2D compressible vortex sheets

Paolo Secchi

Department of Mathematics
Brescia University

Joint work with J.F. Coulombel

EVEQ 2008, International Summer School on Evolution Equations,
Prague, Czech Republic, 16 - 20. 6. 2008
1. **Introduction**
   - Euler’s equations of isentropic gas dynamics
   - Smooth and piecewise smooth solutions
   - Existence results

2. **Compressible Vortex Sheets**
   - Compressible vortex sheets
   - Linear Spectral Stability
   - Formulation of the problem

3. **Main Result**
   - Linear stability: $L^2$ estimate
   - Linear stability: Tame estimate in Sobolev norm
   - Nonlinear stability: Nash-Moser iteration

4. **Related Problems**
   - Weakly stable shock waves
   - Subsonic phase transitions
We consider a **compressible inviscid** fluid described by

- the density $\rho(t, x) \in \mathbb{R}$
- the velocity field $u(t, x) \in \mathbb{R}^d$
- the pressure $p = p(\rho)$, where $p \in C^\infty$, $p' > 0$,

whose evolution is governed by the **Euler** equations

\[
\begin{aligned}
\partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x p(\rho) &= 0,
\end{aligned}
\]

where $t \geq 0$ denotes the time variable, $x \in \mathbb{R}^d$ the space variable.
Plan

1. Introduction
   - Euler’s equations of isentropic gas dynamics
   - Smooth and piecewise smooth solutions
   - Existence results

2. Compressible Vortex Sheets
   - Compressible vortex sheets
   - Linear Spectral Stability
   - Formulation of the problem

3. Main Result
   - Linear stability: $L^2$ estimate
   - Linear stability: Tame estimate in Sobolev norm
   - Nonlinear stability: Nash-Moser iteration

4. Related Problems
   - Weakly stable shock waves
   - Subsonic phase transitions
The Euler equations may be written as a symmetric hyperbolic system. This allows to solve locally in time the Cauchy problem:

- Initial data $\rho_0 \in \overline{\rho} + H^s(\mathbb{R}^d)$, $u_0 \in H^s(\mathbb{R}^d)$ with $s > 1 + d/2$.
  Existence and uniqueness of a solution in the space $C([0, T]; \overline{\rho} + H^s(\mathbb{R}^d)) \times C([0, T]; H^s(\mathbb{R}^d))$ [Kato, 1975]


Local smooth solution of the initial boundary value problem under the slip boundary condition $u \cdot \nu = 0$ (characteristic boundary) [Beirao da Veiga, 1981]
The function

\[(\rho, u) := \begin{cases} 
(\rho^+, u^+) & \text{if } x_d > \varphi(t, x_1, \ldots, x_{d-1}) \\
(\rho^-, u^-) & \text{if } x_d < \varphi(t, x_1, \ldots, x_{d-1})
\end{cases}\]

is a weak solution of the Euler equations if

\((\rho^\pm, u^\pm)\) is a smooth solution on either sides of the interface

\[\Sigma := \{x_d = \varphi(t, x_1, \ldots, x_{d-1})\}\]

and it satisfies the Rankine-Hugoniot jump conditions at \(\Sigma\):

\[
\begin{align*}
\partial_t \varphi [\rho] - [\rho u \cdot \nu] &= 0, \\
\partial_t \varphi [\rho u] - [\rho u \cdot \nu] u - [p] \nu &= 0,
\end{align*}
\]

\(\nu\) is a (space) normal vector to \(\Sigma\); \([q] := q^+ - q^-\) denotes the jump of \(q\) across \(\Sigma\).

\(\Sigma\) is an unknown of the problem. **Free boundary problem**!
**Plan**

1. **Introduction**
   - Euler’s equations of isentropic gas dynamics
   - Smooth and piecewise smooth solutions
   - Existence results

2. **Compressible Vortex Sheets**
   - Compressible vortex sheets
   - Linear Spectral Stability
   - Formulation of the problem

3. **Main Result**
   - Linear stability: $L^2$ estimate
   - Linear stability: Tame estimate in Sobolev norm
   - Nonlinear stability: Nash-Moser iteration

4. **Related Problems**
   - Weakly stable shock waves
   - Subsonic phase transitions

**P. Secchi (Brescia University)**
Existence results

- Existence of two uniformly stable shock waves [Métiévier, 1986]
- Existence of one rarefaction wave [Alinhac, 1989]
- Existence of sound waves [Métiévier, 1991]
- Existence of one small shock wave [Francheteau & Métiévier, 2000]
## Plan

1. **Introduction**
   - Euler’s equations of isentropic gas dynamics
   - Smooth and piecewise smooth solutions
   - Existence results

2. **Compressible vortex sheets**
   - Compressible vortex sheets
   - Linear Spectral Stability
   - Formulation of the problem

3. **Main result**
   - Linear stability: $L^2$ estimate
   - Linear stability: Tame estimate in Sobolev norm
   - Nonlinear stability: Nash-Moser iteration

4. **Related problems**
   - Weakly stable shock waves
   - Subsonic phase transitions

---

**P. Secchi (Brescia University)**

**Compressible vortex sheets**
\((\rho, \mathbf{u})\) is a contact discontinuity if the Rankine-Hugoniot conditions (??) are satisfied in the form

\[
\partial_t \varphi = \mathbf{u}^+ \cdot \nu = \mathbf{u}^- \cdot \nu, \\
\quad p^+ = p^-.
\]

\(p\) monotone gives equivalently

\[
\partial_t \varphi = \mathbf{u}^+ \cdot \nu = \mathbf{u}^- \cdot \nu, \\
\quad \rho^+ = \rho^-.
\]

The front \(\Sigma := \{x_d = \varphi(t,x_1,\ldots,x_{d-1})\}\) is characteristic with respect to either side.

Density and normal velocity are continuous across the front \(\Sigma\).

Jump of tangential velocity \(\implies\) vortex sheet.
We want to show the (local) existence of compressible vortex sheets (contact discontinuities).
Plan

1. Introduction
   - Euler’s equations of isentropic gas dynamics
   - Smooth and piecewise smooth solutions
   - Existence results

2. Compressible Vortex Sheets
   - Compressible vortex sheets
   - Linear Spectral Stability
   - Formulation of the problem

3. Main Result
   - Linear stability: $L^2$ estimate
   - Linear stability: Tame estimate in Sobolev norm
   - Nonlinear stability: Nash-Moser iteration

4. Related Problems
   - Weakly stable shock waves
   - Subsonic phase transitions
Linearize the Euler equations around a \textit{piecewise constant} vortex sheet

\[
(\rho, \mathbf{u}) = \begin{cases} 
(\bar{\rho}, \bar{v}, 0), & \text{if } x_d > 0, \\
(\bar{\rho}, -\bar{v}, 0), & \text{if } x_d < 0.
\end{cases}
\]

- If \( d = 3 \), the linearized equations do not satisfy the Lopatinskii condition (\( \exists \) exponentially exploding modes!) \( \Rightarrow \) violent instability.

- If \( d = 2 \), and \( |[\mathbf{u} \cdot \tau]| < 2\sqrt{2}c(\rho) \) the linearized equations do not satisfy the Lopatinskii condition \( \Rightarrow \) violent instability.

- If \( d = 2 \), and \( |[\mathbf{u} \cdot \tau]| > 2\sqrt{2}c(\rho) \) the linearized equations satisfy the weak Lopatinskii condition \( \Rightarrow \) weak stability,

where \( c(\rho) := \sqrt{p'(\rho)} \) is the sound speed and \( \tau \) a tangential unit vector to \( \Sigma \).
Plan

1. Introduction
   - Euler's equations of isentropic gas dynamics
   - Smooth and piecewise smooth solutions
   - Existence results

2. Compressible vortex sheets
   - Compressible vortex sheets
   - Linear Spectral Stability
   - Formulation of the problem

3. Main result
   - Linear stability: $L^2$ estimate
   - Linear stability: Tame estimate in Sobolev norm
   - Nonlinear stability: Nash-Moser iteration

4. Related problems
   - Weakly stable shock waves
   - Subsonic phase transitions
Formulation of the problem

The interface $\Sigma := \{x_2 = \varphi(t, x_1)\}$ is unknown so that the problem is a free boundary problem.
In order to work in a fixed domain $\{y_2 > 0\}$ we introduce the change of variables

$$(\tau, y_1, y_2) \rightarrow (t, x_1, x_2),$$
$$(t, x_1) = (\tau, y_1),$$
$$x_2 = \Phi(\tau, y_1, y_2),$$

where

$$\Phi : \{ (\tau, y_1, y_2) \in \mathbb{R}^3 \} \rightarrow \mathbb{R},$$
$$\Phi(\tau, y_1, 0) = \varphi(t, x_1), \quad \partial_{y_2} \Phi(\tau, y_1, y_2) \geq \kappa > 0.$$

We write again $(t, x_1, x_2)$ instead of $(\tau, y_1, y_2)$.
Denote $\Phi^{\pm}(t, x_1, x_2) := \Phi(t, x_1, \pm x_2)$. 

P. Secchi (Brescia University)
By the Rankine-Hugoniot conditions the boundary matrix of the system of equations is singular at \( \{ x_2 = 0 \} \), i.e. the interface is a characteristic boundary. The \( 3 + 3 \) equations are not sufficient to determine the unknowns \( U^\pm := (\rho^\pm, u^\pm) = (\rho^\pm, v^\pm, u^\pm) \) and \( \Phi^\pm \).

We may prescribe that \( \Phi^\pm \) solve in the domain \( \{ x_2 > 0 \} \) the eikonal equations

\[
\partial_t \Phi^\pm + v^\pm \partial_{x_1} \Phi^\pm - u^\pm = 0.
\]

This choice has the advantage that the boundary matrix of the system for \( U^\pm \) has constant rank in the whole domain \( \{ x_2 \geq 0 \} \) (uniformly characteristic boundary).
We obtain the first order system:

\[
\begin{align*}
\partial_t \rho^+ + v^+ \partial_{x_1} \rho^+ + (u^+ - \partial_t \Phi^- - v^+ \partial_{x_1} \Phi^-) \frac{\partial x_2 \rho^+}{\partial x_2 \Phi^-} + \rho^+ \partial_{x_1} v^+ \\
+ \rho^+ \frac{\partial x_2 u^+}{\partial x_2 \Phi^-} - \rho^+ \frac{\partial x_1 \Phi^+}{\partial x_2 \Phi^-} \partial_{x_2} v^+ = 0, \\
\partial_t v^+ + v^+ \partial_{x_1} v^+ + (u^+ - \partial_t \Phi^- - v^+ \partial_{x_1} \Phi^-) \frac{\partial x_2 v^+}{\partial x_2 \Phi^-} + \frac{p'(\rho^+)}{\rho^+} \partial_{x_1} \rho^+ \\
- \frac{p'(\rho^+)}{\rho^+} \frac{\partial x_1 \Phi^+}{\partial x_2 \Phi^-} \partial_{x_2} \rho^+ = 0, \\
\partial_t u^+ + v^+ \partial_{x_1} u^+ + (u^+ - \partial_t \Phi^- - v^+ \partial_{x_1} \Phi^-) \frac{\partial x_2 u^+}{\partial x_2 \Phi^-} + \frac{p'(\rho^+)}{\rho^+} \frac{\partial x_2 \rho^+}{\partial x_2 \Phi^-} = 0,
\end{align*}
\]

in the fixed domain \( \{x_2 > 0\} \).

\((\rho^-, v^-, u^-, \Phi^-)\) should solve a similar system.
The boundary conditions are

\[
\Phi^+_|_{x_2=0} = \Phi^-|_{x_2=0} = \varphi, \\
(v^+ - v^-)|_{x_2=0} \partial x_1 \varphi - (u^+ - u^-)|_{x_2=0} = 0, \\
\partial_t \varphi + v^+_|_{x_2=0} \partial x_1 \varphi - u^+_|_{x_2=0} = 0, \\
(\rho^+ - \rho^-)|_{x_2=0} = 0, \quad (t, x) \in [0, T] \times \mathbb{R}_+^2,
\]

that we rewrite in the compact form as

\[
\Phi^+_|_{x_2=0} = \Phi^-|_{x_2=0} = \varphi, \\
\mathbb{B}(U^+_|_{x_2=0}, U^-|_{x_2=0}, \varphi) = 0.
\]
We obtain the (non standard) IBVP

\[
\begin{align*}
\partial_t U^\pm + A_1(U^\pm)\partial_{x_1} U^\pm + A_2(U^\pm, \nabla \Phi^\pm)\partial_{x_2} U^\pm &= 0, \\
\partial_t \Phi^\pm + v^\pm \partial_{x_1} \Phi^\pm - u^\pm &= 0, \\
\Phi^+_{|x_2=0} &= \Phi^-_{|x_2=0} = \varphi, \\
\mathcal{B}(U^+_{|x_2=0}, U^-_{|x_2=0}, \varphi) &= 0, \\
(U^\pm, \Phi^\pm)|_{t=0} &= (U^\pm_0, \Phi^\pm_0), \quad x \in \mathbb{R}^2_+.
\end{align*}
\]
Theorem (Coulombel, S., 2004, 2008)

Let $d = 2$ and consider a piecewise constant weakly stable vortex sheet. Let $T > 0$ and $m \geq 6$.

Consider initial data $(U_0^\pm, \varphi_0)$ that are perturbations in $H^{m+15/2}(\mathbb{R}_+^2) \times H^{m+8}(\mathbb{R})$ of the piecewise constant vortex sheet. The initial data have compact support and satisfy suitable compatibility conditions. If the perturbation is sufficiently small, then there exists a unique solution $(U^\pm, \varphi)$ on $[0, T]$ with initial data $(U_0^\pm, \varphi_0)$. The solution belongs to the space $H^m([0, T[ \times \mathbb{R}_+^2) \times H^{m+1}([0, T[ \times \mathbb{R})$. 

P. Secchi (Brescia University)
# Plan

## 1 Introduction
- Euler’s equations of isentropic gas dynamics
- Smooth and piecewise smooth solutions
- Existence results

## 2 Compressible Vortex Sheets
- Compressible vortex sheets
- Linear Spectral Stability
- Formulation of the problem

## 3 Main Result
- Linear stability: $L^2$ estimate
- Linear stability: Tame estimate in Sobolev norm
- Nonlinear stability: Nash-Moser iteration

## 4 Related Problems
- Weakly stable shock waves
- Subsonic phase transitions
Consider a perturbation of the piecewise constant solution

\[ U_{r,l} = \begin{pmatrix} \bar{\rho} \\ \pm \bar{v} \\ 0 \end{pmatrix} + \dot{U}_{r,l}(t,x), \]

\[ \Phi_{r,l} = \pm x_2 + \dot{\Phi}_{r,l}(t,x), \]

where \( U_{r,l}, \Phi_{r,l} \) are linked by the Rankine-Hugoniot conditions, \( \dot{U}_{r,l} \) and \( \dot{\Phi}_{r,l} \) have compact support, and solve the eikonal equations

\[ \partial_t \Phi_{r,l} + v_{r,l} \partial_{x_1} \Phi_{r,l} - u_{r,l} = 0. \] (3)

Let us consider the linearized equations around \( U_{r,l}, \Phi_{r,l} \):

\[ \mathcal{L}(U_{r,l}, \Phi_{r,l})W = f \quad \text{in } ]0, T[ \times \mathbb{R}^2_+, \]

\[ \mathcal{B}(U_{r,l}, \Phi_{r,l})(W, \psi) = g \quad \text{on } ]0, T[ \times \mathbb{R}. \]
Let $T > 0$. Assume that

(i) the piecewise constant solution $\overline{U}^\pm$ is weakly stable,

(ii) $(\overline{U}^\pm + \dot{U}_{r,l}, \pm x_2 + \dot{\Phi}_{r,l})$ satisfies the Rankine-Hugoniot conditions and the eikonal equations $(??)$,

(iii) the perturbation $(\dot{U}_{r,l}, \dot{\Phi}_{r,l})$ has compact support and is sufficiently small in $W^{3,\infty}(]0, T[ \times \mathbb{R}^2_+)$.

Then there exists a solution of the linearized equations that satisfies the a priori estimate:

$$
\| W \|_{L^2(]0, T[ \times \mathbb{R}^2_+)}^2 + \| W^{nc} \|_{L^2(]0, T[ \times \mathbb{R})}^2 + \| \psi \|_{H^1(]0, T[ \times \mathbb{R})}^2 
\leq C \left( \| f \|_{L^2(\mathbb{R}^+; H^1(]0, T[ \times \mathbb{R}))}^2 + \| g \|_{H^1(]0, T[ \times \mathbb{R})}^2 \right) .
$$
Microlocal analysis of the paralinearized system associated to the linearized equations.

Determination of the roots of the Lopatinskii determinant, the poles and the points of non diagonalization of the symbol.

The singularities of the solution are (micro)localized on bicharacteristic curves propagating from the boundary in the interior domain.

Despite the loss of regularity, the linearized problem is well-posed in $L^2$ with source terms in $H^1$. [Coulombel, 2005]
Paralinearization of the equations.

Using the paradifferential calculus (extension of the pseudodifferential calculus which allows a low regularity of the symbols), we substitute in the equations the paradifferential operators (w.r.t. the tangential variables $\left(t, x_1\right)$) and obtain a system of O.D.E. in $x_2$ with symbols instead of derivatives in $\left(t, x_1\right)$.

This step essentially reduces to the constant coefficient case.

Elimination of the front.

The projected boundary condition onto a suitable subspace of the frequency space gives an elliptic equation of order one for the front $\psi$. One obtains an estimate of the form

$$\|\psi\|_{H^1_\gamma(\omega_T)}^2 \leq C \left( \frac{1}{\gamma^2} \|\mathcal{B}(W^{nc}, \psi)\|_{H^1_\gamma(\omega_T)}^2 + \|W^{nc} \mid_{x_2=0}\|_{L^2_\gamma(\omega_T)}^2 \right) + \text{error terms},$$

with no loss of regularity with respect to the source terms.

Thus, it is enough to estimate $W^{nc} \mid_{x_2=0}$. 
Problem with reduced boundary conditions.

The projection of the boundary condition onto the orthogonal subspace gives a boundary condition involving only $W^{nc}$, i.e. without involving $\psi$. Thus we are left with the (paradifferential version of the) linear problem for $W$

$$A_0^r \partial_t W^+ + A_1^r \partial_{x_1} W^+ + I_2 \partial_{x_2} W^+ + A_0^r C^r W^+ = F^+ , \quad x_2 > 0 ,$$

$$A_0^l \partial_t W^- + A_1^l \partial_{x_1} W^- + I_2 \partial_{x_2} W^- + A_0^l C^l W^- = F^- , \quad x_2 > 0 ,$$

$$\Pi \tilde{M} W|_{x_2=0} = \Pi g , \quad x_2 = 0 , \quad (4)$$

where $\text{diag} (0, 1, 1)$, and $\Pi$ denotes the suitable projection operator.
The boundary is characteristic with constant multiplicity. The problem satisfies a Kreiss-Lopatinski condition in the weak sense and not uniformly. In fact, the Lopatinski determinant associated to the boundary condition vanishes at some points in the frequency space (only simple roots). The proof of the $L^2$ energy estimate is based on the construction of a degenerate Kreiss’ symmetrizer.

In order to explain the main idea, let us consider for simplicity the linearization around the piecewise constant solution (constant coefficients case).
Then, instead of (??), we have a problem of the form ($\hat{W} = $ Fourier transform in $(t, x_1)$)

$$
(\tau A_0 + i\eta A_1)\hat{W} + A_2 \frac{d\hat{W}}{dx_2} = 0, \quad x_2 > 0,
$$
$$
\beta(\tau, \eta)\hat{W}_{nc}(0) = \hat{h}, \quad x_2 = 0.
$$

Because of the characteristic boundary, the two first equations do not involve differentiation with respect to the normal variable $x_2$:

$$
(\tau + ivr\eta)\hat{W}_1^+ - ic^2\eta\hat{W}_2^+ + ic^2\eta\hat{W}_3^+ = 0,
$$
$$
(\tau + ivl\eta)\hat{W}_1^- - ic^2\eta\hat{W}_2^- + ic^2\eta\hat{W}_3^- = 0.
$$

For Re $\tau > 0$, we obtain an expression for $\hat{W}_1^+$ and $\hat{W}_1^-$ that we plug in the other equations.

This operation yields a system of O.D.E. of the form:

$$
\frac{d\hat{W}_{nc}}{dx_2} = A(\tau, \eta)\hat{W}_{nc}, \quad x_2 > 0,
$$
$$
\beta(\tau, \eta)\hat{W}_{nc}(0) = \hat{h}, \quad x_2 = 0.
$$
By microlocalization, the analysis is performed in the neighborhood of points $(\tau, \eta)$ of the following type:

1) Points where $\mathcal{A}(\tau, \eta)$ is diagonalizable and the Lopatinskii condition is satisfied. By using the classical Kreiss’ symmetrizer we obtain an $L^2$ estimate with no loss of derivatives.

2) Points where $\mathcal{A}(\tau, \eta)$ is diagonalizable and the Lopatinskii condition breaks down (the Lopatinskii determinant has simple roots). We construct a degenerate Kreiss’ symmetrizer; this yields an $L^2$ estimate with loss of one derivative.

3) Points where $\mathcal{A}(\tau, \eta)$ is not diagonalizable. In those points, the Lopatinskii condition is satisfied.

4) Poles of $\mathcal{A}$. At those points, the Lopatinskii condition is satisfied. We construct a symmetrizer by working on the original system (??).
Plan

1. Introduction
   - Euler’s equations of isentropic gas dynamics
   - Smooth and piecewise smooth solutions
   - Existence results

2. Compressible vortex sheets
   - Compressible vortex sheets
   - Linear Spectral Stability
   - Formulation of the problem

3. Main Result
   - Linear stability: $L^2$ estimate
   - Linear stability: Tame estimate in Sobolev norm
   - Nonlinear stability: Nash-Moser iteration

4. Related Problems
   - Weakly stable shock waves
   - Subsonic phase transitions
**Theorem**

Let $T > 0$ and let $m \geq 3$ be an integer. Assume that (i) the piecewise constant solution $\overline{U}^\pm$ is weakly stable, (ii) $(\overline{U}^\pm + \dot{U}_{r,l}, \pm x_2 + \dot{\Phi}_{r,l})$ satisfies the Rankine-Hugoniot conditions and the eikonal equations (??), (iii) the perturbation $(\dot{U}_{r,l}, \dot{\Phi}_{r,l})$ has compact support and is sufficiently small in $H^6(\mathbb{R}^2 \times [0, T])$. Then the solution of the linearized equations satisfies the a priori estimate:

\[
\| W \|_{H^m([0, T] \times \mathbb{R}^2)} + \| W_{nc} \|_{x_2=0} \| H^m([0, T] \times \mathbb{R}) + \| \psi \|_{H^{m+1}([0, T] \times \mathbb{R})} \\
\leq C \left\{ \| f \|_{H^{m+1}([0, T] \times \mathbb{R}^2)} + \| g \|_{H^{m+1}([0, T] \times \mathbb{R})} + \\
+ \left( \| f \|_{H^4([0, T] \times \mathbb{R}^2)} + \| g \|_{H^4([0, T] \times \mathbb{R})} \right) \| (\dot{U}_{r,l}, \dot{\Phi}_{r,l}) \|_{H^{m+3}([0, T] \times \mathbb{R}^2)} \right\}.
\]
Apply the $L^2$ energy estimates to the tangential derivatives.

Normal derivatives estimated via the equations and a vorticity equation. No loss of normal regularity inspite of the characteristic boundary. No need to work in spaces with conormal regularity.
1) Estimate of **tangential derivatives** $\partial_t^h \partial_{x_1}^k W$ and the **front function** $\psi$ by differentiation of the equations along the tangential directions and application of the $L^2$ energy estimate given in Theorem 2.

2) Estimate of **normal derivatives**.

Consider the original non-linear equations. On both sides of the interface the solution is smooth, the interface is a streamline and there is continuity of the normal velocity across the interface; this suggests to estimate the vorticity on either part of the front.

We define the **"linearized vorticity"**

$$\dot{\xi}_\pm := \partial_{x_1} \dot{u}_\pm - \frac{1}{\partial_{x_2} \Phi_{r,l}} (\partial_{x_1} \Phi_{r,l} \partial_{x_2} \dot{u}_\pm + \partial_{x_2} \dot{v}_\pm).$$
Then

\[ \partial_t \dot{\xi}_\pm + v_{r,l} \partial_{x_1} \dot{\xi}_\pm = \partial_{x_1} F^\pm_2 - \frac{1}{\partial_{x_2} \Phi_{r,l}} (\partial_{x_1} \Phi_{r,l} \partial_{x_2} F^\pm_2 + \partial_{x_2} F^\pm_1) + \Lambda^{r,l}_1 \partial_{x_1} \dot{U}_\pm + \Lambda^{r,l}_2 \partial_{x_2} \dot{U}_\pm, \]

where

\[ \Lambda^{r,l}_{1,2} = \Lambda^{r,l}_{1,2}(\dot{U}_{r,l}, \nabla \dot{U}_{r,l}, \nabla \dot{\Phi}_{r,l}, \nabla^2 \dot{\Phi}_{r,l}). \]

An energy argument gives the apriori estimate for \( \dot{\xi}_\pm \).
This yields the estimate of the normal derivatives of the characteristic part of the solution.
This allows to obtain the a priori estimate in the standard Sobolev space \( H^m(\Omega_T) \). Otherwise we should work in the anisotropic weighted Sobolev space \( H^m_*(\Omega_T) \), as for current-vortex sheets in MHD.
3) Since

\[ \partial_{x_2} W^\pm_1 = \frac{1}{\langle \partial_{x_1} \Phi_{r,l} \rangle^2} \left\{ \partial_{x_2} \Phi_{r,l} \left( \partial_{x_1} \dot{u}^\pm - \dot{\xi}^\pm \right) ight. \\
- \partial_{x_1} \Phi_{r,l} \left( \partial_{x_2} T_{r,l} W^\pm \right)_3 - \left( \partial_{x_2} T_{r,l} W^\pm \right)_2 \right\}, \]

we may estimate \( \partial_{x_2} W^\pm_1 \) by the previous steps. The estimate of normal derivatives \( \partial_{x_2} W^{nc} \) of the noncharacteristic part of the solution follows directly from the equations:

\[ \mathbf{I}_2 \partial_{x_2} W^\pm = F^\pm - A^r_0 \partial_t W^\pm - A^r_1 \partial_{x_1} W^\pm - A^r_0 C^{r,l} W^\pm, \]

since

\[ \mathbf{I}_2 := \text{diag} \left( 0, 1, 1 \right), \quad W^{nc} := (W^+_2, W^+_3, W^-_2, W^-_3). \]
1 Introduction

- Euler’s equations of isentropic gas dynamics
- Smooth and piecewise smooth solutions
- Existence results

2 Compressible vortex sheets

- Compressible vortex sheets
- Linear Spectral Stability
- Formulation of the problem

3 Main result

- Linear stability: $L^2$ estimate
- Linear stability: Tame estimate in Sobolev norm
- Nonlinear stability: Nash-Moser iteration

4 Related problems

- Weakly stable shock waves
- Subsonic phase transitions
We use a Nash-Moser iteration where we force the Rankine-Hugoniot jump conditions and the eikonal equations at each step:

- Start from an approximate solution.
- Regularize the coefficients of the linearized equations, force the Rankine-Hugoniot conditions and the eikonal equations.
- Solve the linearized equations, for well chosen source terms.
- Regularize the new coefficients, force the Rankine-Hugoniot conditions and the eikonal equations etc.
The nonlinear problem

\[ \mathcal{L}(V, \Psi) := \mathbb{L}(V + U^a, \Psi + \Phi^a) - \mathbb{L}(U^a, \Phi^a) = f^a \quad \text{in } \Omega_T , \]

\[ \mathcal{E}(V, \Psi) := \partial_t \Psi + (v^a + v) \partial_{x_1} \Psi - u + v \partial_{x_1} \Phi^a = 0 , \quad \text{in } \Omega_T , \]

\[ \mathcal{B}(V, \psi) := \mathbb{B}((V + U^a)_{x_2=0}, \psi + \varphi^a) = 0 , \quad \text{on } \omega_T , \quad (6) \]

\[ \Psi^+_{|x_2=0} = \Psi^-_{|x_2=0} =: \psi , \quad \text{on } \omega_T . \]

\[ V(t, \cdot) = 0, \quad \Psi(t, \cdot) = 0, \quad \psi(t, \cdot) = 0 \quad \forall t < 0, \]

where

\[ V = (\rho, v, u)^T , \quad U^a = (\rho^a, v^a, u^a)^T , \]

\[ \begin{cases} f^a := -\mathbb{L}(U^a, \Phi^a) , & t > 0 , \\ f^a := 0 , & t < 0 . \end{cases} \]
The smoothing operators

**Theorem (cfr. Hamilton, Francheteau-Métivier)**

Let $T > 0$, $\gamma \geq 1$, and let $M \in \mathbb{N}$, with $M \geq 4$. There exists a family $\{S_\theta\}_{\theta \geq 1}$ of operators

$$S_\theta : \mathcal{F}^3_\gamma(\Omega_T) \times \mathcal{F}^3_\gamma(\Omega_T) \rightarrow \bigcap_{\beta \geq 3} \mathcal{F}^\beta_\gamma(\Omega_T) \times \mathcal{F}^\beta_\gamma(\Omega_T),$$

where $\mathcal{F}^s_\gamma(\Omega_T) := \{ u \in H^s_\gamma(\Omega_T) u = 0 \text{ for } t < 0 \}$ and a constant $C > 0$ (depending on $M$), such that

$$\| S_\theta U \|_{H^\beta_\gamma(\Omega_T)} \leq C \theta^{(\beta-\alpha)+} \| U \|_{H^\alpha_\gamma(\Omega_T)}, \quad \forall \alpha, \beta \in \{1, \ldots, M\},$$

$$\| S_\theta U - U \|_{H^\beta_\gamma(\Omega_T)} \leq C \theta^{\beta-\alpha} \| U \|_{H^\alpha_\gamma(\Omega_T)}, \quad 1 \leq \beta \leq \alpha \leq M,$$

$$\| \frac{d}{d\theta} S_\theta U \|_{H^\beta_\gamma(\Omega_T)} \leq C \theta^{\beta-\alpha-1} \| U \|_{H^\alpha_\gamma(\Omega_T)},$$

$$\forall \alpha, \beta \in \{1, \ldots, M\}.$$
Moreover, (i) if \( U = (u^+, u^-) \) satisfies \( u^+ = u^- \) on \( \omega_T \), then \( S_\theta u^+ = S_\theta u^- \) on \( \omega_T \), (ii) the following estimate holds:

\[
\left\| (S_\theta u^+ - S_\theta u^-) \right\|_{x_2=0} H^\beta_\gamma(\omega_T) \leq C \theta^{(\beta+1-\alpha)+} \left\| (u^+ - u^-) \right\|_{x_2=0} H^\alpha_\gamma(\omega_T),
\]

\( \forall \alpha, \beta \in \{1, \ldots, M\} \).

There is another family of operators, still denoted \( S_\theta \), that acts on functions that are defined on the boundary \( \omega_T \), and that enjoy the properties (??), with the norms \( \| \cdot \|_{H^\alpha_\gamma(\omega_T)} \).
The Nash-Moser iteration

The iterative scheme starts from $V_0 = 0, \Psi_0 = 0, \psi_0 = 0$. Assume that $V_k, \Psi_k, \psi_k$ are already given for $k = 1, \ldots, n$ and verify

$$V_k = 0, \quad \Psi_k = 0, \quad \psi_k = 0 \quad \text{for } t < 0,$$

$$\Psi_k^+ = \Psi_k^- = \psi_k \quad \text{on } \omega_T, \; k = 1, \ldots, n.$$

Given $\theta_0 \geq 1$, let us set $\theta_n := (\theta_0^2 + n)^{1/2}$ and consider the smoothing operators $S_{\theta_n}$. Let us set

$$V_{n+1} = V_n + \delta V_n, \quad \Psi_{n+1} = \Psi_n + \delta \Psi_n, \quad \psi_{n+1} = \psi_n + \delta \psi_n.$$
We consider the decomposition

\[ \mathcal{L}(V_{n+1}, \Psi_{n+1}) - \mathcal{L}(V_n, \Psi_n) \]

\[ = \mathcal{L}'(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2}) (\delta V_n, \delta \Psi_n) + e'_n + e''_n + e'''_n, \]

\[ \mathcal{B}(V_{n+1}, \psi_{n+1}) - \mathcal{B}(V_n, \psi_n) \]

\[ = \mathcal{B}'(V_{n+1/2}, \psi_{n+1/2}) (\delta V_n, \delta \psi_n) + e'_n + e''_n + e'''_n, \]
where

\[ e'_k := \mathbb{L}(U^a + V_{k+1}, \Phi^a + \Psi_{k+1}) - \mathbb{L}(U^a + V_k, \Phi^a + \Psi_k) \]
\[ -\mathbb{L}'(U^a + V_k, \Phi^a + \Psi_k)(\delta V_k, \delta \Psi_k), \]

\[ \tilde{e}'_k := \mathcal{B}(V_{k+1}, \psi_{k+1}) - \mathcal{B}(V_k, \psi_k) - \mathcal{B}'(V_k, \psi_k)(\delta V_k, \delta \psi_k) \]

are the "quadratic errors" of Newton's scheme,

\[ e''_k := \mathbb{L}'(U^a + V_k, \Phi^a + \Psi_k)(\delta V_k, \delta \Psi_k) \]
\[ -\mathbb{L}'(U^a + S_{\theta_k}V_k, \Phi^a + S_{\theta_k}\Psi_k)(\delta V_k, \delta \Psi_k), \]

\[ \tilde{e}''_k := \mathcal{B}'(V_k, \psi_k)(\delta V_k, \delta \psi_k) - \mathcal{B}'(S_{\theta_k}V_k, S_{\theta_k}\psi_k)(\delta V_k, \delta \psi_k) \]

are the "first substitution errors" involving the smoothing operators,
Compressible vortex sheets

Main result

Related problems

Linear stability: $L^2$ estimate

Linear stability: Tame estimate in Sobolev norm

Nonlinear stability: Nash-Moser iteration

\[ e_k''' := \mathcal{L}'(U^a + S_{\theta_k} V_k, \Phi^a + S_{\theta_k} \Psi_k)(\delta V_k, \delta \Psi_k) \]
\[- \mathcal{L}'(U^a + V_{k+1/2}, \Phi^a + \Psi_{k+1/2})(\delta V_k, \delta \Psi_k), \]

\[ \tilde{e}_k''' := \mathcal{B}'(S_{\theta_k} V_k, S_{\theta_k} \psi_k)(\delta V_k, \delta \psi_k) - \mathcal{B}'(V_{k+1/2}, \psi_{k+1/2})(\delta V_k, \delta \psi_k) \]

are the "second substitution errors" involving the smooth modified state $V_{n+1/2}, \Psi_{n+1/2}, \psi_{n+1/2}$ satisfying the Rankine-Hugoniot conditions and the eikonal equations.
Introducing the **new unknown**

\[
\delta \dot{V}_n := \delta V_n - \delta \Psi_n \frac{\partial_x (U^a + V_{n+1/2})}{\partial_x (\Phi^a + \Psi_{n+1/2})}.
\]

gives

\[
\mathcal{L}(V_{n+1}, \Psi_{n+1}) - \mathcal{L}(V_n, \Psi_n) =
\]

\[
= (L_{n+1/2} + C_{n+1/2}) \delta \dot{V}_n + D_{n+1/2} \delta \Psi_n + e'_n + e''_n + e'''_n,
\]

\[
\mathcal{B}(V_{n+1}, \psi_{n+1}) - \mathcal{B}(V_n, \psi_n) = \mathcal{B}_{n+1/2}'(\delta \dot{V}_n, \delta \psi_n) + \bar{e}'_n + \bar{e}''_n + \bar{e}'''_n,
\]
where

\[
L_{n+1/2} + C_{n+1/2} := L(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2}) + C(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2}),
\]

\[
D_{n+1/2} \delta \Psi_n := \frac{\delta \Psi_n}{\partial x_2 (\Phi^a + \Psi_{n+1/2})} \partial x_2 \left\{ \mathbb{L}(U^a + V_{n+1/2}, \Phi^a + \Psi_{n+1/2}) \right\},
\]

\[
\mathbb{B}'_{n+1/2} = \mathbb{B}'(U^a + V_{n+1/2}, \varphi^a + \psi_{n+1/2}).
\]
Let us set

\[ e_n := D_{n+1/2} \delta \Psi + e'_n + e''_n + e'''_n, \]

\[ \tilde{e}_n := \tilde{e}'_n + \tilde{e}''_n + \tilde{e}'''_n. \]

The iteration proceeds as follows.

Given

\[ V_0 = 0, \quad \Psi_0 = 0, \quad \psi_0 = 0, \]

\[ f_0 = S_0 f^a, \quad g_0 = 0, \quad E_0 = 0, \quad \tilde{E}_0 = 0, \]

\[ V_1, \ldots, V_n, \quad \Psi_1, \ldots, \Psi_n, \quad \psi_1, \ldots, \psi_n, \]

\[ f_1, \ldots, f_{n-1}, \quad g_1, \ldots, g_{n-1}, \]

\[ e_0, \ldots, e_{n-1}, \quad \tilde{e}_0, \ldots, \tilde{e}_{n-1}, \]

first compute for \( n \geq 1 \)

\[ E_n = \sum_{k=0}^{n-1} e_k, \quad \tilde{E}_n = \sum_{k=0}^{n-1} \tilde{e}_k. \]
Then compute $f_n, g_n$ from

$$
\sum_{k=0}^{n} f_k + S_{\theta_n} E_n = S_{\theta_n} f^a, \quad \sum_{k=0}^{n} g_k + S_{\theta_n} \tilde{E}_n = 0,
$$

and solve the problem

$$
(L_{n+1/2} + C_{n+1/2}) \delta \dot{V}_n = f_n \quad \text{in } \Omega_T, \\
\mathbb{B}'_{n+1/2}(\delta \dot{V}_n, \delta \psi_n) = g_n \quad \text{on } \omega_T, \\
\delta \dot{V}_n = 0, \quad \delta \psi_n = 0 \quad \text{for } t < 0,
$$

finding $(\delta \dot{V}_n, \delta \psi_n)$. Then compute $\delta \Psi_n = (\delta \Psi^+_n, \delta \Psi^-_n)$ from a suitable modification of the eikonal equations and consequently $\delta V_n, V_{n+1}, \Psi_{n+1}, \psi_{n+1}$. Finally compute $e_n, \tilde{e}_n$ from

$$
\mathcal{L}(V_{n+1}, \Psi_{n+1}) - \mathcal{L}(V_n, \Psi_n) = f_n + e_n, \\
\mathcal{B}(V_{n+1}, \psi_{n+1}) - \mathcal{B}(V_n, \psi_n) = g_n + \tilde{e}_n.
$$
Adding (??) from 0 to $N$ gives

\[
\mathcal{L}(V_{N+1}, \Psi_{N+1}) = S_{\theta_N} f^a + (I - S_{\theta_N}) E_N + e_N ,
\]

\[
\mathcal{B}(V_{N+1}, \psi_{N+1}) = (I - S_{\theta_N}) \tilde{E}_N + \tilde{e}_N .
\]

Because

\[
S_{\theta_N} \to I \quad \text{as} \quad N \to +\infty
\]

\[
e_N \to 0, \quad \tilde{e}_N \to 0,
\]

we formally obtain the resolution of the problem from

\[
\mathcal{L}(V_{N+1}, \Psi_{N+1}) \to f^a , \quad \mathcal{B}(V_{N+1}, \psi_{N+1}) \to g^a .
\]

The rigorous proof of convergence follows from apriori estimates of $V_k, \Psi_k, \psi_k$ proved by induction for every $k$. 
Plan

1. Introduction
   - Euler's equations of isentropic gas dynamics
   - Smooth and piecewise smooth solutions
   - Existence results

2. Compressible vortex sheets
   - Compressible vortex sheets
   - Linear Spectral Stability
   - Formulation of the problem

3. Main result
   - Linear stability: $L^2$ estimate
   - Linear stability: Tame estimate in Sobolev norm
   - Nonlinear stability: Nash-Moser iteration

4. Related problems
   - Weakly stable shock waves
   - Subsonic phase transitions
The existence of weakly stable shock waves

Consider the Euler equations (??) in $\mathbb{R}^d$ where $d = 2$ or $3$. Shock waves solutions to (??) are smooth solutions on either side of a hypersurface $\Sigma = \{ x_d = \varphi(t, y), t \in [0, T], y \in \mathbb{R}^{d-1} \}$, satisfying at $\Sigma$ the Rankine–Hugoniot conditions

$$
\rho^+ (u^+ - v^+ \cdot \nabla_y \varphi - \partial_t \varphi) = \rho^- (u^- - v^- \cdot \nabla_y \varphi - \partial_t \varphi) =: j,
$$

$$
j (u^+ - u^-) + (p(\rho^+) - p(\rho^-)) \begin{pmatrix} -\nabla_y \varphi \\ 1 \end{pmatrix} = 0,
$$

and the Lax’ shock inequalities for a 1-shock wave (for example)

$$
j > 0, \quad 0 < \frac{u^+ - v^+ \cdot \nabla_y \varphi - \partial_t \varphi}{c(\rho^+) \sqrt{1 + |\nabla_y \varphi|^2}} < 1 < \frac{u^- - v^- \cdot \nabla_y \varphi - \partial_t \varphi}{c(\rho^-) \sqrt{1 + |\nabla_y \varphi|^2}}.
$$

P. Secchi (Brescia University)
Up to Galilean transformations, the planar shock waves have the form

$$
(\rho, v, u) = \begin{cases} 
U_r := (\rho_r, 0, u_r), & \text{if } x_d > 0, \\
U_l := (\rho_l, 0, u_l), & \text{if } x_d < 0,
\end{cases}
$$

where

$$
\rho_r u_r = \rho_l u_l =: j, \quad j = \sqrt{\rho_r \rho_l \frac{p(\rho_r) - p(\rho_l)}{\rho_r - \rho_l}}, \quad 0 < \frac{u_r}{c(\rho_r)} < 1 < \frac{u_l}{c(\rho_l)}.
$$
The (linear) stability of planar shock waves:

**Theorem (Majda 1983)**

*The shock wave (??) is uniformly stable if and only if*

\[
\frac{u_r^2}{c(\rho_r)^2} \left( \frac{\rho_r}{\rho_l} - 1 \right) < 1.
\]

*In particular, when \( p \) is a convex function of \( \rho \), this inequality always holds.*

Majda constructs shock waves that are close to a uniformly stable planar shock.
When

\[ \frac{u_r^2}{c(\rho_r)^2} \left( \frac{\rho_r}{\rho_l} - 1 \right) > 1, \]  

(12)

the planar shock wave (?) is only weakly stable.

Coulombel 2004: the linearized problem around a variable coefficients small perturbation of the planar shock (??) satisfies an a priori estimate with a loss of one tangential derivative.
Theorem (Coulombel, S., 2008)

Consider a planar shock wave \( (??) \) that satisfies the weak stability condition \( (??) \). Let \( T > 0 \), and let \( \mu \in \mathbb{N} \) be sufficiently large. Then there exists an integer \( \tilde{\mu} \geq \mu \), such that if the initial data \( (U_0^\pm, \varphi_0) \) have the form

\[
U_0^\pm = U_{r,l} + \dot{U}_0^\pm,
\]

with \( \dot{U}_0^\pm \in H^{\tilde{\mu} + 1/2}(\mathbb{R}_+^2) \), \( \varphi_0 \in H^{\tilde{\mu} + 3/2}(\mathbb{R}) \), if they are compatible up to order \( \tilde{\mu} - 1 \), have a compact support, and are sufficiently small, then there exists a solution \( U^\pm = U_{r,l} + \dot{U}^\pm, \Phi^\pm, \varphi \) to \( (??), (??), (??) \), on the time interval \( [0, T] \). This solution satisfies \( \dot{U}^\pm \in H^\mu([0, T] \times \mathbb{R}^{d-1} \times \mathbb{R}^+) \), \( \varphi \in H^{\mu+1}([0, T] \times \mathbb{R}^{d-1}) \), and \( (\dot{U}^\pm, \varphi)_{|t=0} = (\dot{U}_0^\pm, \varphi_0) \).
Plan

1 Introduction
   - Euler’s equations of isentropic gas dynamics
   - Smooth and piecewise smooth solutions
   - Existence results

2 Compressible Vortex Sheets
   - Compressible vortex sheets
   - Linear Spectral Stability
   - Formulation of the problem

3 Main result
   - Linear stability: $L^2$ estimate
   - Linear stability: Tame estimate in Sobolev norm
   - Nonlinear stability: Nash-Moser iteration

4 Related problems
   - Weakly stable shock waves
   - Subsonic phase transitions
Subsonic phase transitions in a Van der Waals fluid

Consider the Euler equations (??) in $\mathbb{R}^d$ where $d = 2$ or $3$.

Model of isothermal liquid/vapor phase transitions in a van der Waals fluid:

$$p(\rho) = \pi(v) := \frac{RT}{v - b} - \frac{a}{v^2}, \quad v := 1/\rho.$$ 

Phase transition:
smooth solution of (??) on either side of a hypersurface $\Sigma = \{x_d = \varphi(t, y)\}$, that satisfies the Rankine-Hugoniot jump conditions at each point of $\Sigma$:

$$\rho^+ (u^+ - v^+ \cdot \nabla_y \varphi - \partial_t \varphi) = \rho^- (u^- - v^- \cdot \nabla_y \varphi - \partial_t \varphi) =: j,$$

$$j (u^+ - u^-) + (p(\rho^+) - p(\rho^-)) \begin{pmatrix} -\nabla_y \varphi \\ 1 \end{pmatrix} = 0,$$

$$j > 0, \quad 0 < \frac{u^\pm - v^\pm \cdot \nabla_y \varphi - \partial_t \varphi}{c(\rho^\pm) \sqrt{1 + |\nabla_y \varphi|^2}} < 1,$$

(undercompressive shock waves of type 0, Freistühler 1998, Lax’ shock inequalities are not satisfied)
together with the generalized equal area rule (capillary admissibility criterion):

\[
\int_{v^-}^{v^+} \pi(v) \, dv = \frac{\pi(v^+) + \pi(v^-)}{2} (v^+ - v^-).
\]

(14)

Consider a planar phase transition

\[
(r, v, u) = \begin{cases} 
U_r := (\rho_r, 0, u_r), & \text{if } x_d > 0, \\
U_l := (\rho_l, 0, u_l), & \text{if } x_d < 0,
\end{cases}
\]

(15)

that satisfies \( \rho_r > \rho_M, \rho_l < \rho_m, \) and the jump conditions

\[
\rho_r u_r = \rho_l u_l =: j, \quad j = \sqrt{\rho_r \rho_l \frac{p(\rho_r) - p(\rho_l)}{\rho_r - \rho_l}}, \quad 0 < \frac{u_r}{c(\rho_r)} < 1 < \frac{u_l}{c(\rho_l)},
\]

\[
\int_{v_l}^{v_r} \pi(v) \, dv = \frac{p(\rho_r) + p(\rho_l)}{2} (v_r - v_l).
\]
There exist planar phase transitions (??), with $\rho_{r,l}$ close enough to $\rho_{M,m}$, and these planar phase transitions are weakly stable. In any case, the uniform Lopatinskii condition is not satisfied.
**Theorem (Coulombel, S., 2008)**

Consider a planar phase transition (??), as given in Theorem ??, Let $T > 0$, and let $\mu \in \mathbb{N}$ be sufficiently large. Then there exists an integer $\tilde{\mu} \geq \mu$, such that if the initial data $(U_0^\pm, \varphi_0)$ have the form

$$U_0^\pm = U_{r,l} + \dot{U}_0^\pm,$$

with $\dot{U}_0^\pm \in H^{\tilde{\mu} + 1/2}(\mathbb{R}_+^2)$, $\varphi_0 \in H^{\tilde{\mu} + 3/2}(\mathbb{R})$, if they are compatible up to order $\tilde{\mu} - 1$, have a compact support, and are sufficiently small, then there exists a solution $U^\pm = U_{r,l} + \dot{U}^\pm$, $\Phi^\pm$, $\varphi$ to (??), (??), (??) on the time interval $[0, T]$. This solution satisfies $\dot{U}^\pm \in H^{\mu}([0, T] \times \mathbb{R}^{d-1} \times \mathbb{R}^+)$, $\varphi \in H^{\mu+1}([0, T] \times \mathbb{R}^{d-1})$, and $(\dot{U}^\pm, \varphi)|_{t=0} = (\dot{U}_0^\pm, \varphi_0)$.