th drop

Balance equations of continuum physics

Balance of mass, linear and angular momentum, balance of energy and the second law of thermodynamics

$$\begin{aligned} \varrho_{,t} + \operatorname{div}(\varrho \mathbf{v}) &= 0 \\ (\varrho \mathbf{v})_{,t} + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{T} &= \mathbf{0} \\ \mathbf{T}^T &= \mathbf{T} \\ \left(\varrho(e + |\mathbf{v}|^2/2)\right)_{,t} + \operatorname{div}(\varrho(e + |\mathbf{v}|^2/2)\mathbf{v}) + \operatorname{div} \mathbf{q} &= \operatorname{div}(\mathbf{T}\mathbf{v}) \end{aligned}$$

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\end{aligned}
$$

- ϱ ... density
- $\mathbf{v}$ ... velocity
- e ... internal energy
- $\mathbf{T}$ ... the Cauchy stress
- $\mathbf{q}$ ... heat flux

Balance equations of continuum physics

Balance of mass, linear and angular momentum, balance of energy and the second law of thermodynamics

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Eulerian description - flows of fluid-like bodies
No external sources - for simplicity

Balance equations of continuum physics/2

$B \subset \Omega$ fix for all $t \geq 0$:

$$\frac{d}{dt}\int_B \varrho\,dx = -\int_{\partial B} \varrho\mathbf{v}\cdot\mathbf{n}\,dS \quad \Longrightarrow \quad \text{FVM}$$

$$= -\int_B \operatorname{div}(\varrho\mathbf{v})\,dx \quad \Longrightarrow \quad \varrho_t + \operatorname{div}\varrho\mathbf{v} = 0$$

Balance equations of continuum physics/2

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Choice $B = \{x \in \Omega; \eta(x) > r\}$, where $r \in (0,\infty)$ and $\eta \in \mathcal{D}(\Omega)$

$$\frac{d}{dt}\int_B \varrho\eta\, dx - \int_B \varrho\mathbf{v}\cdot\nabla\eta\, dx = 0 \Longrightarrow \textbf{weak solution}, \text{FEM}$$

Oseen, Leray, ..., Chen, Torres, Ziemer, ... Feireisl:

- weak formulation of balance equations - the **primary setting**
- classical formulation of balance equations - the secondary setting

"Equivalent" formulation of the balance of energy

$$
\begin{aligned}
\varrho_{,t} + \operatorname{div}(\varrho \mathbf{v}) &= 0 \\
(\varrho \mathbf{v})_{,t} + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{T} &= \mathbf{0} \qquad \text{(BLM)} \\
\mathbf{T}^T &= \mathbf{T} \\
\left(\varrho(e + |\mathbf{v}|^2/2)\right)_{,t} + \operatorname{div}(\varrho(e + |\mathbf{v}|^2/2)\mathbf{v}) + \operatorname{div} \mathbf{q} &= \operatorname{div}(\mathbf{T}\mathbf{v})
\end{aligned}
$$

is equivalent, provided that $\mathbf{v}$ is **admissible** test function in (BLM), to

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\end{aligned}
$$

Note that $\mathbf{T} \cdot \nabla \mathbf{v} = \mathbf{T} \cdot \mathbf{D}$ where $\mathbf{D} := \mathbf{D}(\mathbf{v})$ is the symmetric part of the velocity gradient

Entropy

$$(\varrho e)_{,t} + \operatorname{div}(\varrho e \mathbf{v}) + \operatorname{div} \mathbf{q} = \mathbf{T} \cdot \nabla \mathbf{v} \tag{1}$$

Continuum thermodynamics (Callen 1985): there is η (specific entropy density) being a function of state variables, here $\eta = \tilde{\eta}(e)$, fulfilling:

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Continuum thermodynamics (Callen 1985): there is η (specific entropy density) being a function of state variables, here $\eta = \tilde{\eta}(e)$, fulfilling:

- $\tilde{\eta}$ is increasing function of e $\Longrightarrow$ $\frac{1}{\theta} =: \frac{\partial \tilde{\eta}}{\partial e}$ or $e = \tilde{e}(\eta) \Longrightarrow \theta = \frac{\partial \tilde{e}}{\partial \eta}$
- $\eta \to 0+$ as $\theta \to 0+$
- $S(t) := \int_\Omega \varrho^* \eta(t, \cdot) dx$ goes to its maximum as $t \to \infty$ provided that the body is thermally and mechanically isolated

Entropy

$$(\varrho e)_{,t} + \operatorname{div}(\varrho e \mathbf{v}) + \operatorname{div} \mathbf{q} = \mathbf{T} \cdot \nabla \mathbf{v} \tag{1}$$

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(1) is equivalent to

$$\frac{\partial \tilde{\eta}}{\partial e}\Big(\varrho\big[e_{,t} + \mathbf{v} \cdot \nabla e\big]\Big) + \frac{\operatorname{div} \mathbf{q}}{\theta} = \frac{\mathbf{T} \cdot \mathbf{D}(\mathbf{v})}{\theta}$$

$$\varrho\Big[\eta_{,t} + \eta \cdot \nabla \mathbf{v}\Big] + \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) = \frac{1}{\theta}\Big[\mathbf{T} \cdot \mathbf{D}(\mathbf{v})\Big] - \frac{\mathbf{q} \cdot \nabla \theta}{\theta^2}$$

Second law of thermodynamics/1

$$\left(\varrho\eta\right)_{,t} + \operatorname{div}(\varrho\eta\mathbf{v}) + \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) = \xi \qquad \text{with } \theta\xi := \mathbf{T}\cdot\mathbf{D}(\mathbf{v}) - \frac{\mathbf{q}\cdot\nabla\theta}{\theta} \quad (2)$$

Second law of thermodynamics: $\boxed{\xi \geq 0}$

Stronger requirement: $\mathbf{T}\cdot\mathbf{D}(\mathbf{v}) \geq 0$ (entropy production due to work being converted into heat) and $-\frac{\mathbf{q}\cdot\nabla\theta}{\theta} \geq 0$ (entropy production due to heat conduction)

We shall use the constitutive equations that automatically meet these requirements

Minimum principle for e

$$\text{if } e_0 \geq C^* \text{ in } \Omega \text{ then } e(t,\cdot) \geq C^* \text{ in } \Omega \text{ for all } t$$

Second law of thermodynamics/2

$$\left(\varrho\eta\right)_{,t} + \operatorname{div}(\varrho\eta\mathbf{v}) + \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) = \xi \qquad \text{with } \theta\xi \geq \mathbf{T}\cdot\mathbf{D}(\mathbf{v}) - \frac{\mathbf{q}\cdot\nabla\theta}{\theta}$$

In terms of the internal energy $\eta = \tilde{\eta}(e)$

$$e_{,t} + \operatorname{div}(e\mathbf{v}) + \operatorname{div}\mathbf{q} \geq \mathbf{T}\cdot\mathbf{D}(\mathbf{v})$$

or, using the balance of energy,

$$\left(|\mathbf{v}|^2\right)_{,t} - 2\operatorname{div}(\mathbf{T}\mathbf{v}) + \operatorname{div}\left(\mathbf{v}|\mathbf{v}|^2\right) \leq 0$$

Suitable weak solution (in the sense of Caffarelli, Kohn, Nirenberg): In addition to equations representing balance of mass, linear momentum and energy we require that solution satisfies one of the formulations of the second law of thermodynamics

Incompressibility

Definition

Volume of any chosen subset (at initial time $t = 0$) remains constant during the motion.

$$\text{for all } t: \quad |\mathcal{V}_t| = |\mathcal{V}_0| \iff \det \mathbf{F}_{\boldsymbol{\chi}} = 1$$

Taking the derivative w.r.t. time and using the identity

$$\frac{\mathrm{d}}{\mathrm{d}t} \det \mathbf{F}_{\boldsymbol{\chi}} = \operatorname{div} \mathbf{v} \det \mathbf{F}_{\boldsymbol{\chi}}$$

we conclude that

$$\operatorname{div} \mathbf{v} = 0$$

Balance equations for Inhomogeneous incompressible fluids

Balance equations

$$
\begin{aligned}
\varrho_{,t} + \operatorname{div}(\varrho \mathbf{v}) &= 0 \\
(\varrho \mathbf{v})_{,t} + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{T} &= \mathbf{0} \qquad \text{(BLM)} \\
\left(\varrho(e + |\mathbf{v}|^2/2)\right)_{,t} + \operatorname{div}(\varrho(e + |\mathbf{v}|^2/2)\mathbf{v}) + \operatorname{div} \mathbf{q} &= \operatorname{div}(\mathbf{T}\mathbf{v})
\end{aligned}
$$

Consequences of incompressibility

$$
\operatorname{div} \mathbf{v} = 0 \qquad \text{and } \mathbf{T} = -p\mathbf{I} + \mathbf{S}
$$

$$
\boxed{
\begin{aligned}
\operatorname{div} \mathbf{v} &= 0 \\
\varrho_t + \mathbf{v} \cdot \nabla \varrho &= 0 \\
(\varrho \mathbf{v})_t + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} &= -\nabla p \\
\left(\varrho(e + |\mathbf{v}|^2/2)\right)_{,t} + \operatorname{div}(\varrho(e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div} \mathbf{q} &= \operatorname{div}(\mathbf{S}\mathbf{v})
\end{aligned}
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Balance equations for Inhomogeneous incompressible fluids

Balance equations

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S and **q**: additional (the so-called) constitutive equations

Balance equations for Inhomogeneous incompressible fluids

Balance equations

$$\varrho_{,t} + \operatorname{div}(\varrho \mathbf{v}) = 0$$
$$(\varrho \mathbf{v})_{,t} + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{T} = \mathbf{0} \qquad \text{(BLM)}$$
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Consequences of incompressibility

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$\mathbf{S}$ and $\mathbf{q}$: additional (the so-called) constitutive equations

Homogeneous fluids: the density is constant

Balance equations for homogeneous incompressible fluids

$$\operatorname{div} \mathbf{v} = 0 \tag{3}$$

$$\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} = -\nabla p \tag{4}$$

$$(e + |\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div} \mathbf{q} = \operatorname{div} (\mathbf{S}\mathbf{v}) \tag{5}$$

$$e_{,t} + \operatorname{div}(e\mathbf{v}) + \operatorname{div} \mathbf{q} \geq \mathbf{S} \cdot \mathbf{D}(\mathbf{v}) \tag{6}$$

- Constitutive equations for **S** and **q** (next section)
- Boundary conditions (internal flows)
- Initial data

IBVP

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} &= -\nabla p \\ (e + |\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div} \mathbf{q} &= \operatorname{div}(\mathbf{S}\mathbf{v}) \end{aligned}$$

Data

IBVP

$$\operatorname{div} \mathbf{v} = 0$$

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Data

- $\Omega \subset \mathbb{R}^3$ bounded open connected container, $T \in (0, \infty)$ length of time interval
- $\mathbf{v}(0, \cdot) = \mathbf{v}_0$, $e(0, \cdot) = e_0$ in Ω
- α that appears in boundary conditions (thermally and mechanically or energetically isolated body)

IBVP

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} &= -\nabla p \\ (e + |\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div} \mathbf{q} &= \operatorname{div}(\mathbf{S}\mathbf{v}) \end{aligned}$$

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Task Mathematical Consistency of a Model - for any set of data to find uniquely defined, smooth, solution (*notion of solution, its existence, uniqueness, regularity*)

IBVP

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} &= -\nabla p \\ (e + |\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div} \mathbf{q} &= \operatorname{div}(\mathbf{S}\mathbf{v}) \end{aligned}$$

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Task Mathematical Consistency of a Model - for any set of data to find uniquely defined, smooth, solution (*notion of solution, its existence, uniqueness, regularity*)

Weak solution - solution dealing with averages

Boundary conditions

$$(e + |\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div}\mathbf{q} - \operatorname{div}(\mathbf{S}\mathbf{v}) = 0$$

$$\frac{d}{dt}\left(\int_\Omega E(t,x)\,dx\right) + \int_{\partial\Omega} [(E+p)\mathbf{v}\cdot\mathbf{n} + \mathbf{q}\cdot\mathbf{n} - \mathbf{S}\mathbf{v}\cdot\mathbf{n}]\;dS = 0$$

Mechanically and thermally isolated body, Navier's slip on $[0, T] \times \Omega$:

Boundary conditions

$$(e+|\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e+|\mathbf{v}|^2/2+p)\mathbf{v}) + \operatorname{div}\mathbf{q} - \operatorname{div}(\mathbf{S}\mathbf{v}) = 0$$

$$\frac{d}{dt}\left(\int_\Omega E(t,x)\,dx\right) + \int_{\partial\Omega}\left[(E+p)\mathbf{v}\cdot\mathbf{n} + \mathbf{q}\cdot\mathbf{n} - \mathbf{S}\mathbf{v}\cdot\mathbf{n}\right]\,dS = 0$$

Mechanically and thermally isolated body, Navier's slip on $[0,T]\times\Omega$:

- $\mathbf{v}\cdot\mathbf{n} = 0 \qquad \mathbf{q}\cdot\mathbf{n} = 0$
- $\lambda(\mathbf{S}\mathbf{n})_{\boldsymbol{\tau}} + (1-\lambda)\mathbf{v}_{\boldsymbol{\tau}} = \mathbf{0}$ for $\lambda\in(0,1) \qquad \mathbf{u}_{\boldsymbol{\tau}} := \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}$
- $\lambda = 0 \implies$ no-slip $\qquad \lambda = 1 \implies$ slip

Boundary conditions

$$(e+|\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e+|\mathbf{v}|^2/2+p)\mathbf{v}) + \operatorname{div}\mathbf{q} - \operatorname{div}(\mathbf{S}\mathbf{v}) = 0$$

$$\frac{d}{dt}\left(\int_\Omega E(t,x)\,dx\right) + \int_{\partial\Omega} [(E+p)\mathbf{v}\cdot\mathbf{n} + \mathbf{q}\cdot\mathbf{n} - \mathbf{S}\mathbf{v}\cdot\mathbf{n}]\, dS = 0$$

Mechanically and thermally isolated body, Navier's slip on $[0,T]\times\Omega$:

- $\mathbf{v}\cdot\mathbf{n} = 0 \qquad \mathbf{q}\cdot\mathbf{n} = 0$
- $\lambda(\mathbf{S}\mathbf{n})_{\boldsymbol{\tau}} + (1-\lambda)\mathbf{v}_{\boldsymbol{\tau}} = \mathbf{0}$ for $\lambda\in(0,1) \qquad \mathbf{u}_{\boldsymbol{\tau}} := \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}$
- $\lambda = 0 \implies$ no-slip $\qquad \lambda = 1 \implies$ slip

Energetically isolated body, Navier's slip on $[0,T]\times\Omega$:

Boundary conditions

$$(e + |\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div}\mathbf{q} - \operatorname{div}(\mathbf{S}\mathbf{v}) = 0$$

$$\frac{d}{dt}\left(\int_\Omega E(t,x)\,dx\right) + \int_{\partial\Omega} [(E+p)\mathbf{v}\cdot\mathbf{n} + \mathbf{q}\cdot\mathbf{n} - \mathbf{S}\mathbf{v}\cdot\mathbf{n}]\,dS = 0$$

Mechanically and thermally isolated body, Navier's slip on $[0, T] \times \Omega$:

- $\mathbf{v}\cdot\mathbf{n} = 0 \qquad \mathbf{q}\cdot\mathbf{n} = 0$
- $\lambda(\mathbf{S}\mathbf{n})_{\boldsymbol{\tau}} + (1-\lambda)\mathbf{v}_{\boldsymbol{\tau}} = \mathbf{0}$ for $\lambda \in (0,1) \qquad \mathbf{u}_{\boldsymbol{\tau}} := \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}$
- $\lambda = 0 \implies$ no-slip $\qquad \lambda = 1 \implies$ slip

Energetically isolated body, Navier's slip on $[0, T] \times \Omega$:

- $\mathbf{v}\cdot\mathbf{n} = 0 \qquad \mathbf{q}\cdot\mathbf{n} = -\alpha|\mathbf{v}_{\boldsymbol{\tau}}|^2$
- $(\mathbf{S}\mathbf{n})_{\boldsymbol{\tau}} + \alpha\mathbf{v}_{\boldsymbol{\tau}} = \mathbf{0} \qquad \alpha := (1-\lambda)/\lambda$

"Equivalent" formulation of the balance of energy/1

$$
\begin{aligned}
\operatorname{div} \mathbf{v} &= 0\\
\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) - \operatorname{div}\mathbf{S} &= -\nabla p\\
(e + |\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div}\mathbf{q} &= \operatorname{div}(\mathbf{S}\mathbf{v})
\end{aligned}
$$

is equivalent (if $\mathbf{v}$ is admissible test function in BM) to

$$
\begin{aligned}
\operatorname{div} \mathbf{v} &= 0\\
\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) - \operatorname{div}\mathbf{S} &= -\nabla p\\
e_{,t} + \operatorname{div}(e\mathbf{v}) + \operatorname{div}\mathbf{q} &= \mathbf{S}\cdot\mathbf{D}(\mathbf{v})
\end{aligned}
$$

Helmholtz decomposition $\mathbf{u} = \mathbf{u}_{\mathrm{div}} + \nabla g^{\mathbf{v}}$
Leray's projector $\mathbb{P} : \mathbf{u} \mapsto \mathbf{u}_{\mathrm{div}}$

"Equivalent" formulation of the balance of energy/2

$$
\begin{aligned}
\operatorname{div} \mathbf{v} &= 0 \\
\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} &= -\nabla p \\
(e + |\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div} \mathbf{q} &= \operatorname{div}(\mathbf{S}\mathbf{v})
\end{aligned}
$$

is equivalent (if $\mathbf{v}$ is admissible test function in BM) to

$$
\begin{aligned}
\operatorname{div} \mathbf{v} &= 0 \\
\mathbf{v}_{,t} + \mathbb{P} \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \mathbb{P} \operatorname{div} \mathbf{S} &= \mathbf{0} \\
e_{,t} + \operatorname{div}(e\mathbf{v}) + \operatorname{div} \mathbf{q} &= \mathbf{S} \cdot \mathbf{D}(\mathbf{v})
\end{aligned}
$$

Advantages/Disadvantages

$$\operatorname{div} \mathbf{v} = 0$$
$$\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} = -\nabla p$$
$$(e + |\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div} \mathbf{q} = \operatorname{div}(\mathbf{S}\mathbf{v})$$

is equivalent (if $\mathbf{v}$ is admissible test function in BM) to

$$\operatorname{div} \mathbf{v} = 0$$
$$\mathbf{v}_{,t} + \mathbb{P} \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \mathbb{P} \operatorname{div} \mathbf{S} = \mathbf{0}$$
$$e_{,t} + \operatorname{div}(e\mathbf{v}) + \operatorname{div} \mathbf{q} = \mathbf{S} \cdot \mathbf{D}(\mathbf{v})$$

Advantages/Disadvantages

- + pressure is not included into the 2nd formulation
- + minimum principle for e if $\mathbf{S} \cdot \mathbf{D}(\mathbf{v}) \geq 0$
- $-$ $\mathbf{S} \cdot \mathbf{D}(\mathbf{v}) \in L^1$ while $\mathbf{S}\mathbf{v} \in L^q$ with $q > 1$

Part #3

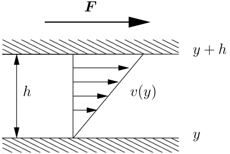
Constitutive equations

Newtonian fluids

Definition

The viscosity: the coefficient of the proportionality between the shear rate and the shear stress.

Simple shear flow: $\mathbf{v}(x, y, z) = (v(y), 0, 0)$



Newton: *The resistance arising from the want of lubricity in parts of the fluid,* **other things being equal**, *is proportional to the velocity with which the parts are separated from one another.*

$$\mathbf{T}_{xy} = \nu v'(y) \qquad\qquad g(\mathbf{T}_{xy}, v'(y)) = 0$$

Generalized Newtonian fluids

Experimental data show that the viscosity may depend on the pressure, shear rate, temperature, concentration, ..., density (if fluid is inhomogeneous)

$$\mathbf{T}_{xy} = \nu v'(y) \qquad \nu = \nu(p, \theta, |v'(y)|) \qquad \mathbf{S} = \nu(p, \theta, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}$$

Examples:

- $\mathbf{T} = -p\mathbf{I} + 2\mu_0 \mathbf{D}, \qquad \operatorname{tr} \mathbf{D} = 0$
- $\mathbf{T} = -p\mathbf{I} + 2\mu_0 |\mathbf{D}|^{r-2}\mathbf{D} \qquad r \in [1, \infty)$
- $\mathbf{T} = -p\mathbf{I} + 2\mu_0 \left(1 + |\mathbf{D}|^2\right)^{\frac{r-2}{2}} \mathbf{D}$
- $\mathbf{T} = -p\mathbf{I} + 2\mu_0 \exp(\alpha p)\mathbf{D}$ or $\mathbf{T} = -p\mathbf{I} + \left(1 + \alpha\mu(p, \theta) + |\mathbf{D}|^2\right)^{\frac{r-2}{2}} \mathbf{D}$
- $\mathbf{T} = -p\mathbf{I} + 2\nu(p, \varrho, \theta)\mathbf{D} = -p\mathbf{I} + A\sqrt{\varrho} \exp\left(\frac{B(p+D\varrho^2)}{\theta}\right) \mathbf{D}$
- $\mathbf{T} = -p\mathbf{I} + 2\mu_0 \exp(1/\theta - 1/\theta_0) \left(1 + \alpha\mu(p, \theta) + |\mathbf{D}|^2\right)^{\frac{r-2}{2}} \mathbf{D}$

Implicitely constituted fluids

More general **implicit** relations

$$G(\mathbf{T}_{xy}, v'(y)) = 0 \quad \text{or } G(p, \theta, \mathbf{T}_{xy}, v'(y)) = 0 \qquad \mathbf{G}(p, \theta, \mathbf{S}, \mathbf{D}) = \mathbf{0}$$

have the ability to capture complicated responses of materials without any need to introduce (non-physical) internal variable constitutive theories, etc. Implicit relations

- algebraic
- rate type
- integral

Newtonian versus non-Newtonian fluids

Incompressible Newtonian fluid

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D}, \qquad \mathrm{tr}\,\mathbf{D} = 0$$

Departures from Newtonian behavior (at a simple shear flow)

- Dependence of the viscosity on the shear rate
- Dependence of the viscosity on the pressure
- The presence of the yield stress (or other activation or deactivation criteria)
- The presence of the normal stress differences
- Stress relaxation
- Nonlinear creep

Fourier fluids - heat conducting fluids

Definition

The heat conductivity: the coefficient of the proportionality between the heat flux $\mathbf{q}$ and the temperature gradient $\nabla\theta$.

Landau, Lifschitz: *The heat flux is* **related** *to the variation of temperature through the fluid. ... We can then expand* $\mathbf{q}$ *as a series of powers of temperature gradient, taking only the first terms of the expansion. The constant term is evidently zero since* $\mathbf{q}$ *must vanish when* $\nabla\theta$ *does so. Thus we have*

$$\mathbf{q} = -\kappa\nabla\theta$$

The coefficient κ *is in general a function of temperature and pressure.*

Examples:

- $\mathbf{q} = -\kappa\nabla\theta$
- $\mathbf{q} = -\kappa(\theta, p)\nabla\theta$
- $\mathbf{q}(\nabla\theta) = \mathbf{q}(\mathbf{0}) + \partial_{\mathbf{z}}(\mathbf{0})\nabla\theta + 1/2\partial_{\mathbf{z}}^{(2)}(\mathbf{0})\nabla\theta \otimes \nabla\theta$

Implicitely constituted heat conducting fluids

More general **implicit** relations

$$\mathbf{r}(\mathbf{q}, \nabla\theta) = \mathbf{0} \qquad\qquad \mathbf{r}(\mathbf{q}, p, \theta, \nabla\theta, \mathbf{D}) = \mathbf{0}$$

Implicit relations

- algebraic
- rate type
- integral

Part #4

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