On Solvability of a non-linear heat equation with non-integrable convective term and the right-hand side involving measures

Josef Málek

Mathematical Intitute of the Charles University
Sokolovská 83, 186 75 Prague 8, Czech Republic

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References

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[F3] **M. Bulíček**, **L. Consiglieri**, J. Málek: Slip boundary effects on unsteady flows of incompressible viscous heat conducting fluids with a nonlinear internal energy-temperature relationship

[Q1] **M. Bulíček**, **L. Consiglieri**, J. Málek: On Solvability of a non-linear heat equation with a non-integrable convective term and the right-hand side involving measures

Problem formulation/1

$$
\begin{aligned}
e_{,t} + \operatorname{div}(e\mathbf{v}) + \operatorname{div}\mathbf{q}(\cdot, e, \nabla e) &= f \geq 0 && \text{in } Q := (0,T) \times \Omega \\
e(0,x) &= e_0(x) \geq c > 0 && \text{in } \Omega \qquad (*) \\
\mathbf{q}(t,x,e(t,x),\nabla e(t,x)) \cdot \mathbf{n}(x) &= 0 && (0,T) \times \partial\Omega
\end{aligned}
$$

- for all $(e, \mathbf{u}) \in \mathbb{R} \times \mathbb{R}^d$: $\mathbf{q}(\cdot, e, \mathbf{u})$ is measurable,
- for almost all $(t,x) \in Q$: $\mathbf{q}(t,x,\cdot,\cdot)$ is continuous in $\mathbb{R} \times \mathbb{R}^d$,
- there are $C_1, C_2 > 0$ such that for all $(e, \mathbf{u}) \in \mathbb{R} \times \mathbb{R}^d$

  $\mathbf{q}(\cdot, e, \mathbf{u}) \cdot \mathbf{u} \geq C_1 |\mathbf{u}|^q$ and $|\mathbf{q}(\cdot, e, \mathbf{u})| \leq C_2 |\mathbf{u}|^{q-1}$,
- for all $e \in \mathbb{R}$ and for all $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^d$, $\mathbf{u}_1 \neq \mathbf{u}_2$

  $(\mathbf{q}(\cdot, e, \mathbf{u}_1) - \mathbf{q}(\cdot, e, \mathbf{u}_2)) \cdot (\mathbf{u}_1 - \mathbf{u}_2) > 0$.

Problem formulation/2

$$
\begin{aligned}
e_{,t} + \operatorname{div}(e\mathbf{v}) + \operatorname{div} \mathbf{q}(\cdot, e, \nabla e) = f \geq 0 \qquad & \text{in } Q := (0,T) \times \Omega \\
e(0,x) = e_0(x) > 0 \qquad & \text{in } \Omega \qquad (\mathcal{P}) \\
\mathbf{q}(t,x,e(t,x),\nabla e(t,x)) \cdot \mathbf{n}(x) = 0 \qquad & (0,T) \times \partial\Omega
\end{aligned}
$$

Data: $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary, $\quad T \in (0,\infty)$

$$
\begin{aligned}
& e_0 \in L^1(\Omega) \\
& f \in L^1(Q) \quad \text{or} \quad M(Q) := (C(\overline{Q}))^* \\
& \mathbf{v} \in L^r(0,T;\mathbf{L}^s(\Omega)) \qquad (1 \leq r, s \leq \infty) \\
& \qquad \operatorname{div} \mathbf{v} = 0 \text{ in } Q, \quad \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } (0,T) \times \partial\Omega
\end{aligned}
$$

Task: Large data mathematical theory (notion of solution, its existence, uniqueness, ...) to Problem $\mathcal{P}$, for any set of data and for largest class of constitutive relations

Approximations and apriori estimates/1

$$
\begin{aligned}
e^n_{,t} + \operatorname{div}(e^n \mathcal{H}_n(\mathbf{v})) + \operatorname{div} \mathbf{q}(\cdot, e^n, \nabla e^n) &= f^n \geq 0 \\
e^n(0,\cdot) &= e^n_0 > 0 && \text{[ic]} \qquad (\mathcal{P}_n) \\
\mathbf{q}(\cdot, e^n, \nabla e^n) \cdot \mathbf{n}(x) &= 0 && \text{[bc]}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}_n(\mathbf{v}) := (\chi_n \mathbf{v}) * \omega_n - \nabla \eta_n \ &\Longrightarrow \ \operatorname{div} \mathcal{H}_n(\mathbf{v}) = 0 \quad \text{and} \quad \mathcal{H}_n(\mathbf{v}) \cdot \mathbf{n} = 0 \\
&\Longrightarrow \mathcal{H}_n(\mathbf{v}) \in L^\infty(0,T;\mathbf{L}^k(\Omega)) \quad \forall k \in [1,\infty) \\
&\Longrightarrow \mathcal{H}_n(\mathbf{v}) \to \mathbf{v} \in L^r(0,T;\mathbf{L}^s(\Omega)) \\
f^n \in L^\infty(Q) \qquad & \qquad f^n \to f \text{ in } M(Q) \text{ or in } L^1(Q) \\
0 < e^n_0 \in L^\infty(\Omega) \qquad & \qquad e^n_0 \to e_0 \text{ in } L^1(\Omega)
\end{aligned}
$$

Approximations and apriori estimates/2

$$
\begin{aligned}
e^n_{,t} + \operatorname{div}(e^n \mathcal{H}_n(\mathbf{v})) + \operatorname{div} \mathbf{q}(\cdot, e^n, \nabla e^n) &= f^n \geq 0 \\
e^n(0,\cdot) &= e^n_0 > 0 \qquad \text{[ic]} \qquad (\mathcal{P}_n) \\
\mathbf{q}(\cdot, e^n, \nabla e^n) \cdot \mathbf{n}(x) &= 0 \qquad \text{[bc]}
\end{aligned}
$$

Truncation operators

$$
T_k(z) := \begin{cases} z & \text{if } |z| \leq k, \\ \operatorname{sign}(z) k & \text{if } |z| > k, \end{cases}
$$

$$
T_{k,\delta}(z) := \begin{cases} z & \text{if } |z| \leq k, \\ \operatorname{sign}(z)(k + \delta/2) & \text{if } |z| > k + \delta \end{cases},
$$

such that $T_{k,\delta} \in \mathcal{C}^2(\mathbb{R})$, $0 \leq T'_{k,\delta} \leq 1$.

$$
\Theta_k(s) := \int_0^s T_k(t)\, dt, \qquad \Theta_{k,\delta}(s) := \int_0^s T_{k,\delta}(t)\, dt.
$$

Approximations and apriori estimates/3

$$e^n_{,t} + \operatorname{div}(e^n \mathcal{H}_n(\mathbf{v})) + \operatorname{div} \mathbf{q}(\cdot, e^n, \nabla e^n) = f^n \geq 0$$

$$e^n(0,\cdot) = e^n_0 > 0 \qquad [\text{ic}] \qquad (\mathcal{P}_n)$$

$$\mathbf{q}(\cdot, e^n, \nabla e^n) \cdot \mathbf{n}(x) = 0 \qquad [\text{bc}]$$

For any $\lambda > 0$

$$\mathcal{E} := \left\{ e \geq 0; \quad e \in L^\infty(0,T; L^1(\Omega)), \quad \nabla(1+e)^{\frac{q-1-\lambda}{q}} \in L^q(0,T; L^q(\Omega)^d) \right\}$$

$$\|e^n\|_{\mathcal{E}} \leq C \implies \| \, |e^n|^{q-1} \, \|_{L^1(Q)} \leq C \quad \text{if } q > \frac{2d+1}{d+1}$$

$$\|\nabla T_k(e^n)\|_{L^q(Q)} \leq C.$$

$$\|T_k(e^n)_{,t}\|_{L^1(0,T;(W^{1,z})^*)} \leq C, \quad \text{for sufficiently large } z.$$

Consequently,

$$e^n \to e \qquad \text{almost everywhere in } Q$$

Weak Solution

Let $q > \frac{2d+1}{d+1}$ and $\mathbf{v} \in L^r(0, T; \mathbf{L}^s(\Omega))$ with

$$\frac{r'}{s} < \frac{q(d+1) - 2d}{d} \quad \text{and} \qquad s > \frac{d(q-1)}{q(d+1) - 2d}$$

We say that:

$e \in \mathcal{E}$ is a *weak solution* to Problem $(\mathcal{P})$ if for all $\varphi \in \mathcal{D}(-\infty, T; \mathcal{C}^\infty(\overline{\Omega}))$

$$-(e, \varphi_{,t})_Q + (\mathbf{q}(\cdot, e, \nabla e), \nabla\varphi)_Q = \langle f, \varphi\rangle + (e\mathbf{v}, \nabla\varphi)_Q + (e_0, \varphi(0))_\Omega$$

Theorem (**Bulíček**, **Consiglieri**, Málek)

There exists a weak solution to Problem $(\mathcal{P})$.

Entropy solution

Let $q > 1$ and $\mathbf{v} \in L^1(0, T; \mathbf{L}^1(\Omega))$ and $f \in L^1(Q)$.
We say that:

$e \in \mathcal{E}$ is an *entropy solution* to Problem $(\mathcal{P})$ if for a.a. $t \in (0, T)$

$$\langle \varphi_{,t}, T_k(e - \varphi)\rangle_{Q_t} + \int_\Omega \Theta_k(e(t) - \varphi(t)) + (\mathbf{q}(\cdot, e, \nabla e), \nabla T_k(e - \varphi))_{Q_t}$$

$$\leq (T_k(e - \varphi)\mathbf{v}, \nabla\varphi)_{Q_t} + (f, T_k(e - \varphi))_{Q_t} + \int_\Omega \Theta_k(e(0) - \varphi(0))\, dx$$

$$\text{for all } \varphi \in L^\infty(0, T; W^{1,\infty}(\Omega)) \text{ with } \varphi_{,t} \in L^{q'}(0, T; W^{-1,q'}(\Omega))$$

Theorem (**Bulíček**, **Consiglieri**, Málek)

There exists an entropy solution to Problem $(\mathcal{P})$. This solution is unique in the class of entropy solutions provided that $\mathbf{v} \in L^{q'}(Q)$ and $\mathbf{q}$ does not explicitly depends on e.

Results and their relation to earlier studies

$$e_{,t} + \mathrm{div}(e\mathbf{v}) + \mathrm{div}\,\mathbf{q}(\cdot, e, \nabla e) = f \geq 0 \quad \mathbf{v} \text{ given with } \mathrm{div}\,\mathbf{v} = 0$$

Theorem W/a. (**Bocardo, Murat '92**)
$\mathrm{div}(\mathbf{v}\theta) \in L^1$, f non-negative measure $\Longrightarrow$ existence of weak solution.

Theorem W/b. (**Diening, Růžička, Wolf '08**)
$\mathbf{v}\theta \in L^1$, $f \in L^{q'}(0, T; W^{-1,q'})$ $\Longrightarrow$ existence of weak solution.

Theorem W/c. (**Bulíček, Consiglieri**, Málek '08)
$\mathbf{v}\theta \in L^1$, f non-negative measure $\Longrightarrow$ existence of weak solution.

Theorem E/a. (**Prignet '97**)
$\mathbf{v} = \mathbf{0}$, $f \in L^1(Q)$ $\Longrightarrow$ existence and uniqueness of entropy solution.

Theorem E/b. (**Bulíček, Consiglieri**, Málek '08)
$\mathbf{v} \in L^1(Q)$, $f \in L^1(Q)$ $\Longrightarrow$ existence of entropy solution.
$\mathbf{v} \in L^{q'}(Q)$, $\mathbf{q} = \mathbf{q}(\cdot, \nabla e)$ and $f \in L^1(Q)$ $\Longrightarrow$ uniqueness.

Key step: almost everywhere convergence of $\{e^n\}$

Theorem

Let given $\mathbf{q}$ *fulfil the assumptions with* $q > 1$ *and* $\mathbf{v} \in L^1(Q)$. *Assume that* $\{|e^n|\}_{n=1}^{\infty}$ *is bounded in* $\mathcal{E}$, $\{f^n\}_{n=1}^{\infty}$ *is bounded in* $L^1(0, T; L^1(\Omega))$, *and*

$$\begin{aligned}\langle T_{k,\delta}(e^n)_{,t}, \varphi\rangle + (\mathbf{q}(\cdot, e^n, \nabla e^n), \nabla(T'_{k,\delta}(e^n)\varphi))_Q \\ = (f^n T'_{k,\delta}(e^n), \varphi)_Q + (e^n \mathcal{H}_n(\mathbf{v}), \nabla(T'_{k,\delta}(e^n)\varphi))_Q, \\ \textit{for all } \varphi \in L^\infty(0, T; W_0^{1,\infty}(\Omega)) \textit{ and all } k, \delta \in \mathbb{R}_+.\end{aligned}$$

Then there exists a subsequence e^n *and* e:

$$|e| \in \mathcal{E} \textit{ and } \nabla e^n \to \nabla e \textit{ a.e. in } Q$$

Key tool: Lipschitz approximations of Bochner functions/1

Lemma. Let for $1 < q < \infty$

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^q(0,T;W^{1,q}(\Omega)) \qquad f \in L^1(Q) \qquad \mathbf{q} \in L^{q'}(0,T;\mathbf{L}^{q'}$$

fulfil

$$u_{,t} = \operatorname{div} \mathbf{q} + f \quad \text{in } \mathcal{D}'(Q)\,.$$

Moreover, let $E \subset\subset Q$ be an open set such that

$$\mathcal{M}^\alpha(|\nabla u|) + \alpha \mathcal{M}^\alpha(|\mathbf{q}|) + \alpha \mathcal{M}^\alpha(|f|) \le C < +\infty, \quad \text{a.e. in } Q \setminus E. \quad (1)$$

Then there holds

$$\nabla \mathcal{L}_E^\alpha u \in L^\infty(0,T;L^\infty(\Omega))$$
$$\partial_t (\mathcal{L}_E^\alpha u)(\mathcal{L}_E^\alpha u - u) \in L^1_{loc}(Q)$$

Key tool: Lipschitz approximations of Bochner functions/2

and for all $\phi_1 \in \mathcal{C}_0^\infty(\Omega)$ and all $\phi_2 \in \mathcal{C}_0^\infty(0, T)$

$$
\begin{aligned}
\int_0^T \langle \partial_t u, T_\varepsilon(\mathcal{L}_E^\alpha u)\phi_1 \rangle \phi_2 \, dt = &- \int_Q \Theta_\varepsilon(\mathcal{L}_E^\alpha u)\phi_1(\partial_t \phi_2) \, dx \, dt \\
&- \int_Q (u - \mathcal{L}_E^\alpha u) \, \partial_t \left(T_\varepsilon(\mathcal{L}_E^\alpha u) \right) \phi_1 \phi_2 \, dx \, dt \\
&- \int_Q (u - \mathcal{L}_E^\alpha u) \, T_\varepsilon(\mathcal{L}_E^\alpha u)\phi_1 \left(\partial_t \phi_2 \right) \, dx \, dt
\end{aligned}
$$

Proof is a minor (important) generalization (due to BCM) of the assertion due to **Diening, Růžička and Wolf (2008)**.

Ads: Lipschitz approximations of Sobolev function/1

Theorem. (**Diening**, Málek, **Steinhauer** '08 inspired by **Frehse**, Málek, **Steinhauer** '03)
Let $1 < q < \infty$ and $\Omega \in \mathcal{C}^{0,1}$. Let

$$\mathbf{u}^n \in W_0^{1,q}(\Omega)^d \quad \text{and } \mathbf{u}^n \rightharpoonup \mathbf{0} \text{ weakly in } W_0^{1,q}(\Omega)^d.$$

Set

$$K := \sup_n \|\mathbf{u}^n\|_{1,q} < \infty,$$
$$\gamma_n := \|\mathbf{u}^n\|_q \to 0 \qquad (n \to \infty).$$

Let $\theta_n > 0$ be such that (e.g. $\theta_n := \sqrt{\gamma_n}$)

$$\theta_n \to 0 \qquad \text{and} \qquad \frac{\gamma_n}{\theta_n} \to 0 \qquad (n \to \infty).$$

Let $\mu_j := 2^{2^j}$.

Ads: Lipschitz approximations of Sobolev function/2

Then there exists a sequence $\lambda_{n,j} > 0$ with

$$\mu_j \leq \lambda_{n,j} \leq \mu_{j+1},$$

and a sequence $\mathbf{u}^{n,j} \in W_0^{1,\infty}(\Omega)^d$ such that for all $j, n \in \mathbb{N}$

$$\left\| \mathbf{u}^{n,j} \right\|_\infty \leq \theta_n \to 0 \qquad (n \to \infty),$$
$$\left\| \nabla \mathbf{u}^{n,j} \right\|_\infty \leq c\, \lambda_{n,j} \leq c\, \mu_{j+1}$$

and

$$\{\mathbf{u}^{n,j} \neq \mathbf{u}^n\} \subset \Omega \cap \big(\{M\mathbf{u}^n > \theta_n\} \cup \{M(\nabla \mathbf{u}^n) > 2\,\lambda_{n,j}\}\big),$$

and for all $j \in \mathbb{N}$ and $n \to \infty$

$$\mathbf{u}^{n,j} \to \mathbf{0} \quad \text{strongly in } L^s(\Omega)^d \text{ for all } s \in [1, \infty],$$
$$\mathbf{u}^{n,j} \rightharpoonup \mathbf{0} \quad \text{weakly in } W_0^{1,s}(\Omega)^d \text{ for all } s \in [1, \infty),$$
$$\nabla \mathbf{u}^{n,j} \overset{*}{\rightharpoonup} \mathbf{0} \quad \text{weakly-}* \text{ in } L^\infty(\Omega)^{d \times d}.$$

Ads: Lipschitz approximations of Sobolev function/3

Furthermore, for all $n, j \in \mathbb{N}$

$$|\{\mathbf{u}^{n,j} \neq \mathbf{u}^n\}|_d \leq \frac{c\|\mathbf{u}^n\|_{1,q}^q}{\lambda_{n,j}^q} + c\left(\frac{\gamma^n}{\theta^n}\right)^q$$

and

$$\|\nabla \mathbf{u}^{n,j}\, \chi_{\{\mathbf{u}^{n,j} \neq \mathbf{u}^n\}}\|_q \leq c\,\|\lambda_{n,j}\chi_{\{\mathbf{u}^{n,j} \neq \mathbf{u}^n\}}\|_q \leq c\,\frac{\gamma_n}{\theta_n}\,\mu_{j+1} + c\,\epsilon_j,$$

where $\epsilon_j := K\,2^{-j/q}$ vanishes as $j \to \infty$. The constant c depends on Ω.

Ads: Lipschitz approximations of Sobolev function/4

The gradient of any function $\phi \in W^{1,1}_{\mathrm{loc}}(\Omega)$, that is constant on some measurable subset of Ω, vanishes on this set. Consequently for $\phi := \mathbf{u}^{n,j}$

$$\begin{aligned}\nabla \mathbf{u}^{n,j} &= \nabla(\mathbf{u}^{n,j} - \mathbf{u}^n) + \nabla \mathbf{u}^n = (\nabla \mathbf{u}^{n,j} - \nabla \mathbf{u}^n)\chi_{\{\mathbf{u}^{n,j} \neq \mathbf{u}^n\}} + \nabla \mathbf{u}^n \\ &= \nabla \mathbf{u}^{n,j} \chi_{\{\mathbf{u}^{n,j} \neq \mathbf{u}^n\}} + \nabla \mathbf{u}^n \chi_{\{\mathbf{u}^{n,j} = \mathbf{u}^n\}} .\end{aligned}$$

In particular this implies that

$$\text{if } \operatorname{div} \mathbf{u}^n = 0 \text{ then } \operatorname{div} \mathbf{u}^{n,j} = \operatorname{div} \mathbf{u}^{n,j} \chi_{\{\mathbf{u}^{n,j} \neq \mathbf{u}^n\}} .$$

Relation to analysis of unsteady flows of heat-conducting incompressible fluids/1

$$\operatorname{div} \mathbf{v} = 0 \tag{2}$$

$$\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} = -\nabla p \tag{3}$$

$$(e + |\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div} \mathbf{q} = \operatorname{div} (\mathbf{S}\mathbf{v}) \tag{4}$$

- $\mathbf{v}$ ... velocity
- e ... internal energy   total energy $E := e + |\mathbf{v}|^2/2$
- p ... pressure
- $\mathbf{S}$ ... a part of the Cauchy stress $\mathbf{T} = -p\mathbf{I} + \mathbf{S}$, $\mathbf{S} = \mathbf{S}^T$
- $\mathbf{q}$ ... heat flux

Nonlinear system of PDEs

Constitutive equations

$$\operatorname{div} \mathbf{v} = 0$$

$$\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} = -\nabla p$$

$$(e + |\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div} \mathbf{q} = \operatorname{div}(\mathbf{S}\mathbf{v})$$

Constitutive equations $\quad 2\mathbf{D}(\mathbf{v}) := \nabla \mathbf{v} + (\nabla \mathbf{v})^T$

$$\mathbf{S} = \nu(p, e, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}) \tag{5}$$

$$\mathbf{q} = -\kappa(p, e, \nabla e, |\mathbf{D}(\mathbf{v})|^2)\nabla e \tag{6}$$

- Linear (Navier-Stokes and Fourier) relations
- Non-Linear constitutive equations (power-law, etc.)

Constitutive Equations - examples

- $\nu(|\mathbf{D}(\mathbf{v})|^2) = \nu_0 |\mathbf{D}(\mathbf{v})|^{r-2}$ Power-law fluids $r \in [1, \infty)$
- $\nu(|\mathbf{D}(\mathbf{v})|^2) = \nu_0 + \nu_1 |\mathbf{D}(\mathbf{v})|^{r-2}$ Generalized NS fluids $r \in [1, \infty)$
- $\nu(p) = \nu_0 \exp(\alpha p)$ Barus (1893)
- $\nu(\theta) = \nu_0 \exp\left(\frac{a}{b+\theta}\right)$ Vogel (1922)
- $\nu(p, \theta) = A \exp\left(\frac{Bp+D}{\theta}\right)$ Andrade's (1929), Bridgman (1931)
- $\nu(p, |\mathbf{D}(\mathbf{v})|^2) = \frac{\nu_0 p}{|\mathbf{D}(\mathbf{v})|}$ Schaeffer (1987)

IBVP

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} &= -\nabla p \\ (e + |\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div} \mathbf{q} &= \operatorname{div}(\mathbf{S}\mathbf{v}) \end{aligned}$$

Data

- $\Omega \subset \mathbb{R}^3$ bounded open connected container, $T \in (0, \infty)$ length of time interval
- $\mathbf{v}(0, \cdot) = \mathbf{v}_0$, $e(0, \cdot) = e_0$ in Ω
- α that appears in boundary conditions (thermally and mechanically or energetically isolated body)

Task Mathematical Consistency of a Model - for any set of data to find uniquely defined, smooth, solution (*notion of solution, its existence, uniqueness, regularity*)

Weak solution - solution dealing with averages

Boundary conditions

$$(e + |\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div} \mathbf{q} - \operatorname{div} (\mathbf{S}\mathbf{v}) = 0$$

$$\frac{d}{dt}\left(\int_\Omega E(t,x)\, dx\right) + \int_{\partial\Omega} [(E+p)\mathbf{v}\cdot\mathbf{n} + \mathbf{q}\cdot\mathbf{n} - \mathbf{S}\mathbf{v}\cdot\mathbf{n}]\; dS = 0$$

Mechanically and thermally isolated body, Navier's slip on $[0, T] \times \Omega$:

- $\mathbf{v}\cdot\mathbf{n} = 0 \qquad \mathbf{q}\cdot\mathbf{n} = 0$
- $\lambda(\mathbf{S}\mathbf{n})_{\boldsymbol{\tau}} + (1-\lambda)\mathbf{v}_{\boldsymbol{\tau}} = \mathbf{0}$ for $\lambda \in (0,1) \qquad \mathbf{u}_{\boldsymbol{\tau}} := \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}$
- $\lambda = 0 \implies$ no-slip $\qquad \lambda = 1 \implies$ slip

Energetically isolated body, Navier's slip on $[0, T] \times \Omega$:

- $\mathbf{v}\cdot\mathbf{n} = 0 \qquad \mathbf{q}\cdot\mathbf{n} = -\alpha|\mathbf{v}_{\boldsymbol{\tau}}|^2$
- $(\mathbf{S}\mathbf{n})_{\boldsymbol{\tau}} + \alpha\mathbf{v}_{\boldsymbol{\tau}} = \mathbf{0} \qquad \alpha := (1-\lambda)/\lambda$

"Equivalent" formulation of the balance of energy/1

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} &= -\nabla p \\ (e + |\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div} \mathbf{q} &= \operatorname{div}(\mathbf{S}\mathbf{v}) \end{aligned}$$

is equivalent (if $\mathbf{v}$ is admissible test function in BM) to

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} &= -\nabla p \\ e_{,t} + \operatorname{div}(e\mathbf{v}) + \operatorname{div} \mathbf{q} &= \mathbf{S} \cdot \mathbf{D}(\mathbf{v}) \end{aligned}$$

Helmholtz decomposition $\mathbf{u} = \mathbf{u}_{\mathrm{div}} + \nabla g^{\mathbf{v}}$
Leray's projector $\mathbb{P} : \mathbf{u} \mapsto \mathbf{u}_{\mathrm{div}}$

"Equivalent" formulation of the balance of energy/2

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} &= -\nabla p \\ (e + |\mathbf{v}|^2/2)_{,t} + \operatorname{div}((e + |\mathbf{v}|^2/2 + p)\mathbf{v}) + \operatorname{div} \mathbf{q} &= \operatorname{div}(\mathbf{S}\mathbf{v}) \end{aligned}$$

is equivalent (if $\mathbf{v}$ is admissible test function in BM) to

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ \mathbf{v}_{,t} + \mathbb{P} \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \mathbb{P} \operatorname{div} \mathbf{S} &= \mathbf{0} \\ e_{,t} + \operatorname{div}(e\mathbf{v}) + \operatorname{div} \mathbf{q} &= \mathbf{S} \cdot \mathbf{D}(\mathbf{v}) \end{aligned}$$

Advantages/Disadvantages

- $+$ pressure is not included into the 2nd formulation
- $+$ minimum principle for e if $\mathbf{S} \cdot \mathbf{D}(\mathbf{v}) \geq 0$
- $-$ $\mathbf{S} \cdot \mathbf{D}(\mathbf{v}) \in L^1$ while $\mathbf{S}\mathbf{v} \in L^q$ with $q > 1$

Assumptions on $\mathbf{S} = \nu(e, |\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v})$ and $\mathbf{q} = \kappa(e, \nabla e)\nabla e$

(C1) given $\boxed{r > 1}$ there are $C_1 > 0$ and $C_2 > 0$ such that for all symmetric matrices $\mathbf{B}$, $\mathbf{D}$ and $e \in \mathcal{R}^+$

$$C_1(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}}|\mathbf{B}|^2 \leq \frac{\partial\left[\nu(e, |\mathbf{D}|^2)\mathbf{D}\right]}{\partial \mathbf{D}} \cdot (\mathbf{B} \otimes \mathbf{B}) \leq C_2(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}}|\mathbf{B}|^2$$

(C2) given $\boxed{q > 1}$ there are $C_3 > 0$ and $C_4 > 0$ such that for all vectors $\mathbf{u}$, $\mathbf{w}$ and $e \in \mathcal{R}^+$

$$C_3(1 + |\mathbf{u}|^2)^{\frac{q-2}{2}}|\mathbf{w}|^2 \leq \frac{\partial\left[\kappa(e, \mathbf{u})\mathbf{u}\right]}{\partial \mathbf{u}} \cdot (\mathbf{w} \otimes \mathbf{w}) \leq C_4(1 + |\mathbf{u}|^2)^{\frac{q-2}{2}}|\mathbf{w}|^2$$

Result

Theorem 4. (**M. Bulíček, L. Consiglieri**, J. Málek '07)
Let **(C1)–(C2)** hold and r and q fulfil

$$\boxed{r > \frac{9}{5} \text{ and } q > \frac{7}{4}}$$

Assume that

- $\partial\Omega \in C^{1,1}$
- $\mathbf{v}_0 \in L^2_{\mathbf{n},div}$ and $e_0 \in L^1$, $e_0 \geq C^* > 0$ a.a. in Ω

Then for all $T > 0$ (and any $\alpha \in (0,1]$) and any $(\mathbf{v}_0, e_0)$ there exists at least one suitable weak solution $(\mathbf{v}, p, e)$ of the system relevant system completed by Navier's slip boundary conditions (mechanically and thermally isolated domain).

Concluding remarks/1

- General mathematical theory for internal unsteady flows of incompressible heat conducting fluids - mathematical self-consistency of IBVP
- Implicit constitutive theory
- "Equivalent" forms of the balance of energy
- The role of boundary conditions at tangent directions to the boundary

Concluding remarks/2

Methods to take the limit in nonlinearities (three groups)

- Convective terms: products of weakly and strongly converging sequences, Aubin-Lions compactness lemma for $\mathbf{v}$ and e
- Material nonlinearities: monotone operator theory, L^∞-truncation and Lipschitz truncation method, perturbations of strictly monotone operators
- Term representing the dissipation energy: energy equality method (if $\mathbf{v}$ is admissible test function in BLM), otherwise use a primary form of energy balance
- Entropy, renormalized, suitable, dissipative solutions: use maximum information that is in place

Concluding remarks/3

Open problems

- $\nu(p, e)$ or $\nu(p)$
- BC's: no-slip, inflow, outflow
- Qualitative theory: uniqueness, regularity
- More complicated constitutive relations (stress relaxation, normal stress differences, nonlinear creep), discontinuous (fully implicit) relationships