NONLINEAR WATER WAVES

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Motivation and Aims

Layman motivation Roughly 70 % of the Earth's surface is covered by water and waves can develop whenever the surface of the water is free to move, providing us with beautiful sights. The study of water waves has had practical applications throughout history; its recent significance was demonstrated dramatically by the tsunami of 2004. Mathematics is essential in seeking to understand this fascinating field of physical reality and adds immensely to the enjoyment obtained from the observation of the wealth of water wave phenomena nature confronts us with.

Physical aspects Given a physical situation, under certain simplifying assumptions established laws from physics can be applied to obtain a model of the ongoing process. Physical insight determines the crucial factors governing a given phenomenon and identify the factors that can safely be neglected. The conclusions reached will reflect reality (that is, specific situations which may be observed in nature or experimentally) only insofar as the accuracy of the model permits: the value of a model depends on the number of physically useful deductions which can be made from it. The "truth" of the model is meaningless as all experiments contain inaccuracies of measurement and effects other than those accounted for cannot be totally excluded.

Role of mathematics "Hydrodynamics is one of those fundamental areas where progress at any moment can be regarded as a standard to measure the real success of mathematical science" cf. [Arnold & Khesin]. The study of water waves draws freely from various mathematical disciplines but one can not expect to use mathematical tools mechanically. Even at simple levels subtle use of mathematical concepts is needed and many important achievements in the study of water waves are based on profound theories, some of which were pioneered in this context and "... stimulated developments in the domains of pure mathematics, such as complex analysis, topology ..." cf. [Arnold & Khesin]. Also, cf. Peregrine "water waves present sufficiently challenging problems that much of the development of the mathematical theory of wave motion over the past 200 years has been pioneered on their theory". Without mathematics the study of water waves, though possibly informative, leads to a very limited understanding as the elimination of mathematics leaves little indication of the justification/degree of credibility of a theory and diminishes the possibility to extend observations on one's own account. A superficial treatment cannot offer the sense of satisfaction and involvement produced by genuine understanding while, at the other extreme, focusing on advanced topics can produce a false sophistication which emphasizes technique rather than ideas — abstruse theoreticians eventually loose themselves in the arid labyrinths of pure technique.

FIELD DATA & EXPERIMENTS Both were and are key areas in the quest to understand and predict water waves, offering invaluable insight and providing motivation for further studies. Theory and experiment interlace: theory interprets to some extent experiments and builds upon this basis, while experiments confirm or discard theories. However, portraying scientific discovery as resulting from the confrontation of theoretical predictions with experiments, with perfect matching establishing unassailable scientific truth is misleading — we will see a concrete example when discussing particle trajectories beneath Stokes waves.

COMPUTATIONAL ASPECTS Many theoretical developments were stimulated and supported by numerical computations (for example, solitons — nonlinear waves that can recover their shape after colliding — were discovered as a result of numerical simulations) and often such simulations are compelling even if they do not yet relate to rigorous theory. While numerical simulations are important to unveil possibilities that guide the subsequent analysis, rigorous theory is necessary to guide such investigations and their interpretation — without theory the computations are capable of producing realms of indigestible information that is practically useless. Theory is a conceptual tool rather than merely a device to express numerical calculations.

AIMS

Most predictions for water wave motions use linear approximations but this is adequate only for waves of small amplitude — for waves of moderate and large amplitude nonlinear effects can not be neglected. We will discuss redwave-current interactions with emphasis on the existence theory of steady two-dimensional water waves propagating at the surface of water with a flat bed in a flow with a general vorticity distribution. The theme offers a wide range of open questions, for some of which conjectures supported by numerical or experimental evidence were already formulated and will be presented.

The thrust of the presentation is analytical but throughout the discussions we take advantage of input from experimental evidence and numerical simulations.

Simplifying assumptions

In the study of water wave flows, just like in most scientific endeavours, one uses theories that are fairly simple to specify but have complex implications, being thus able to explain accurately a large class of observations and to make definite predictions about future observations. We now specify the simplifying assumptions proper to the investigation of water waves of moderate and large amplitude, indicating the grounds for neglecting certain factors:

- **the macroscopic viewpoint**

  We consider the behaviour of matter in the large (on a scale large compared to the distance between molecules) in order to evade the difficulties inherent in the conception of a fluid as consisting of a granular structure of discrete molecules. This idealisation amounts to the assumption that the water is distributed continuously (the continuum hypothesis). This hypothesis is in complete accord with the everyday observation of water where the idea of regarding its structure as anything other than continuous would be unnatural. This inevitably implies that it is possible to attach a definite meaning to the notion of the value of some property of the water at a point.

- **constant density**

  The density of water is effectively a function of depth, temperature and salinity cf. [Lighthill] so that the following numerical data are relevant: temperature variations in the ocean, varying from the freezing point of seawater, $-2^{\circ} C$, to values around $27^{\circ} C$, are responsible for density changes of up to 0.5%, salinity changes account for variations in density of up to 0.2% while a descent of 1 km in depth increases the density by no more than 0.5%. This great resistance to changes in density allows us to regard water for most purposes as homogeneous to a very high degree of accuracy, with density 1000 kg/m^3. Of course, at great depths of the ocean the change in density of the water is significant but the variation of water density in the ocean is limited to about 4%, the maximum being attained at depths of around 10 km.

- **inviscid flow**

  We neglect viscosity. Although water is a viscous fluid, "an inviscid approximation is realistic since the velocity profile in the water, whether due to laminar viscosity or turbulent mixing, is usually established over length scales which are long compared with a wave length" cf. [Da Silva & Peregrine]. For example, it is possible to observe waves traveling over oceanic distances of 1000 km in the time it takes to lose one-tenth of their amplitude cf. [Milne-Thompson].

- ## the motion of the air

The influence exerted by the air above on the water's surface atmosphere is in the form of pressure acting on the surface, taken to be a constant. This assumption decouples the motion of the air from that of the water so that in analyzing the motion of the water we neglect the motion of the air above. While any disturbance of the free surface clearly implies some motion of the air, the argument is made that the change in the pressure in the air due to this motion is negligible so that the air pressure may be approximated by its undisturbed value P_{atm} (constant atmospheric pressure). Notice that the mechanism mainly responsible for the most rapid stage of growth of sea waves under the action of the wind still remains to be found. For this reason we suppose that at some moment in the past a disturbance was created on the surface of the still water in some fashion (by the wind, for example but other causes are also possible, e.g. earthquakes in the case of tsunamis), and we investigate the subsequent motion of the water.

- ## gravity water waves

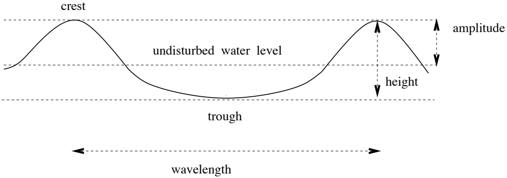
The balance between a restoring force and the inertia of the system governs the evolution of the disturbance of the surface of water — this is the so-called free surface (as its form is not prescribed *a priori*) and its determination is our primary objective. We will study gravity waves arising through the restoring action of gravity on water displaced from the equilibrium level (still water) without investigating internal waves (occuring at the interface between two layers of water of different density), planetary waves (tides related to the effects of the Moon or of the Sun on oceanic movements, effects of the Earth's rotation), capillary waves (for which the flatness-restoring force is not gravity but surface tension) or capilary-gravity waves (for which the effects of surface tension are comparable with those of gravity). Surface tension is related to the unusually strong cohesive properties of water molecules, making it behave as though it were coated with an invisible film. The effects of surface tension can however be neglected for wave lengths or wave amplitudes greater than a few *cm*, respectively *mm*, cf. [Lighthill].

- ## a rigid flat bed

The topography of the bed has a great influence on the surface waves. In view of the present limited understanding of the fundamental processes involved, we will restrict our attention to a rigid flat bed. While a moving bed is of interest in the generation of tsunami waves by an undersea earthquake and for sandy beds the water flow induces a motion of the sand as well as the percolation of the water through the bed, these phenomena are not investigated.

▶ two-dimensional flows

We will mainly consider regular wave trains of plane waves (steady periodic two-dimensional waves) for which the motion is identical in any direction parallel to the crest line and which propagate at constant speed in a fixed direction. For a wave train the crest is the most elevated part of the wave about the undisturbed water level, the lowest part of the wave that is below this surface being called the trough, while the vertical distance from the wave crest to wave trough is called the wave height. The amplitude of the wave is the maximum deviation of the wave from undisturbed water surface, being thus either the distance from the wave crest or the distance from the wave trough to the undisturbed water level — for sinusoidal profiles these two distances are the same but periodic sea waves tend to have sharper elevations and flatter depressions so that the maximal elevation above the sea level usually exceeds the maximal depression below it. Wave amplitudes, measuring a few *mm* at the lower end, can attain a few tens of *m* at the upper end. The wave period is the time required for two successive crests or two successive troughs to pass a fixed point in space, while the horizontal distance between two successive crests (or troughs) is the wavelength, reaching from *cm* to hundreds of *km*. The ratio of wavelength to wave period is the wave speed, with the fastest waves traveling at hundreds of km/h whereas the slowest travel below 1 km/h.



The occurence of such waves can be observed in a range of scenarios. Indeed, ocean waves are classified as either sea or swell. Irregular patterns made up of various waves with different speeds wavelengths and heights are called sea. When these waves move past the area of influence of the generating winds, they sort themselves into groups with similar speed and wavelengths. This process produces swell that is characteristically a regular pattern of undulation of the ocean surface and serves as a prototype for steady periodic two-dimensional water waves. Swell often moves thousands of miles away from a storm to a shore somewhere (for example, swell originating from Antarctic storms has been recorded close to the Alaskan coast after more than 8000 *km*).

Photograph — swell

Linear wave theory predicts sinuosoidal wave profiles but in the photograph taken from the Great Ocean Road, Victoria, Australia [From WAVES by Steve Hawk; © 2005 by Steve Hawk. Used with permission from Chronicle Books LLC, San Francisco] the crest is higher and narrower and the trough broader and less deep. Only nonlinear theory can capture these features.

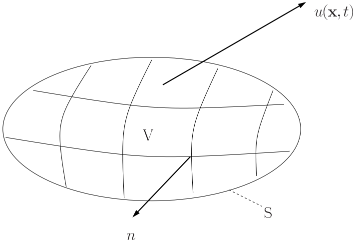
Photograph — regular wave train

This wave train in water of small depth with a flat bed near the coast of California [From WAVES by Steve Hawk; © 2005 by Steve Hawk. Used with permission from Chronicle Books LLC, San Francisco] is even further from a sinusoidal wave pattern, being almost flat near the trough and presenting pronounced elevations near the crest.

The governing equations

- The equation of mass conservation

Imagine a volume V, bounded by a C^1 surface S, within water of constant density ρ. If n is the outward unit normal on S and $\mathbf{u} \in C^1$ is the velocity of the water, the outward velocity component across S is $\mathbf{u} \cdot n$. Decomposing $\mathbf{u}$ into normal and tangential components, the tangential ones do not contribute to the outflow, so that the rate at which mass flows out of V is $\int_S \rho\, \mathbf{u} \cdot n\, dS$. The rate of change of mass in V is $\frac{d}{dt} \int_V \rho\, d\mathbf{x} = \int_V \frac{d\rho}{dt}\, d\mathbf{x} = 0$ in view of the homogeneity assumption (constant density).

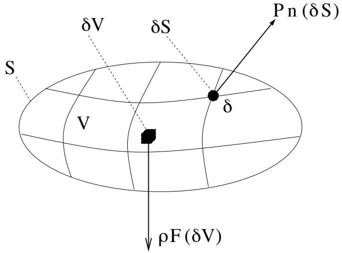


Since matter (mass) is neither created nor destroyed anywhere in the water, the rate of change of mass in V is brought about only by the rate of mass flowing into V across S, so $\mathbf{0} = -\int_S \rho\, \mathbf{u} \cdot n\, dS$ or $\int_V \nabla \cdot \mathbf{u}\, d\mathbf{x} = 0$, by the divergence theorem. This has to be true for any volume V, yielding the equation of mass conservation

$$\nabla \cdot \mathbf{u} = 0. \qquad (mc)$$

▶ The equation of motion: Euler's equation

There are two types of forces that are relevant in fluid mechanics: one distinguishes between body forces (external forces), having their source exterior to the fluid and being practically the same for all particles, and internal forces (local forces) exerted on a fluid element by other elements nearby — the latter are the forces which obey Newton's third law (equality of action and reaction). For gravity water waves, the body force is due to gravity so that, in Cartesian coordinates $\mathbf{x} = (x, y, z)$ with z measured upwards, $F = (0, 0, -g)$. The internal forces are pressure forces, whereby a lump of water is acted on across its surface by the rest of the continuum in a purely normal direction to the surface (there are no tangential components): the internal forces acting on an infinitesimally small area δS within the water region take the form of the thrust due to the pressure (a scalar quantity P depending on the position of the point at which it is measured and on time) — a force of magnitude $P\,(\delta S)$ in the direction of the inward normal. This characterization of the internal forces is valid in an ideal fluid, a concept supposing the matter to be continuously distributed and in inviscid flow. The concept of pressure in an incompressible fluid presents certain difficulties to the physical understanding — in a compressible fluid the pressure determines the density but here the variable of state P is denied any influence upon the way the fluid ocupies the space. The hydrodynamic pressure P has to be regarded as a reaction to the constraint of incompressibility (mc).



Consider now a volume V having as boundary the C^1 surface S within the water. We assume $\mathbf{u}$ and P of class C^1 throughout the fluid.

The total force (body + local) acting on the water in V is

$$\int_V \rho F \, d\mathbf{x} - \int_S P \, n \, dS,$$

n being the outward unit normal on S. Using the divergence theorem, we transform the above expression to

$$\int_V (\rho F - \nabla P) \, d\mathbf{x}.$$

We balance the forces on the water body inside S (this is Newton's law of motion)

$$\int_V \rho \, \frac{D\mathbf{u}}{Dt} \, d\mathbf{x} = \int_V (\rho F - \nabla P) \, d\mathbf{x}. \qquad (*)$$

The left-hand side is the sum of the acceleration times mass (recall that density is mass/volume) for each particle in the volume V while $\dfrac{D}{Dt}$ is the total differentiation following the particular particle. Indeed, the velocity vector of a particle is $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, so that the particle follows the path $\mathbf{x}(t)$ on which $\dfrac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}(t), t)$ and thus the acceleration of the particle is

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{D\mathbf{u}}{Dt},$$

which explains the interpretation of the left-hand side of $(*)$. Since V in $(*)$ is arbitrary, we obtain the Euler equation of motion

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \, \nabla P + (0, 0, -g). \qquad (ee)$$

The pressure P appears as the fourth unknown besides the components of the velocity $\mathbf{u}$. Note the nonlinear character of Euler's equations caused by the presence of the term $\mathbf{u} \cdot \nabla \mathbf{u}$. This prohibits the use of the principle of superposition and makes the analysis of the equations a considerably difficult mathematical problem.

▶ The boundary conditions

There are two boundaries of the fluid region: the rigid flat bed and the water's free surface. The kinematic boundary conditions (called this way because they do not involve the action of forces) express the fact that a particle on the boundary remains there at all times. If $[S(\mathbf{x},\ t)=0]$ is the equation of the C^1 free surface, choosing an arbitrary particle $\mathbf{x}_0$ lying at time $t=t_0$ on the free surface, the subsequent location $\mathbf{x}(t)$ is given by the unique solution of the differential equation $\mathbf{x}'(t)=\mathbf{u}(\mathbf{x}(t),\,t)$ with data $\mathbf{x}(t_0)=\mathbf{x}_0$, and should satisfy $S(\mathbf{x}(t),\ t)=0$ for $t\geq t_0$. Differentiation with respect to t shows that a necessary condition is $\dfrac{DS}{Dt}=0$ along the surface. This condition is also sufficient in view of differential geometry considerations: it expresses the fact that the flow $(\mathbf{x}(t),\ t)$ is everywhere tangential to the time-dependent free surface and the absence of a normal component prevents particles from leaving the free surface. On the free surface $[z-h(x,y,t)=0]$ we therefore have the kinematic boundary condition

$$u_3=h_t+u_1\,h_x+u_2\,h_y\quad\text{on}\quad z=h(x,y,t),\tag{tk}$$

while on the flat bed $[z+d=0]$ the kinematic boundary condition is

$$u_3=0\quad\text{on}\quad z=-d.\tag{bk}$$

For gravity waves we ignore the effects of surface tension, the influence exerted by the atmosphere on the water surface is in the form of pressure acting on the surface, taken to be the (constant) atmospheric pressure P_{atm}. This gives the dynamic boundary condition

$$P=P_{atm}\quad\text{on}\quad z=h(x,y,t),\tag{d}$$

which reflects the external actions on the free surface (at the bottom we do not encounter this type of condition as no interactions occur).

The general description of the water-wave problem for gravity waves is encompassed by Euler's equation (ee) and the equation of mass conservation (mc) together with the boundary conditions (kt)-(kb) and (d): these are the governing equations for water waves. A distinctive feature in water wave theory is that one of the boundaries — the free surface — is not known and must be determined as part of the solution. This is a free-boundary problem: the shape of the surface waves and the motion of the fluid are interdependent.

Well-posedness

The first issue that needs to be addressed in connection with the governing equations for water waves is their well-posedness. If these equations are a genuine mathematical model of the physical phenomena, they should have the properties that if appropriate data are given initially (say, at time $t = 0$), then, at least in some small time-interval $[0, T)$ with $T > 0$:

(i) a solution exists

(ii) the solution is unique

(iii) the solution depends continuously on the data, in some reasonable topology.

The necessity of the existence of a solution, condition (i), is motivated by physical reality. It is not the only requirement that is necessary. Experience suggests the hypothesis of determinism: if the same initial occurence repeats itself exactly, we should get the same outcome, that is, (ii) must hold. The last requirement, (iii), is also needed for a satisfactory model. Since one can never be certain that an occurence can be repeated exactly the same way, if the data are almost the same we expect the outcome to be almost identical, at least for some time. The reason why the formulation of well-posedness is local in character (that is, for some small time-interval) is that in the context of water waves one has to make provision for wave breaking. While no physical and little qualitative insight is to be gained from the study of well-posedness — and for this reason, rather than pursuing an in-depth study of this problem, we will only point out the essential features without going into technical details — this aspect of the problem is of fundamental importance. Lack of well-posedness invalidates the model.

The mathematics behind the well-posedness of the governing equations is deep and interesting, and still not completely resolved despite remarkable recent advances. While a satisfactory general well-posedness result remains elusive, for the important category of irrotational flows modeling waves without swirls it is known that the governing equations are well-posed.

Let us first formulate the well-posedness problem in more precise terms before briefly discussing the most relevant results. Given the rigid bed $z = -d$, the data (at time $t = 0$) consists of the initial surface $z = h_0(x, y)$ and of the initial divergence-free irrotational velocity field $\mathbf{u}_0(\mathbf{x})$ with $(h_0, \mathbf{u}_0) \in H^{s+1}(\mathbb{R}^2) \times H^{s+1}(\mathbb{R}^3)$ with the Sobolev index $s \geq 3$. Then there is a unique solution $(h_0, \mathbf{u}_0) \in C^1([0, t);\, H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^3))$ with initial data $(h_0, \mathbf{u}_0)$, and such that $\mathbf{u}(\cdot, t) : \mathbb{R}^3 \to \mathbb{R}^3$ is divergence-free and irrotational at any fixed instant $t \in [0, T)$. Notice that $s \geq 3$ ensures that both h and $\mathbf{u}$ are C^1. In order to include the largest variety of possible water flows one would like to ask the minimal regularity for these functions (continuously differentiable functions come naturally to mind), the choice of Sobolev functions being due to technicalities arising in the mathematical analysis.

Notice that we have not specified anything about the pressure. The reason for this is that the mathematical role of the pressure inside the fluid domain is to keep the evolution of the velocity within the space of divergence-free vectors: the role of the pressure within the fluid is to guarantee that the left-hand side of the Euler equation (ee) is a gradient, a condition that can be expressed exclusively but in a less elegant form in terms of the components of the velocity field. The dynamic boundary condition (d) can also be expressed as a condition on $(\mathbf{u}, h)$: at any fixed instant $t \in [0, T)$, knowing the free surface $z = h(x, y, t)$ and the velocity field $\mathbf{u}(\mathbf{x}, t)$, the condition that the pressure is constant on $z = h(x, y, t)$ can be expressed exclusively in terms of $\mathbf{u}$ and h by using Euler's equation (ee) to write the condition that any derivative of $P(\cdot, t)$ in a tangential direction to the free surface vanishes, and the possible time-dependence of the constant-value $P_0(t)$ of P restricted to the free surface that one obtains is eliminated by requiring $P_0'(t) = 0$, a relation that again can be expressed as a condition on $\mathbf{u}$ and h. A hint that the pressure P is not to be regarded as an independent variable in the governing equations, being determined by the configuration of the fluid velocity and by the shape of the free surface, is given by observing that in the governing equations P is defined up to an additive irrelevant constant: for any constant $P_0 \in \mathbb{R}$, the function $P_0 + P$ can act as a pressure provided that we allow the adjustment of (d). Despite all this, the pressure P is not merely an ornamental device for the purpose of elegant formulations: it is physically relevant and of invaluable use in certain mathematical considerations, as we shall see.

A water flow without swirls is called irrotational, the mathematical expression of this condition taking the simple form of imposing that the velocity field is curl-free. Irrotational water waves are common physical occurences and there are considerable mathematical advantages in the analysis of these flows, as we shall see. For our present discussion the most relevant information is that an initially irrotational flow remains so at later times (see the discussion of vorticity). Lannes proved the well-posedness of the governing equations for irrotational water waves without imposing the condition that the motion is a small perturbation of still water, as it was the case in earlier work. As shown by Wu, the result goes even beyond our framework, allowing even for initial wave profiles that are overhanging (and therefore not representable in the form $z = h(x, y, t)$ for some function h), provided the water's surface is nonself-intersecting. Thus the governing equations for irrotational water waves have a firm mathematical foundation.

For general flows, that is, allowing for swirls, the well-posedness issue for the governing equations for gravity water waves is not yet settled. Nevertheless, if initially all along the water's free surface the exterior normal derivative of the pressure $\dfrac{\partial P}{\partial n}$ is negative, that is, if there is some $c_0 > 0$ such that

$$\frac{\partial P}{\partial n}\,(\mathbf{x}, 0) \leq -\, c_0 < 0 \quad \text{on} \quad z = h(x, y, 0),$$

Lindblad and Coutand & Shkoller proved that the governing equations are well-posed (and throughout the existence time a condition of this type on the pressure will hold). This condition is not required in the irrotational case because in this setting one can prove that it holds automatically. However, for water flows with swirls this condition has to be imposed initially, as failure to do so might result in the governing equations being ill-posed, as proved by Ebin.

We conclude our discussion of the well-posedness of the governing equations for water waves by pointing up an important gap in the theory. This concerns the issue of breaking waves: characterizing how a solution fails to exist after some finite time. There are solutions that exist for all times:

– for three-dimensional flows Germain & Masmoudi & Shatah as well as Wu proved recently that if the irrotational data have small size in a certain weighted Sobolev norm, then the solution exists for all times $t \geq 0$. The key ingredients of their proof are high-order energy estimates and dispersive estimates which are obtained by the analysis of the spatial and temporal resonances (the dimension having a certain advantage because the basic decay rate is faster and this contributes to improved dispersive estimates).

– for two-dimensional flows we will prove the existence of steady traveling waves, allowing even the presence of swirls in the water.

However, the wave breaking phenomenon is commonly observed while watching waves near the shore or in the open sea. It seems intuitively obvious that solutions do not exist for very long after they overturn (that is, after the slope of the free surface becomes infinite so that viewing it as the graph of a function is no longer possible) but a rigorous mathematical proof of this fact is not available. Nor are there special examples to guide us and the physical contention is not even supported by some partial geometrical results (for example of the type that the curvature of the surface might diverge at some point). This remains a challenging problem of great importance.

Vorticity

A fundamental property of a fluid flow is the vorticity,

$$\omega = \text{curl}\, \mathbf{u} = \nabla \wedge \mathbf{u},$$

measuring the local spin or rotation of a fluid element (that is, the rotational motion).

To see that the vorticity ω acts like a measure of the local rotation of the particles, consider for simplicity a time-independent flow with a smooth velocity field $\mathbf{u}(\mathbf{x})$ and let us show that to linear order the flow in a neighborhood of some fixed point $\mathbf{x_0}$ of the fluid domain can be viewed as a combination of infinitesimal translation, rotation, and deformation, with the vorticity accountable for the rotational part. The smooth velocity field has a Taylor series expansion near $\mathbf{x_0}$:

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x_0}) + [(D\mathbf{u})\,(\mathbf{x}_0)]\,(\mathbf{x} - \mathbf{x_0}) + O(|\mathbf{x} - \mathbf{x_0}|^2),$$

for $|\mathbf{x} - \mathbf{x_0}|$ small, where we denoted by $(D\mathbf{u})\,(\mathbf{x}) = \left(\dfrac{\partial u_i}{\partial x_j}\right)_{1\leq i,\,j\leq 3}(\mathbf{x})$ the differential of $\mathbf{u}$ at $\mathbf{x}$. If we write $(D\mathbf{u})(\mathbf{x_0})$ as the sum of its symmetric and anti-symmetric parts, $(D\mathbf{u})(\mathbf{x_0}) = D_S(\mathbf{x_0}) + D_A(\mathbf{x_0})$, with

$$D_S(\mathbf{x_0}) = \frac{(D\mathbf{u})(\mathbf{x_0}) \;+\; [(D\mathbf{u})(\mathbf{x_0})]^T}{2}, \qquad D_A(\mathbf{x_0}) = \frac{(D\mathbf{u})(\mathbf{x_0}) \;-\; [(D\mathbf{u})(\mathbf{x_0})]^T}{2},$$

where J^T denotes the transposed matrix of J, a straightforward calculation yields

$$2\, D_A(\mathbf{x_0})\mathbf{h} = \omega(\mathbf{x_0}) \wedge \mathbf{h}, \qquad \mathbf{h} \in \mathbb{R}^3,$$

so that to linear order in $|\mathbf{h}|$, where $\mathbf{h} = \mathbf{x} - \mathbf{x_0}$, the velocity field is the (unique) sum of three terms:

$$\mathbf{u}(\mathbf{x_0} + \mathbf{h}) \approx \mathbf{u}(\mathbf{x_0}) + D_S(\mathbf{x_0})\,\mathbf{h} + \frac{1}{2}\,\omega(\mathbf{x_0}) \wedge \mathbf{h}. \qquad (tdr)$$

The three terms have a natural physical interpretation in terms of translation, deformation and rotation.

If $\mathbf{x}(0)$ is a particle location near $\mathbf{x_0}$ at time $t=0$, then its trajectory is $\mathbf{x}(t)=\mathbf{x_0}+\mathbf{h}(t)$, so that at linear order we are interested in understanding the behavior near $\mathbf{0}$ of the solutions to the differential equation

$$\frac{d\mathbf{h}}{dt} = \mathbf{u}(\mathbf{x_0}) + D_S(\mathbf{x_0})\,\mathbf{h} + \frac{1}{2}\,\omega(\mathbf{x_0}) \wedge \mathbf{h} \tag{lf}$$

with initial data $\mathbf{h}(0)=\mathbf{x}(0)-\mathbf{x_0}$. We think of each of the three terms in the sum appearing on the right-hand side of (lf) as a vector field generating a flow near $\mathbf{0}$.

Retaining only the first term in (lf), we solve the differential equation $\dfrac{d\mathbf{h}(t)}{dt}=\mathbf{u}(\mathbf{x_0})$, with the solution $\mathbf{h}(t)=\mathbf{h}(0)+\mathbf{u}(\mathbf{x_0})\,t$ describing an infinitesimal translation.

The flow generated by the second term in (lf) encodes a stretching of the flow along the eigenspaces of the symmetric matrix $D_S(\mathbf{x_0})$, called the deformation or rate-of-strain matrix cf. [Majda & Bertozzi]. Indeed, this diagonalizable matrix has three real eigenvalues $\lambda_1 \le \lambda_2 \le \lambda_3$, not necessarily distinct, such that the corresponding eigenvectors $\mathbf{e}_i$ $(i=1,\,2,\,3)$ span $\mathbb{R}^3$, the stretching along these directions being simply the information encoded in the defining relations

$$D_S(\mathbf{x_0})\,\mathbf{e}_i = \lambda_i\,\mathbf{e}_i \quad \text{for} \quad i=1,\,2,\,3.$$

To see this, observe that if we ignore the first and third term in (lf), we obtain

$$\frac{d\mathbf{h}}{dt} = D_S(\mathbf{x_0})\,\mathbf{h}.$$

Writing $\mathbf{h}(t)=h_1(t)\,\mathbf{e}_1+h_2(t)\,\mathbf{e}_2+h_3(t)\,\mathbf{e}_3$, the above equation is equivalent to three simple linear equations $\dfrac{dh_i}{dt}=\lambda_i\,h_i$ for $i=1,\,2,\,3$. The rate of change along the $\mathbf{e}_i$ axis is λ_i, as

$$h_i(t) = h_i(0)\,e^{\lambda_i t} \quad \text{for} \quad t \ge 0.$$

Thus the local effect of the vector field $D_S(\mathbf{x_0})$ is merely an expansion or a contraction along each of the axes $\mathbf{e_i}$ (according to whether λ_i is positive or negative), hence the name "stretching" or "deformation". Furthermore, if $\mathbf{u}$ satisfies the equation of mass conservation (mc), then $0=\text{trace}\,(D\mathbf{u}(\mathbf{x_0}))=\text{trace}\,(D_S(\mathbf{x_0}))$ so that $\lambda_1+\lambda_2+\lambda_3=0$. Consequently the deformation associated to $D_S(\mathbf{x_0})$ is volume preserving as a fluid flowing with velocity $\mathbf{u}$ is being compressed at the rate $\nabla\cdot\mathbf{u}$ (so that the velocity field of an incompressible fluid is divergence free).

The third flow, obtained by solving

$$\frac{d\mathbf{h}}{dt} = \frac{1}{2}\,\omega(\mathbf{x_0}) \wedge \mathbf{h}, \tag{r}$$

defines a rigid rotation (in this analogy with rigid body motion the factor $\frac{1}{2}$ is a flaw but its removal by absorbtion in the definition of the curl symbol would have uncomfortable consequences for vector analysis) with angular velocity $\frac{1}{2}\,\omega(\mathbf{x_0})$. Indeed, assuming $\omega(\mathbf{x_0}) \neq \mathbf{0}$, the solution $\mathbf{h}(t)$ to (r) describes at constant angular speed $\theta = \frac{|\omega(\mathbf{x_0})|}{2}$ a circle lying in a plane orthogonal to the vector $\omega(\mathbf{x_0})$. To see this, choose the unit vectors $\mathbf{f}$ and $\mathbf{j}$ such that $\mathbf{e} = \frac{\omega(\mathbf{x_0})}{|\omega(\mathbf{x_0})|}$, $\mathbf{f}$ and $\mathbf{j}$ form an orthonormal basis for $\mathbb{R}^3$. Let

$$\mathbf{h}(0) = d\,\mathbf{e} + L\,\cos(\varphi)\,\mathbf{f} + L\,\sin(\varphi)\,\mathbf{j},$$

with φ being the angular coordinate in the $(\mathbf{f},\mathbf{j})$-plane, and observe that

$$\mathbf{h}(t) = d\,\mathbf{e} + L\,\cos(\varphi + \theta\,t)\,\mathbf{f} + L\,\sin(\varphi + \theta\,t)\,\mathbf{j},$$

satisfies

$$\mathbf{h}'(t) = -L\,\theta\,\sin(\varphi + \theta\,t)\,\mathbf{f} + L\,\theta\cos(\varphi + \theta\,t)\,\mathbf{j} = \frac{1}{2}\,\omega(\mathbf{x_0}) \wedge \mathbf{h}(t).$$

By the uniqueness theorem for the linear differential equation (r) this must be the solution with initial data $\mathbf{h}(0)$. Thus the flow (r) is a rotation with constant angular velocity, the axis of rotation being given by $\omega(\mathbf{x_0})$.

By splitting the vector field on the right-hand side of (lf) into these three components we described in each case the local flow behavior (at the linear level). It is tempting to conclude that the flow (lf) can therefore be viewed as a combination of a translation, a deformation and a rigid rotation. This is true to some extent.

Recall that if A is an 3×3 matrix and $\mathbf{b_0} \in \mathbb{R}^3$ is a vector, the unique solution of the initial-value problem

$$\mathbf{z}'(t) = A\,\mathbf{z}(t) + \mathbf{b_0}, \qquad \mathbf{z}(0) = \mathbf{z_0} \in \mathbb{R}^3,$$

is given by

$$\mathbf{z}(t;\, \mathbf{z_0}) = e^{At}\mathbf{z_0} + \Big(\int_0^t e^{A(t-s)}\, ds \Big)\, \mathbf{b_0}, \qquad t \in \mathbb{R},$$

where $e^{At} = \displaystyle\sum_{k=1}^{\infty} \frac{A^k t^k}{k!}$, the convergence being uniform on compact intervals of $\mathbb{R}$. Given another 3×3 matrix B, it is known that $e^{(A+B)\,t} = e^{At}e^{Bt}$ for all $t \in \mathbb{R}$ if and only if A and B commute. The solution to (lf) is

$$\mathbf{h}(t) = e^{\big(\, D_S(\mathbf{x_0}) + D_A(\mathbf{x_0})\big)\, t}\ \mathbf{h}(0) + \Big(\int_0^t e^{\big(\, D_S(\mathbf{x_0}) + D_A(\mathbf{x_0})\big)\,(t-s)}\, ds \Big)\ \mathbf{u}(\mathbf{x_0}),$$

and before we established that the maps $\mathbf{h} \mapsto e^{D_S(\mathbf{x_0})\,t}\mathbf{h}$ and $\mathbf{h} \mapsto e^{D_A(\mathbf{x_0})\,t}\mathbf{h}$ represent a deformation along three fixed axes, respectively a rotation with a fixed axis. Therefore, if the matrices $D_S(\mathbf{x_0})$ and $D_A(\mathbf{x_0})$ commute, then the flow (tf) represents a rotation with a fixed axis followed by a deformation along three fixed axes (and *vice-versa*, by commutativity),

$$\mathbf{h}(0) \longrightarrow e^{D_S(\mathbf{x_0})\,t}\ \Big(e^{D_A(\mathbf{x_0})\,t}\ \mathbf{h}(0) \Big),$$

and succeeded by a translation by the vector

$$\Big(\int_0^t e^{(\, D_S(\mathbf{x_0}) + D_A(\mathbf{x_0}))\,(t-s)}\ \Big)\ \mathbf{u}(\mathbf{x_0}).$$

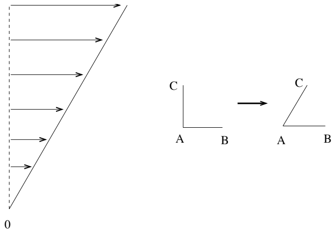
Unfortunately, an easy calculation shows that the matrices $D_S(\mathbf{x_0})$ and $D_A(\mathbf{x_0})$ commute if and only if $D\mathbf{u}(\mathbf{x_0})$ commutes with its transpose, and this is generally not true even if (mc) holds. Therefore it is not true that locally the flow can always be decomposed into a rotation with a fixed axis, a deformation along three fixed axes, and a translation. We emphasize that these considerations have a physical meaning only locally in space and time since they depend on linearizing. However, performing n pairs of such rotations combined with deformations, all being succeeded by a translation, gives a reasonable approximation of the flow if n is large enough, in view of Trotter's product formula

$$e^{(A+B)t} = \lim_{n \longrightarrow \infty} \Big(e^{A\,t/n}\, e^{B\,t/n} \Big)^n \quad \text{uniformly on compact intervals,}$$

which is valid even for matrices that do not commute cf. [Chicone].

We would like to emphasize the local nature of the rotation measured by the vorticity — from the knowledge of vorticity nothing can be concluded with respect to a global rotation of the water:

i) In $\mathbb{R}^3$, for the two-dimensional flow with velocity $\mathbf{u} = (y, 0, 0)$ the fluid is not rotating in any sense despite having vorticity $\omega = (0, 0, -1)$. The path of a particle located at (x_0, y_0, z_0) at time $t = 0$ is $t \mapsto (x_0 + t\, y_0,\ y_0,\ z_0)$ so that all particles move horizontally in straight lines. Clearly the fluid does not rotate in any sense but the relative positions of three particles A, B, C with the two momentarily perpendicular fluid line elements AB and AC oriented as shown below in a horizontal plane $[z = z_0]$ change in time, indicating the presence of some local spin.



Computing $(D\mathbf{u})(\mathbf{x_0})$ explicitly at an arbitrary but fixed point $\mathbf{x_0} \in \mathbb{R}^3$, one can easily check that $(D\mathbf{u})(\mathbf{x_0})$ does not commute with its transpose. Consequently, in view of the above considerations, we do not expect a simple decomposition of the flow into a rotation around the negative z-axis, a deformation along the axes

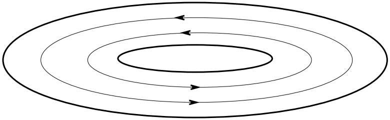
$$\mathbf{e_1} = (0, 0, 1), \quad \mathbf{e_2} = \frac{1}{\sqrt{2}}\,(1, 1, 0), \quad \mathbf{e_3} = \frac{1}{\sqrt{2}}\,(1, -1, 0),$$

given by the eigenvectors of the matrix $D_S(\mathbf{x_0}) = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ corresponding to the eigenvalues $\lambda_1 = 0$, $\lambda_2 = \dfrac{1}{2}$, respectively $\lambda_3 = -\dfrac{1}{2}$, succeeded by a translation. As a matter of fact, there is no global rotation around the z-axis. This situation illustrates the previous comments: it is not always easy to visualize the local rotational effect of vorticity.

ii) In the annulus $1 < x^2 + y^2 < 4$, for the two-dimensional flow with velocity

$$\mathbf{u}(x, y, t) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

each particle rotates with angular velocity r^{-2} depending on the distance $r = \sqrt{x^2+y^2}$ to the centre of the annulus. However the flow has zero vorticity.



Notice that the variable angular velocity of the particles makes it possible for the flow to be irrotational. For example, a particle on the outer boundary $r = 2$ returns to its initial position in time $t = 8\pi$, whereas one on the inner boundary $r = 1$ needs time $t = 2\pi$ to do so. Thus, while each particle describes a circle, the variable angular velocity accounts for the fact that there is no local spin. If, in contrast to this, we consider the velocity field

$$\mathbf{u}(x, y, t) = (-y, x)$$

in the same region, passing to polar coordinates, one can see that now $r'(t) = 0$ and $\theta'(t) = 1$, so that the particle paths are

$$t \mapsto (x(t),\ y(t)) = (r\ \cos(t + t_0),\ r\ \sin(t + t_0)),$$

with t_0 determined by the location at $t = 0$. The new flow is not irrotational: its vorticity can be computed as $\omega = (0, 0, 2)$.

Using the identity

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla(\frac{1}{2}\,\mathbf{u}\cdot\mathbf{u}) + (\nabla\wedge\mathbf{u})\wedge\mathbf{u}$$

in Euler's equation (ee), we obtain

$$\mathbf{u}_t + (\nabla\wedge\mathbf{u})\wedge\mathbf{u} = -\nabla\Big(\frac{1}{2}\,\mathbf{u}\cdot\mathbf{u} + P + gz\Big).$$

Taking the curl we get

$$\frac{\partial\omega}{\partial t} + \nabla\wedge(\omega\wedge\mathbf{u}) = 0,$$

in view of the identity $\nabla\wedge(\nabla f) = 0$ for $f \in C^2(\mathbb{R}^3,\mathbb{R})$. Using the vector identity

$$\nabla\wedge(F\wedge G) = (G\cdot\nabla)F - (F\cdot\nabla)G + F(\nabla\cdot G) - G(\nabla\cdot F)$$

for all $F, G \in C^2(\mathbb{R}^3,\mathbb{R}^3)$, this becomes

$$\frac{\partial\omega}{\partial t} + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u + \omega(\nabla\cdot u) - u(\nabla\cdot\omega) = 0.$$

In the previous equality the fourth term vanishes because water was assumed to be incompressible, while the fifth term vanishes because of the vector identity $\nabla\cdot(\nabla\wedge F) = 0$ for all $F \in C^2(\mathbb{R}^3,\mathbb{R}^3)$. We deduce that

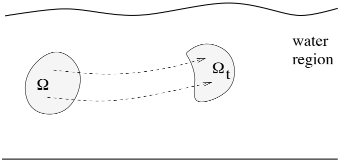
$$\frac{\partial\omega}{\partial t} + (u\cdot\nabla)\omega = (\omega\cdot\nabla)u,$$

or, alternatively

$$\frac{D\omega}{Dt} = (\omega\cdot\nabla)\mathbf{u}, \tag{ve}$$

the vorticity equation. In analyzing water flows with circulation the vorticity equation is extremely valuable because the hydrodynamical pressure has been eliminated: equation (ve) involves only $\mathbf{u}$ and ω which are related by $\omega = \nabla\wedge\mathbf{u}$. In particular, for a two-dimensional water flow (independent on the y-coordinate), we have $\omega_1 = \omega_3 = 0$ so that $(\omega\cdot\nabla)\mathbf{u} = \omega_2\dfrac{\partial\mathbf{u}}{\partial y} = 0$ and $\dfrac{D\omega}{Dt} = 0$, that is in a two-dimensional flow the vorticity of each individual water particle is conserved as the particle moves about.

For three-dimensional water flows with circulation, a rather complicated behaviour is due to the lack of conservation of vorticity — this proves to be a major obstacle to the understanding of crucial properties of the flow. To investigate the vorticity for three-dimensional water flows, it is useful to introduce the flow map $\mathbf{x} \mapsto \varphi(t, \mathbf{x})$: with fixed t, this map advances each particle in the water region from its position $\mathbf{x}$ at time $t = 0$ to its position at time t. For fixed t, φ is an invertible C^1 mapping — this is the Lagrangian viewpoint (there are two standard coordinate systems used in classical fluid dynamics: in Lagrangian coordinates, one describes the fluid as seen from any one of the particles of the fluid — the observer follows the fluid by picking out any particular particle and keeping track of where it goes, that is, one follows the fluid particles — whereas in Eulerian coordinates one describes the fluid from the viewpoint of a fixed observer watching whatever happens by; there exists a precise Eulerian state corresponding to a Lagrangian state and viceversa).



Ω_t is the image of Ω as water particles inΩ flow for time t

From the vorticity equation we infer that

$$\omega(t, \varphi(t, \mathbf{x})) = J(t, \mathbf{x})\, \omega(0, \mathbf{x}), \qquad (ve1)$$

where

$$J(t, \mathbf{x}) = \left(\frac{\partial \varphi_i}{\partial x_j} \right)_{\{1 \le i,j \le 3\}}$$

is the Jacobian matrix of the flow map.

Indeed, if we define

$$F(t, \mathbf{x}) = \omega(t, \varphi(t, \mathbf{x})), \qquad G(t, \mathbf{x}) = J(t, \mathbf{x})\, \omega(0, \mathbf{x}),$$

by (ve1) we have

$$\frac{\partial F}{\partial t} = (F \cdot \nabla)\mathbf{u}.$$

On the other hand, with $\nabla \mathbf{u} = \left(\dfrac{\partial u_i}{\partial x_j} \right)_{\{1 \leq i, j \leq 3\}}$, we have

$$\frac{\partial G}{\partial t} = \nabla[\partial_t \varphi(t, \mathbf{x})]\, \omega(0, \mathbf{x}) = \nabla[u(t, \varphi(t, \mathbf{x}))]\, \omega(0, \mathbf{x}) = (\nabla u)\, J(t, \mathbf{x})\, \omega(0, \mathbf{x}) = (G \cdot \nabla)\mathbf{u},$$

so that F and G satisfy the same linear ordinary differential equation with the same initial data — througout this argument $\mathbf{u}$ is presumed to be known. Thus, by unicity, they are equal and therefore (ve1) holds. An immediate consequence of (ve1) is that in three-dimensional flows a particle which has no vorticity never acquires it and conversely, a particle which is moving rotationally will continue so to move.

The above result says that fluid particles which were originally vorticity-free will continue to be so. An important special case is concerned with the development of water motions which start from rest (for example motions of initially still water generated by swell from a distant storm travelling into such undisturbed water). Then curl $\mathbf{u} \equiv 0$ initially and, therefore (ve1) ensures that such motions are irrotational at all later times. However, in general, part of the water body may be moving irrotationally and other parts rotationally. As the water moves about, the irrotational part may occupy different regions of space; $\omega \equiv 0$ is a property of those parts of the fluid which are moving irrotationally, not of the regions of space they may temporarily occupy: if a fluid flow has regions that are irrotational and others where vorticity is present, the spatial region originally occupied by fluid in irrotational motion may pick up vorticity at later times — in general the irrotational region will move around with the flow.

Irrotational flows

Since the fluid domains D we consider are simply connected, curl-free vector fields $\mathbf{u} : D \longrightarrow \mathbb{R}^3$ are gradients of some function $\phi : D \longrightarrow \mathbb{R}$: for any velocity field $\mathbf{u}$ with zero vorticity there exists a velocity potential ϕ, given up to an additive function of time by the line integral

$$\phi(\mathbf{x}) = \int_{\mathbf{x_0}}^{\mathbf{x}} u_1\, dx + u_2\, dy + u_3\, dz,$$

(for which the independence upon the path of integration joining the fixed basis point $\mathbf{x_0} \in D$ to the arbitrary point $\mathbf{x} \in D$ is ensured by the vanishing of the vorticity ω in view of Stokes' theorem) such that

$$\mathbf{u} = \nabla\phi.$$

Notice that the equation for mass conservation (mc) can be re-stated as $\Delta\phi = 0$: harmonic functions, and analytic functions in the two-dimensional setting, are relevant to irrotational incompressible flows.

While for simply connected domains $D \subset \mathbb{R}^2$ there is no velocity potential for two-dimensional water flows $\mathbf{u} = (u_1, u_2) : D \longrightarrow \mathbb{R}^2$ with non-zero vorticity, one can nevertheless find a stream function $\psi : D \longrightarrow \mathbb{R}$ such that

$$u_1 = \psi_y, \qquad u_2 = -\psi_x.$$

Up to a function of time

$$\psi(x, y) = \int_{(x_0, y_0)}^{(x,y)} u_1\, dy - u_2\, dx,$$

with the path of integration being any curve in D joining the fixed point $(x_0, y_0) \in D$ to the arbitrary point $(x, y) \in D$, the independence of the above line integral on the path being ensured by the incompressibility condition (mc).

▶ Currents

By a current we understand water in a flow with a flat surface. For two-dimensional steady currents moving at constant speed $c > 0$ the vorticity and the mass flux specify the entire flow. Indeed, consider the two-dimensional flow $\Big(u_1(x-ct,z), 0, u_3(x-ct,z)\Big)$ in a layer of water with the free surface $[z=0]$ and the flat bed $[z=-d]$, where $d>0$ is the constant depth. Introducing the stream function $\psi(x-ct,z)$ with $\psi_x=-u_3$ and $\psi_z=u_1-c$, the mass flux m is defined as

$$m = \int_{-d}^{0} u_1(x-ct,z)\,dz.$$

Notice that $m=\psi(x-ct,0)-\psi(x-ct,-d)$ is a constant since the kinematic boundary conditions (tk) and (bk) in this setting simply state that $u_3=0$ on $[z=0]$ and on $[z=-d]$ so that $\psi(x-ct,0)$ as well as $\psi(x-ct,-d)$ are constants. Since ψ is uniquely determined up to an additive constant, and it is constant on both boundaries, we may set $[\psi=0]$ on $[z=0]$ and $[\psi=-m]$ on $[z=-d]$. The vorticity ω has at most the second component non-zero and by abuse of notation it is identified with this scalar. Consequently we can express the vorticity as $\Delta\psi=\omega$, obtaining thus the Dirichlet boundary problem

$$\begin{cases} \psi(x,0)=0 \quad \text{and} \quad \psi(x,-d)=-m \quad \text{for} \quad x\in\mathbb{R}, \\ \Delta\psi(x,y)=\omega(x,y) \quad \text{for} \quad -d<z<0,\ x\in\mathbb{R}, \end{cases}$$

with a periodicity condition in the x-variable. If $\omega\in C^1$ is 2π-periodic in the x-variable throughout the strip $\mathbb{R}\times[-d,0]$, the above problem has a unique solution $\psi\in C^2$ (uniqueness is ensured by maximum principles and existence of $\psi\in C^{2+\alpha}\subset C^2$ if $\omega\in C^1\subset C^\alpha$ from Schauder estimates; both aspects will be addressed later on). It is of interest to provide a relatively explicit formula for the solution: for

$$\omega(x,z)=\sum_{n=0}^{\infty}\omega_n(z)\,e^{inx}$$

we seek

$$\psi(x,z)=\sum_{n=0}^{\infty}\psi_n(z)\,e^{inx}.$$

For $n = 0$ we have $\psi_0''(z) = \omega_0(z)$ for $-d < z < 0$, with $\psi_0(0) = 0$ and $\psi_0(-d) = -m$, so that

$$\psi_0(z) = z\left(\frac{m}{d} + \frac{1}{d}\int_{-d}^{0} s\omega_0(s)\,ds + \int_{-d}^{z}\omega_0(s)\,ds\right) + \int_{z}^{0} s\omega_0(s)\,ds,$$

and for every fixed $n \geq 1$ we have to solve the boundary value problem

$$\begin{cases} \psi_n''(z) \;-\; n^2\,\psi(z) = \omega_n(z), & -d < z < 0, \\ \psi_n(0) = \psi_n(-d) = 0. \end{cases} \qquad (b_n)$$

Writing the second-order differential equation in (b_n) as a first-order system for the vector $\Psi_n = \begin{pmatrix} \psi_n \\ \frac{1}{n}\,\psi_n' \end{pmatrix}$, by the variation of constants formula we have

$$\Psi_n(z) = e^{Az}\Psi_n(0) + \int_0^z e^{A(z-s)}\begin{pmatrix} 0 \\ \frac{1}{n}\,\omega_n(s) \end{pmatrix} ds, \qquad z \in [-d, 0],$$

where

$$A = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}, \qquad e^{Az} = \sum_{k=0}^{\infty}\frac{A^k z^k}{n!} = \begin{pmatrix} \cosh(nz) & \sinh(nz) \\ \sinh(nz) & \cosh(nz) \end{pmatrix}.$$

Imposing on the general solution the two boundary conditions, we obtain

$$\psi_n(z) = \frac{\sinh(nz)}{\sinh(nd)}\int_{-d}^{0}\sinh\Big(n(s+d)\Big)\,\omega_n(s)\,ds + \frac{1}{n}\int_{-d}^{0}\sinh\Big(n(z-s)\Big)\,\omega_n(s)\,ds, \quad n \geq 1.$$

Knowledge of ψ provides us with the flow velocity, and the pressure is easily determined since its value on $[z = 0]$ is P_{atm} by the dynamic boundary condition, and its gradient in the fluid is found using the Euler equation.

Vorticity is thus adequate for the specification of a current, and the previous Fourier series representation shows that if in a two-dimensional steady current ω is independent of x (or, equivalently, is time-independent), so will be ψ and thus u_1, while $u_3 \equiv 0$. Irrotational flows of zero vorticity model still water (in which case there is no current) as well as currents which are uniform with depth, and the simplest rotational setting is that of linearly sheared currents of constant nonzero vorticity. Constant vorticity gives a good description of tidal currents cf. [Peregrine]. These are the most regular and predictable currents, and on areas of the continental shelf and in many coastal inlets they are the most significant currents cf. [Jonsson].

When swell originating from a distant disturbance of the sea propagates through a region with an underlying current the wave-current interaction can generally not be overlooked. Spectacular examples are the wave-current interactions at the Columbia River entrance (where appreciable tidal currents make it one of the most hazardous navigational regions in the world since here the wave height can easily be doubled in just a few hours) and those found off the eastern coast of South Africa (where 6 m high sea waves from southwest meeting the Agulhas current can be tripled in height over a few wavelengths, leading to many wreckages of oil tankers) cf. [Jonsson].

As pointed out above, constant vorticity gives a good description of tidal currents: in the absence of waves, these flows above the flat bed $z = -d$ have a flat free surface $z = 0$ and the fluid velocity is of the form

$$\mathbf{u}(x, y, z, t) = \Big(u_0 + \omega\, z, 0, 0 \Big)$$

for some constants u_0 and ω with $\omega \neq 0$. The vorticity in this case can be easily computed as $(0, \omega, 0)$, being identified with the scalar ω. The positive vorticity case $\omega > 0$, in which the velocity at the surface exceeds that on the bed, is appropriate for the ebb current and negative vorticity (that is, $\omega < 0$) is appropriate for the flood current cf. [Yanagi]. Notice that tides refer to the vertical motion of water (the rise and fall of the tide) caused by the gravitational forces due to the relative motions of Moon, Sun and Earth, whereas tidal currents refer to the alternating, horizontal movement of water associated with the rise and fall of the tide: the current associated with a rising tide is called the flood, whereas the current associated with a falling tide is called the ebb.

On the other hand, the prime source of the major ocean currents is long duration winds and constant vorticity does not give a good description of these currents, a depth-dependent vorticity being more adequate. Also, at the mouth of an estuary the out-flowing current generally exhibits a non-uniform vorticity distribution.