HOMEOMORPHISMS IN THE SOBOLEV SPACE $W^{1,n-1}$

MARIANNA CSÖRNYEI, STANISLAV HENCL AND JAN MALÝ

Abstract. Let $\Omega \subset \mathbb{R}^n$ be open. We show that each homeomorphism $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ satisfies $f^{-1} \in BV_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$. If we moreover assume that f has finite distortion, then $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$ and f^{-1} has finite distortion. The main ingredient is a new result on change of variables in integral (area and coarea formula) for such mappings.

1. Introduction

Suppose that $\Omega \subset \mathbb{R}^n$ is an open set and let $f : \Omega \to f(\Omega) \subset \mathbb{R}^n$ be a homeomorphism. In this paper we continue the study of conditions which guarantee that $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$ or $BV_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$ (see Preliminaries for the definition of these spaces).

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a homeomorphism. Then $f^{-1} \in BV_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$.*

Note, that under these assumptions we cannot expect that $f^{-1} \in W^{1,1}_{\mathrm{loc}}$. Indeed, consider $g(x) = x + u(x)$ on the real line, where u is the usual Cantor ternary function. Let $h = g^{-1}$. Then h^{-1} fails to be absolutely continuous. By setting $f(x) = (h(x_1), x_2, \cdots, x_n)$ we obtain a Lipschitz homeomorphism whose inverse fails to be of the class $W^{1,1}_{\mathrm{loc}}$. On the other hand it is possible to prove that $f^{-1} \in W^{1,1}_{\mathrm{loc}}$ if we add the requirement that f has finite distortion.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a homeomorphism of finite distortion. Then $f^{-1} \in W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$ and f^{-1} is a mapping of finite distortion.*

Above, a homeomorphism $f \in W^{1,1}_{\mathrm{loc}}$ is of (or has) finite distortion if its Jacobian J_f is strictly positive almost everywhere on the set where $|\nabla f|$ does not vanish.

Our results are known in the planar case (see [9, Theorem 1.2] and [11, Theorem 1.1]). In the space, i.e. $n \geq 3$, they are known under

---

2000 *Mathematics Subject Classification.* 26B10, 30C65, 28A5, 46E35.

Key words and phrases. Sobolev mapping, Bounded variation, mapping of finite distortion, Jacobian, inverse.

The second author was supported in part by the research project MSM 0021620839 of the Czech Ministry MŠMT and in part by the grant GA ČR 201/06/P100. The third author was supported in part by the MSM 0021620839 and in part by the grant GA ČR 201/06/0198.

the stronger assumption that $f \in W^{1,p}_{\mathrm{loc}}$ for some $p > n-1$. (See [10], [11] and [18]; with some additional assumptions we may find the result already in [23].) For some other results in this direction consult e.g. [8] and [2]. Let us note that for each $n \geq 3$ and $1 \leq p < n-1$ there is a mapping of finite distortion in $W^{1,p}$ whose inverse fails to be in BV_{loc} (see [11, Example 3.1]) and therefore our regularity assumptions on f are optimal.

The condition $p > n-1$ was concerning the Lebesgue scale of spaces. In fact, this was used mainly to establish the area theorem for almost all hyperplanes, and thus the result, as well as the area formula on $\mathbb{R}^{n-1}$, holds if ∇f belongs to the Lorentz space $L_{n-1,1}$ (for the area and coarea formula in the Lorentz setting see [12], [17]).

Recall that the area formula on $\mathbb{R}^m$ holds for mappings in the Sobolev space $W^{1,p}(\mathbb{R}^m, \mathbb{R}^m)$ if $p > m$ [15], whereas it fails if $p \leq m$ (for description of a counterexample due to Cesari and further discussion see [14]). If $p < m$, there exists even a homeomorphism in $W^{1,p}(\mathbb{R}^m, \mathbb{R}^m)$ such that the area formula fails ([19], for improvements see [13]), but for homeomorphisms of class $W^{1,m}$ the area formula holds [20]. The last result looks promising for our purposes, but it highly relies on the assumption that the homeomorphism is between Euclidean spaces of the same dimension. We want to apply it to mappings from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n$, and then the area formula can fail, see Example 5.1.

Fortunately, we are able to prove that if $f \in W^{1,n-1}((-1,1)^n, \mathbb{R}^n)$ is a homeomorphism, then the area formula holds on almost all hyperplanes. We believe that the new area formula may be of independent interest and it might find applications elsewhere. As a consequence we also derive a related coarea formula (Theorem 4.4).

The area formula on almost all hyperplanes was clear in the Lorentz setting: if $|\nabla f| \in L_{n-1,1}((-1,1)^n)$, then $|\nabla f_{\lfloor H}|$ is in $L_{n-1,1}(H)$ for almost every hyperplane H and the area formula holds then on H. In our new situation, if $|\nabla f| \in L^{n-1}((-1,1)^n)$, Example 5.1 indicates that the finiteness of the norm may not be a sufficient argument. We consider a slightly stronger condition which holds on almost all hyperplanes (see Theorem 3.1).

Theorem 1.3. *Let $f \in W^{1,n-1}_{\mathrm{loc}}((-1,1)^n, \mathbb{R}^n)$ be a homeomorphism. Then for almost every $y \in (-1,1)$ the mapping $f\lfloor_{(-1,1)^{n-1}\times\{y\}}$ satisfies the Luzin (N) condition, i.e., for every $A \subset (-1,1)^{n-1} \times \{y\}$, $\mathcal{H}^{n-1}(A) = 0$ implies $\mathcal{H}^{n-1}(f(A)) = 0$.*

In the previous paper [10] it was moreover possible to prove that both f and f^{-1} are differentiable almost everywhere under the stronger regularity assumption $|\nabla f| \in L_{n-1,1}$. In the last section we use ideas from [22] to give an example of $W^{1,n-1}$ homeomorphism of finite distortion such that both f and f^{-1} are nowhere differentiable.

2. Preliminaries

We say that $g : \Omega \to \mathbb{R}^n$ belongs to the Sobolev space $W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n)$, $1 \le p < \infty$, if g is locally p-integrable and if the coordinate functions of g have locally p-integrable distributional derivatives.

A mapping $f \in L^1(\Omega; \mathbb{R}^n)$ is of bounded variation, $f \in BV(\Omega; \mathbb{R}^n)$, if the coordinate functions of f belong to the space $BV(\Omega)$. This means that the distributional partial derivatives of each coordinate function h of f are measures with finite total variation in Ω : there are Radon (signed) measures $\mu_1, \cdots, \mu_n$ defined in Ω so that for $i = 1, \cdots, n$, $|\mu_i|(\Omega) < \infty$ and

$$\int_\Omega h D_i\varphi \, dx = -\int_\Omega \varphi \, d\mu_i$$

for all $\varphi \in C_0^\infty(\Omega)$.

For $x \in \mathbb{R}^n$ we denote by x_i, $i \in \{1, \dots, n\}$, its coordinates, i.e. $x = [x_1, x_2 \dots x_n]$. Fix $y \in \mathbb{R}$. Since $\mathbb{R}^{n-1} \times \{y\}$ is in fact a copy of $\mathbb{R}^{n-1}$, the Hausdorff measure on it can be identified with the Lebesgue measure and we can write dz instead of $d\mathcal{H}^{n-1}(z)$.

The euclidean norm of $x \in \mathbb{R}^n$ is denoted by $|x|$. By $B(x, r)$ we denote an euclidean ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$ and by $S(x, r)$ we denote the corresponding sphere. We call the product of n one dimensional intervals an n-dimensional interval.

Given a square matrix $P \in \mathbb{R}^{n\times n}$, we define the norm $|P|$ as the supremum of $|Px|$ over all vectors x of unit euclidean norm. The adjugate $\operatorname{adj} P$ of a regular matrix P is defined by the formula

$$P \operatorname{adj} P = I \det P,$$

where $\det P$ denotes the determinant of P and I is the identity matrix. The operator adj is then continuously extended to $\mathbb{R}^{n\times n}$. We also denote by $\operatorname{cof} P$ the transpose of $\operatorname{adj} P$.

We use the symbol $|E|$ for the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$. A mapping $f : \Omega \to \mathbb{R}^n$ is said to satisfy the Luzin condition (N) on E if $|f(A)| = 0$ for every $A \subset E$ such that $|A| = 0$.

Let $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ and $E \subset \Omega$ be a measurable set. The multiplicity function $N(f, E, y)$ of f is defined as the number of preimages of y under f in E. We say that the area formula holds for f on E if

$$\int_E \eta(f(x))\, |J_f(x)|\, dx = \int_{\mathbb{R}^n} \eta(y)\, N(f, E, y)\, dy \tag{2.1}$$

for any nonnegative Borel measurable function η on $\mathbb{R}^n$. It is well known that there exists a set $\Omega' \subset \Omega$ of full measure such that the area formula holds for f on Ω'. Also, the area formula holds on each set on which the Luzin condition (N) is satisfied. This follows from [1, 3.1.4, 3.1.8, 3.2.5], namely, it can be found there that Ω can be covered up to a set of measure zero by countably many sets the restriction to which of f is Lipschitz continuous. For more explicit statements see e.g. [5], [7].

3. Luzin (N) condition on hyperplanes

In this section we will prove Theorem 1.3. If fact, we will show a stronger statement which specifies on which hyperplanes we may be sure that the Luzin (N)-condition is satisfied.

The scheme of argumentation is the following:

$$\text{Lemma 3.2} \to \text{Lemma 3.3} \to \text{Theorem 3.1} \to \text{Theorem 1.3}$$

In what follows, we fix a smooth mollification kernel ψ_1 on $\mathbb{R}^n$ such that the family $(\psi_\delta)_\delta$,

$$\psi_\delta(x) = \delta^{-n}\psi_1\big(\tfrac{x}{\delta}\big), \tag{3.1}$$

is a standard family of mollifiers. Recall that this means (in addition to (3.1)) that each ψ_δ is a smooth nonnegative function with support in $B(0,\delta)$ and $\int_{\mathbb{R}^n} \psi_\delta\,dx = 1$. We also assume in the sequel that $\psi_\delta > 0$ on $B(0,\delta)$.

Also, we introduce a "crude family" of mollification kernels

$$\tilde{\psi}_\delta = (2\delta)^{-n}\chi_{Q(0,\delta)},$$

here χ_M denotes the characteristic function of a set M which is 1 on M and zero elsewhere, and $Q(z,\delta)$ the open cube with center at z and radius δ:

$$Q(z,\delta) = \{x \in \mathbb{R}^n \colon x - z \in (-\delta,\delta)^n\}.$$

Theorem 3.1. *Suppose that $f : (-1,1)^n \to \mathbb{R}^n$ is a homeomorphism of the class $W^{1,n-1}$. Let $t \in (-1,1)$ be such that $f \in W^{1,n-1}((-1,1)^n \times \{t\})$ and*

$$\liminf_{\delta\to 0+} \int_{(-1+\delta,1-\delta)^{n-1}\times\{t\}} \psi_\delta * |\nabla f|^n \le \int_{(-1,1)^{n-1}\times\{t\}} |\nabla f|^n. \tag{3.2}$$

Then f satisfies the Luzin (N)-condition on $(-1,1)^{n-1} \times \{t\}$.

Proof of Theorem 1.3. By Theorem 3.1, we only need to show that (3.2) holds for almost every $t \in (-1,1)$. Set

$$\Phi_\delta(t) = \int_{(-1+\delta,1-\delta)^{n-1}\times\{t\}} \Big|\,|\nabla f|^n - \psi_\delta * |\nabla f|^n\Big|$$

Since the mollifications of an L^1 function converge in L^1, by Fatou's lemma and Fubini's theorem we have

$$\begin{aligned}\int_{-1}^{1} \liminf_{\delta\to 0+} \Phi_\delta &\le \liminf_{\delta\to 0+} \int_{-1}^{1} \Phi_\delta \\ &= \liminf_{\delta\to 0+} \int_{(-1+\delta,1-\delta)^n} \Big|\,|\nabla f|^n - \psi_\delta * |\nabla f|^n\Big| = 0.\end{aligned}$$

This means that $\liminf \Phi_\delta(t) = 0$ for almost every $t \in (-1,1)$, and each such a t satisfies (3.2). $\square$

Proof of Theorem 3.1. Suppose that (3.2) holds for t. Then, by Lemma 3.3 below,

$$\mathcal{H}^{n-1}_\infty(f(Q\times\{t\})) \le C\int_{Q\times\{t\}} |\nabla f|^{n-1}$$

holds for all cubes $Q \subset (-1,1)^{n-1}$. Let $E \subset (-1,1)^{n-1}$ be a set of measure zero. Given $\varepsilon > 0$, we find an open set $G \subset (-1,1)^{n-1}$ such that $E \subset G$ and

$$\int_{G\times\{t\}} |\nabla f|^{n-1} < \varepsilon.$$

Let $\{Q_j\}_j$ be a sequence of nonoverlapping closed cubes such that $G = \bigcup_j Q_j$. Then

$$\begin{aligned}\mathcal{H}^{n-1}_\infty(f(E\times\{t\})) \le \mathcal{H}^{n-1}_\infty(f(G\times\{t\})) &\le \sum_j \mathcal{H}^{n-1}_\infty((f(Q_j\times\{t\})) \\ &\le C\sum_j \int_{Q_j\times\{t\}} |\nabla f|^{n-1} = C\int_{G\times\{t\}} |\nabla f|^{n-1} < C\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain the assertion. $\square$

The following lemma is the main step towards the area formula. In view of its eventual own interest, we prove it in a form which is more precise than what we actually need.

Lemma 3.2. *Suppose that $f : (-1,1)^n \to \mathbb{R}^n$ is a homeomorphism and $U \subset\subset (-1,1)^n$ is an open n-dimensional interval. If the restriction of f to ∂U is of class $W^{1,n-1}$, then*

$$\mathcal{H}^{n-1}_\infty(f(U)) \le C\int_{\partial U} |\operatorname{cof} \nabla f\ \boldsymbol{\nu}_U|\, d\mathcal{H}^{n-1},$$

where $\boldsymbol{\nu}_U$ is the outer normal to U.

Proof. Suppose that there is a sequence $\{f_k\}_k$ of smooth approximations of f such that $f_k \to f$ in $W^{1,n-1}(\partial U;\mathbb{R}^n)$ and $f_k \to f$ uniformly in $\overline{U}$. Then the functions f_k are not homeomorphisms anymore, but this failure will not be important in the sequel. Since f is a homeomorphism, we know that $\deg(f,U,\cdot) = 1$ in $f(U)$ (or -1, but without loss of generality we may agree on 1). Set $G_k = \{y \in f(-1,1)^n : \deg(f_k,U,\cdot) = 1\}$. Then, by Gustin's boxing lemma [6]

$$\mathcal{H}^{n-1}_\infty(G_k) \le C\mathcal{H}^{n-1}(\partial G_k).$$

Since $\deg(f_k,U,\cdot)$ is constant on components of $\mathbb{R}^n \setminus f_k(\partial U)$, ∂G_k is contained in the image of ∂U under f_k, and thus

$$\mathcal{H}^{n-1}_\infty(G_k) \le C\mathcal{H}^{n-1}(f_k(\partial U)) = C\int_{\partial U} |\operatorname{cof} \nabla f_k\ \boldsymbol{\nu}|\, d\mathcal{H}^{n-1}$$

by the smooth coarea formula. Letting $k \to \infty$ we obtain the assertion, as the integrand is continuous in $W^{1,n-1}$ and

$$f(U) \subset \bigcup_{k=1}^{\infty} \bigcap_{j\geq k} G_j$$

(for passage to limit with $\mathcal{H}^{n-1}_\infty$ see [1, 2.10.22]). This argument is enough to establish the estimate for "almost every" interval if $f \in W^{1,n-1}((-1,1)^n;\mathbb{R}^n)$, since in this case the standard mollifications converge on "almost every boundary". The reader interested primarily in the main theorem of the section can now skip to the next lemma. In order to establish the assertion as it is stated, we assume for simplicity that the center of U is at the origin. Let p be the Minkowski seminorm associated with U, namely

$$p(x) = \inf\{s > 0 \colon \tfrac{x}{s} \in U\}.$$

Then for $\varepsilon > 0$ small enough we set

$$f^*(x) = \begin{cases} f\left(\frac{x}{1-\varepsilon}\right), & p(x) < 1-\varepsilon, \\ f\left(\frac{x}{p(x)}\right), & 1-\varepsilon \leq p(x) \leq 1+\varepsilon, \\ f\left(\frac{x}{1+\varepsilon}\right), & p(x) > 1+\varepsilon. \end{cases}$$

Then f^* is uniformly close to f on $\overline{U}$, and since f^* is constant along rays on a neighborhood of ∂U, the mollifications of f^* tend to f in $W^{1,n-1}(\partial U,\mathbb{R}^n)$. This way one can find the required approximations. □

Lemma 3.3. *Suppose that $f : (-1,1)^n \to \mathbb{R}^n$ is a homeomorphism of the class $W^{1,n-1}$. Let $t \in (-1,1)$ be such that $f \in W^{1,n-1}((-1,1)^{n-1} \times \{t\})$ and the condition* (3.2) *is satisfied. Then*

$$\mathcal{H}^{n-1}_\infty(f(Q \times \{t\})) \leq C \int_{Q\times\{t\}} |\nabla f|^{n-1}$$

for each $(n-1)$-dimensional cube $Q \subset (-1,1)^{n-1}$.

Proof. For simplicity of notation let us assume that $t = 0$ and $Q = [-R,R]^{n-1}$. Let us denote

$$\begin{aligned} J &= Q \times \{0\}, \\ I_r &= (-R-r, R+r)^{n-1} \times (-r,r). \end{aligned}$$

Consider $\rho > 0$ such that $I_{n\rho} \subset (-1,1)^n$. By Lemma 3.2,

$$\mathcal{H}^{n-1}_\infty(f(J)) \leq C \int_{\partial I_r} |\operatorname{cof} \nabla f \; \boldsymbol{\nu}_{I_r}| \, d\mathcal{H}^{n-1} \leq C \int_{\partial I_r} |\nabla f|^{n-1} \, d\mathcal{H}^{n-1}.$$

for almost every $r \in (0,\rho)$. The estimate of the "jacobian" by $|\nabla f|^{n-1}$ is crude but sufficient for our purposes. We integrate by r and using the Fubini theorem "on each side" we obtain

$$\rho\mathcal{H}^{n-1}_\infty(f(J)) \le C\int_0^\rho\int_{\partial I_r}|\nabla f|^{n-1}\,d\mathcal{H}^{n-1} \le C\int_{I_\rho}|\nabla f|^{n-1}. \tag{3.3}$$

Let $x \in I_\rho$. Then

$$\mathcal{H}^{n-1}(J \cap Q(x,\rho)) \ge \rho^{n-1}.$$

Hence using the Fubini theorem we obtain

$$\begin{aligned}\rho^{n-1}\int_{I_\rho}|\nabla f|^{n-1}(x)\,dx &\le \int_{I_\rho}\Big(\int_{J\cap Q(x,\rho)}|\nabla f|^{n-1}(x)\,d\mathcal{H}^{n-1}(y)\Big)\,dx\\ &= \int_J\Big(\int_{Q(y,\rho)}|\nabla f|^{n-1}(x)\,dx\Big)\,d\mathcal{H}^{n-1}(y)\\ &= 2^n\rho^n\int_J(\tilde{\psi}_\delta * |\nabla f|^{n-1})(y)\,d\mathcal{H}^{n-1}(y).\end{aligned}\tag{3.4}$$

Since $\tilde{\psi}_\rho \le C\psi_{n\rho}$, from (3.3) and (3.4) we infer

$$\mathcal{H}^{n-1}_\infty((f(J)) \le C\int_J \psi_{n\rho} * |\nabla f|^{n-1}.$$

Now it is clear that the condition (3.2) implies the estimate. □

4. Weak differentiability of the inverse

In this section we use the notation $\pi_r(x) = |x|$ for radial projection and $\pi_S(x) = \frac{x}{|x|}$ for projection to the unit sphere. The following Lemma easily follows from Theorem 1.3.

Lemma 4.1. *Let $f \in W^{1,n-1}(B(x,r_0),\mathbb{R}^n)$ be a homeomorphism. Then for almost every $r \in (0,r_0)$ the mapping $\pi_S \circ f : S(x,r) \to S(0,1)$ satisfies the Luzin (N) condition, i.e.*

$$\mathcal{H}^{n-1}(\pi_S \circ f(A)) = 0 \textit{ for every } A \subset S(x,r) \textit{ such that } \mathcal{H}^{n-1}(A) = 0.$$

The following coarea formula is crucial for our proof of Theorem 1.1. In what follows, $\Omega \subset \mathbb{R}^n$ will be an open set.

Lemma 4.2. *Let $f \in W^{1,n-1}(\Omega,\mathbb{R}^n)$ be a homeomorphism. Set $h = \pi_S \circ f$ and let $E \subset \Omega$ be a measurable set. Then*

$$\int_{\partial B(0,1)}\mathcal{H}^1\big(\pi_r\big(\{x\in E:\ h(x)=z\}\big)\big)\,d\mathcal{H}^{n-1}(z) \le \int_E|\operatorname{adj}\nabla h|\,dx.$$

Proof. If f is Lipschitz, we can use coarea formula from Federer [1, 3.2.12] to obtain

$$\begin{aligned}&\int_{\partial B(0,1)}\mathcal{H}^1\big(\pi_r\big(\{x\in E:\ h(x)=z\}\big)\big)\,d\mathcal{H}^{n-1}(z) \le\\ &\qquad\le \int_{\partial B(0,1)}\mathcal{H}^1\big(\{x\in E:\ h(x)=z\}\big)\,d\mathcal{H}^{n-1}(z) = \int_E|\operatorname{adj}\nabla h|\,dx.\end{aligned}$$

In the general case, we cover the domain of f up to a set of measure zero by countably many sets of the type $\{f = f_j\}$ with f_j Lipschitz.

It remains to consider the case that $E = N$ with $|N| = 0$. For $z \in S(0,1)$ we denote $S_z = \pi_S^{-1}(z)$. To obtain a contradiction suppose that there is a set $P \subset S(0,1)$ such that $\mathcal{H}^{n-1}(P) > 0$ and for every $z \in P$ we have $\mathcal{H}^1\big(\pi_r(f^{-1}(S_z) \cap E)\big) > 0$. Consider the following set $A \subset (0,\infty) \times S(0,1)$:

$$[r,z] \in A \Leftrightarrow z \in P \text{ and } r \in \pi_r\big(f^{-1}(S_z) \cap E\big).$$

By the Fubini theorem we obtain

$$|A| = \int_P \mathcal{H}^1\big(\pi_r(f^{-1}(S_z) \cap E)\big) d\mathcal{H}^{n-1}(z) > 0.$$

Set $E_r = E \cap S(x,r)$. For almost every r we have $\mathcal{H}^{n-1}(E_r) = 0$ and therefore we obtain $\mathcal{H}^{n-1}(\pi_S \circ f(E_r)) = 0$ for almost every r by Lemma 4.1. Now the Fubini theorem implies

$$|A| = \int_0^\infty \mathcal{H}^{n-1}(\pi_S \circ f(E_r)) dr = 0$$

which gives us a contradiction. □

The following lemma will give us the regularity of f^{-1}. In its proof we use ideas from [10, Proof of Lemma 3.2].

Lemma 4.3. *Let $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a homeomorphism. Then*

$$\int_B |f^{-1}(y) - c|\, dy \le C r_0 \int_{f^{-1}(B)} |\operatorname{adj} \nabla f(x)|\, dx, \tag{4.1}$$

for each ball $B = B(y_0, r_0) \subset f(\Omega)$, where

$$c = \fint_B f^{-1}(y)\, dy$$

and $C = C(n)$.

Proof. We fix $y' = f(x') \in B$ and for simplicity of notation (without loss of generality) we assume that $x' = 0$. Denote

$$h(x) = \frac{f(x) - y'}{|f(x) - y'|}.$$

If $y'' = f(x'') \in B$ and $\mathrm{co}(\{y'', y'\})$ is the line segment connecting y' and y'', then $f^{-1}(\mathrm{co}(\{y'', y'\}))$ is a curve connecting x' and x'' and thus

$$|x'' - x'| \le \mathcal{H}^1\Big(\pi_r \circ f^{-1}(\mathrm{co}(\{y'', y'\}))\Big). \tag{4.2}$$

We have

$$y \in \mathrm{co}\{y'', y'\} \implies \frac{y - y'}{|y - y'|} = \frac{y'' - y'}{|y'' - y'|}. \tag{4.3}$$

Hence, if $r = |y'' - y'|$, then

$$\begin{aligned} |f^{-1}(y'') - f^{-1}(y')| &\leq \mathcal{H}^1\Big(\pi_r \circ f^{-1}(\mathrm{co}(\{y'', y'\}))\Big) \\ &\leq \mathcal{H}^1\Big(\pi_r\big(\{x \in f^{-1}(B) :\ h(x) = \tfrac{y''-y'}{r}\}\big)\Big). \end{aligned}$$

Given $r > 0$, using Lemma 4.2 for the mapping $f(x) - y'$ we estimate

$$\begin{aligned} (4.4)\quad &\int_{B\cap\partial B(y',r)} |f^{-1}(y'') - f^{-1}(y')|\, d\mathcal{H}^{n-1}(y'') \\ &\quad\leq \int_{B\cap\partial B(y',r)} \mathcal{H}^1\Big(\pi_r\big(\{x \in f^{-1}(B) :\ h(x) = \tfrac{y''-y'}{r}\}\big)\Big)\, d\mathcal{H}^{n-1}(y'') \\ &\quad\leq r^{n-1}\int_{\partial B(0,1)} \mathcal{H}^1\Big(\pi_r\big(\{x \in f^{-1}(B) :\ \ h(x) = z\}\big)\Big)\, d\mathcal{H}^{n-1}(z) \\ &\quad\leq r^{n-1}\int_{f^{-1}(B)} |\operatorname{adj}\nabla h(x)|\, dx \\ &\quad\leq Cr^{n-1}\int_{f^{-1}(B)} \frac{|\operatorname{adj}\nabla f(x)|}{|f(x) - f(x')|^{n-1}}\, dx, \end{aligned}$$

where the last inequality follows using the chain rule, the formula $|\operatorname{adj}(PQ)| \leq C|\operatorname{adj} P||\operatorname{adj} Q|$ and the estimate

$$\Big|\operatorname{adj}\nabla \frac{z - y'}{|z - y'|}\Big| \leq \frac{C}{|z - y'|^{n-1}}.$$

Hence

$$\begin{aligned} |B|\,|f^{-1}(y') - c| &\leq \int_B |f^{-1}(y'') - f^{-1}(y')|\, dy'' \\ &= \int_0^{2r_0} \Big(\int_{B\cap\partial B(y',r)} |f^{-1}(y'') - f^{-1}(y')|\, d\mathcal{H}^{n-1}(y'')\Big)\, dr \\ &\leq C\int_0^{2r_0} r^{n-1}\Big(\int_{f^{-1}(B)} \frac{|\operatorname{adj}\nabla f(x)|}{|f(x) - f(x')|^{n-1}}\, dx\Big)\, dr \\ &\leq Cr_0^n \int_{f^{-1}(B)} \frac{|\operatorname{adj}\nabla f(x)|}{|f(x) - f(x')|^{n-1}}\, dx. \end{aligned}$$

Integrating with respect to y' and then using Fubini's theorem we obtain

$$\begin{aligned} \int_B |f^{-1}(y') - c|\, dy' &\leq C\int_{f^{-1}(B)} |\operatorname{adj}\nabla f(x)|\Big(\int_B \frac{dy'}{|f(x) - y'|^{n-1}}\Big)\, dx \\ &\leq Cr_0 \int_{f^{-1}(B)} |\operatorname{adj}\nabla f(x)|\, dx. \end{aligned}$$

□

Proof of Theorem 1.1. It follows from Lemma 4.3 that there is a locally finite Radon measure μ such that f^{-1} satisfies 1-Poincaré inequality in $f(\Omega)$

$$\int_B |f^{-1}(y) - c|\, dy \leq r_0 \mu(B)$$

for every ball $B \subset f(\Omega)$. Thus $f^{-1} \in BV_{\text{loc}}$ (see [16, Proposition 1.1]). $\square$

We will introduce some notation needed in the sequel. We write $\mathbb{H}_i$ for the i-th coordinate hyperplane

$$\mathbb{H}_i = \{x \in \mathbb{R}^n :\ x_i = 0\}$$

and denote by π_i the orthogonal projection to $\mathbb{H}_i$, so that

$$\pi_i(x) = x - x_i \mathbf{e}_i, \qquad x \in \mathbb{R}^n.$$

By π^j we denote the projection to the j-th coordinate $\pi^j(x) = x_j$.

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in W^{1,n-1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ is a homeomorphism. Then for each measurable set $E \subset \Omega$ we have*

$$\int_E |\operatorname{adj} \nabla(\pi_i \circ f)| = \int_{\pi_i(\mathbb{R}^n)} \mathcal{H}^1(E \cap (\pi_i \circ f)^{-1}(y))\, dy.$$

Proof. As in the proof of Lemma 4.5, we can rely on the Lipschitz coarea formula [1, 3.2.12] and restrict then our attention to the case when E is Lebesgue null. We need only to show that the set of all points $y \in \pi_i(\mathbb{R}^n)$ such that $\mathcal{H}^1(E \cap (\pi_i \circ f)^{-1}(y)) > 0$ is of measure zero. Since we already know that f^{-1} is of bounded variation, it follows that for almost every y the preimage $(\pi_i \circ f)^{-1}(y))$ is a rectifiable curve. Hence, if this is of positive one-dimensional measure, there exists a one-dimensional projection of this set which is also of positive one-dimensional measure. Now we can obtain contradiction exactly as in the proof of Lemma 4.2. $\square$

Theorem 4.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in W^{1,n-1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ be a homeomorphism such that $f^{-1} \in W^{1,1}_{\text{loc}}(f(\Omega), \mathbb{R}^n)$. Then f^{-1} is a mapping of finite distortion.*

Proof. Suppose that f^{-1} is not a mapping of finite distortion. Then we can find a set $\tilde{A} \subset f(\Omega)$ such that $|\tilde{A}| > 0$ and for every $y \in \tilde{A}$ we have $J_{f^{-1}}(y) = 0$ and $|Df^{-1}(y)| > 0$. Since f^{-1} is of the class $W^{1,1}_{\text{loc}}$, we may assume without loss of generality that f^{-1} is absolutely continuous on all lines parallel to coordinate axes that intersect $\tilde{A}$ and that f^{-1} has classical partial derivatives at every point of $\tilde{A}$ (see e.g. [24, Th. 2.1.4]).

We claim that we can find a Borel set $A \subset \tilde{A}$ such that $|A| > 0$ and $|f^{-1}(A)| = 0$. Since $f^{-1} \in W^{1,1}$ we know that f^{-1} is approximately differentiable almost everywhere (see [1, 3.1.4]). Thus we can find a Borel set $A \subset \tilde{A}$ such that $|A| > 0$ and f^{-1} is approximatively differentiable at A. The area formula holds on a set of approximate differentiability (see [1, 3.2.1]) and hence

$$|f^{-1}(A)| = \int_{\Omega} \chi_{f^{-1}(A)}(x)\, dx = \int_{f(\Omega)} \chi_A(y) J_{f^{-1}}(y)\, dy = 0.$$

Clearly, there is $i \in \{1\dots,n\}$ such that the subset of A where $\frac{\partial f^{-1}(y)}{\partial y_i} \neq 0$ has positive measure. Without loss of generality we will assume that $\frac{\partial f^{-1}(y)}{\partial y_i} \neq 0$ for every $y \in A$. Set $E := f^{-1}(A)$ and recall that $|E| = 0$. Using Theorem 4.4 we obtain

$$\int_{\mathbb{H}_i} \mathcal{H}^1\big(\pi^j(\{x \in E : \pi_i \circ f(x) = z\})\big) dz = 0, \tag{4.5}$$

for each $j \in \{1, \dots, n\}$. By the Fubini theorem,

$$\int_{\mathbb{H}_i} \mathcal{H}^1(A \cap \pi_i^{-1}(z))\, dz = |A| > 0.$$

Therefore there exists $z \in \mathbb{H}_i$ with

$$\mathcal{H}^1\big(\pi^j(E \cap f^{-1}(\pi_i^{-1}(z)))\big) = \mathcal{H}^1\big(\pi^j(\{x \in E : \pi_i \circ f(x) = z\})\big) = 0,$$

and

$$\mathcal{H}^1(A \cap \pi_i^{-1}(z)) > 0.$$

Clearly

$$0 < \int_{A\cap\pi_i^{-1}(z)} \Big|\frac{\partial f^{-1}}{\partial y_i}(y)\Big|\, d\mathcal{H}^1(y)$$

and therefore we can find j such that for $h = \pi^j \circ f^{-1}$ we have

$$0 < \int_{A\cap\pi_i^{-1}(z)} \Big|\frac{\partial h}{\partial y_i}(y)\Big|\, d\mathcal{H}^1(y).$$

Applying the one-dimensional area formula to the absolutely continuous mapping

$$t \mapsto h(z + t\mathbf{e}_i)$$

we obtain

$$\begin{aligned} 0 &< \int_{A\cap\pi_i^{-1}(z)} \Big|\frac{\partial h}{\partial y_i}(y)\Big|\, d\mathcal{H}^1(y) \\ &= \int_{\mathbb{R}} N(h, A \cap \pi_i^{-1}(z), x)\, dx \\ &= \int_{\pi^j(E\cap f^{-1}(\pi_i^{-1}(z)))} N(h, A \cap \pi_i^{-1}(z), x)\, dx \\ &= 0\ , \end{aligned}$$

which is a contradiction.

□

Proof of Theorem 1.2. We claim that there is a function $g \in L^1_{\rm loc}(f(\Omega))$ such that

$$\int_{f^{-1}(B)} |\operatorname{adj} \nabla f| = \int_B g. \tag{4.6}$$

This and Lemma 4.3 imply that the pair f, g satisfies a 1-Poincaré inequality in $f(\Omega)$. From [3, Theorem 9] we then deduce that $f^{-1} \in W^{1,1}_{\rm loc}(f(\Omega), \mathbb{R}^n)$.

There is a set $\Omega' \subset \Omega$ of full measure such that the area formula (2.1) holds for f on Ω'. We define a function $g\colon f(\Omega) \to \mathbb{R}$ by setting

$$g(f(x)) = \begin{cases} \frac{|\operatorname{adj} \nabla f(x)|}{J_f(x)} & \text{if } x \in \Omega' \text{ and } J_f(x) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since f is a mapping of finite distortion, we have

$$|\operatorname{adj} \nabla f(x)| = g(f(x))\, J_f(x) \text{ a.e. in } \Omega.$$

Hence for every $A \subset f(\Omega)$

$$\begin{aligned} \int_{f^{-1}(A)} |\operatorname{adj} \nabla f(x)|\, dx &= \int_{f^{-1}(A)\cap\Omega'} g(f(x))\, J_f(x)\, dx \\ &= \int_A g(y)\, dy. \end{aligned} \tag{4.7}$$

For $A = B$ this gives (4.6) and for other sets A it also implies $g \in L^1_{\rm loc}$.

From Theorem 4.5 we now obtain that f^{-1} has finite distortion. $\square$

5. Examples

Example 5.1. We construct a $W^{1,2}$-homeomorphism of an open set $\Omega \subset \mathbb{R}^2$ into $\mathbb{R}^3$ which does not have the Luzin N-property. We follow an idea due to Reshetnyak [21]. We use a conformal mapping g of the square $(0,1)\times(0,1)$ onto a Jordan domain U with $|\partial U| > 0$. Then g extends to a homeomorphism (labeled again as g) of $[0,1]\times[0,1]$ onto $\overline{U}$. There is at least one side L of $[0,1]\times[0,1]$, say $L = \{0\}\times[0,1]$, such that $|g(L)| > 0$. On $\Omega := (-1,1)\times(0,1)$ we define the $W^{1,2}$-mapping f as

$$f(x_1, x_2) = \big(g_1(|x_1|, x_2),\ g_2(|x_1|, x_2),\ x_1\big).$$

Example 5.2. *There is a homeomorphism of finite distortion $f \in W^{1,n-1}_{\rm loc}((-1,1)^n, \mathbb{R}^n)$ such that both f and f^{-1} are nowhere differentiable.*

Proof. Let $U = (-1,1)^{n-1}$. There exists a nowhere differentiable continuous function $\varphi \in W^{1,n-1}(U)$. For the convenience of the reader we indicate the construction. Recall that given a point z and a small number δ, there is a smooth function η such that $0 < \eta \le 1$, $\eta(z) = 1$, the support of η is in $B(z,\delta)$ and the $W^{1,n-1}$-norm of η is less than δ. By a linear combination of such functions, for given parameters $a, r > 0$, we can construct a Lipschitz function $\varphi_{a,r}$ with the following properties:

$$\|\varphi_{a,r}\|_{W^{1,n-1}} < a, \quad -a \le \varphi \le a$$

and for each $x \in U$ there exists $y \in U$ such that

$$|y - x| < r \text{ and } |\varphi_{a,r}(y) - \varphi_{a,r}(x)| \ge a. \tag{5.1}$$

Now, we define

$$\varphi = \sum_{k=1}^{\infty} \varphi_k,$$

where $\varphi_k = \varphi_{a_k,r_k}$, $a_k = 5^{-k}$ and $r_k < 2^{-k}a_k$ is chosen in such a way that

$$\sum_{j<k} \beta_j < \frac{a_k}{4r_k}, \tag{5.2}$$

where β_k is the Lipschitz constant of φ_k. Then obviously φ is continuous and of class $W^{1,n-1}$. If $x \in U$, then (5.1) and (5.2) give us that for each k there exists $y_k \in U$ such that

$$\begin{aligned} \sum_{j<k} |\varphi_j(y_k) - \varphi_j(x)| &\le \sum_{j<k} \beta_j |y_k - x| \le \frac{a_k}{4}, \\ |\varphi_k(y_k) - \varphi_k(x)| &\ge a_k, \\ \sum_{j>k} |\varphi_j(y_k) - \varphi_j(x)| &\le 2 \sum_{j>k} 5^{k-j} a_k = \frac{a_k}{2}, \end{aligned}$$

and thus

$$|\varphi(x) - \varphi(y_k)| \ge \frac{a_k}{4} \ge 2^{k-2} r_k \ge 2^{k-2} |x - y_k|.$$

It follows that φ is not differentiable at x.

We set

$$f(x_1, \dots, x_n) = \big(x_1, \dots, x_{n-1}, x_n + \varphi(x_1, \dots, x_{n-1})\big).$$

It is easy to check that f is a homeomorphism in $W^{1,n-1}$ which is nowhere differentiable. Analogously we obtain that

$$f^{-1}(y_1, \dots, y_n) = \big(y_1, \dots, y_{n-1}, y_n - \varphi(y_1, \dots, y_{n-1})\big)$$

is nowhere differentiable. □

References

[1] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer-Verlag, New York, 1969 (Second edition 1996).

[2] I. Fonseca and W. Gangbo, *Degree Theory in Analysis and Applications*, Clarendon Press, Oxford, 1995.

[3] B. Franchi, P. Hajlasz and P. Koskela, Definitions of Sobolev classes on metric spaces, *Ann. Inst. Fourier* **49** (1999), no. 6, 1903–1924.

[4] F. W. Gehring, Rings and quasiconformal mappings in space, *Proc. Nat. Acad. Sci U.S.A* **47** (1961), 98–105.

[5] M. Giaquinta, G. Modica and J. Souček, Area and the area formula, *Rend. Sem. Mat. Fis. Milano* **62** (1992), 53–87.

[6] W. Gustin, Boxing inequalities, *J. Math. Mech.* **9** (1960), 229–239.

[7] P. Hajłasz, Change of variables formula under minimal assumptions, *Colloq. Math.* **64** (1993), no. 1, 93–101.

[8] J. Heinonen and P. Koskela, Sobolev mappings with integrable dilatations, *Arch. Rational Mech. Anal.* **125** (1993), no. 1, 81–97.

[9] S. Hencl and P. Koskela, Regularity of the inverse of a planar Sobolev homeomorphism, *Arch. Rational Mech. Anal* **180** (2006), 75–95.

[10] S. Hencl, P. Koskela and J. Malý, Regularity of the inverse of a Sobolev homeomorphism in space, *Proc. Roy. Soc. Edinburgh Sect. A* **136** (2006), no. 6, 1267–1285.

[11] S. Hencl, P. Koskela and J. Onninen, *Homeomorphisms of bounded variation*, to appear in Arch. Rational Mech. Anal.

[12] J. Kauhanen, P. Koskela and J. Malý, On functions with derivatives in a Lorentz space, *Manuscripta Math.* **100** (1999), no. 1, 87–101.

[13] J. Kauhanen, P. Koskela and J. Malý, Mappings of finite distortion: Condition N, *Michigan Math. J.* **49** (2001), 169–181.

[14] J. Malý and O. Martio, Lusin's condition (N) and mappings of the class $W^{1,n}$, *J. Reine Angew. Math.* **458** (1995), 19–36.

[15] M. Marcus and V. Mizel, Transformations by functions in Sobolev spaces and lower semicontinuity for parametric variational problems, *Bull. Amer. Math. Soc.* **79** (1973), 790–795.

[16] M. Jr. Miranda, Functions of bounded variation on "good" metric spaces, *J. Math. Pures Appl.* **82** (2003), 975–1004.

[17] J. Malý, D. Swanson and W. P. Ziemer, Coarea formula for Sobolev mappings, *Trans. Amer. Math. Soc.* **355** (2003), no. 2, 477–492.

[18] J. Onninen, Regularity of the inverse of spatial mappings with finite distotion, *Calc. Var. Partial Differential Equations* **26** (2006), no. 3, 331–341.

[19] S. Ponomarev, Examples of homeomorphisms in the class $ACTL^p$ which do not satisfy the absolute continuity condition of Banach (Russian), *Dokl. Akad. Nauk USSR* **201** (1971), 1053-1054.

[20] Yu. G. Reshetnyak, Some geometrical properties of functions and mappings with generalized derivatives (Russian), *Sibirsk. Math. Zh.* **7** (1966), 886–919.

[21] Yu. G. Reshetnyak, The condition (N) for $W^{1,n}_{\mathrm{loc}}$ space mappings (Russian), *Sibirsk. Math. Zh.* **28** (1987), 149–153.

[22] J. Serrin, On the differentiability of functions of several variables, *Arch. Rational Mech. Anal* **7** (1961), 359-372.

[23] W. P. Ziemer, Change of variables for absolutely continuous functions, *Duke Math. J.* **36** (1969), no. 1, 171–178.

[24] W. P. Ziemer, *Weakly differentiable functions. Sobolev spaces and functions of bounded variation*, Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989.

Department of Mathematics, University College London, Gower Street, London WC1E 6BT, United Kingdom

E-mail address: `mari@math.ucl.ac.uk`

Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic

E-mail address: `hencl@karlin.mff.cuni.cz`

Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic, and
Department of Mathematics, J. E. Purkyně University, České mládeže 8, 400 96 Ústí nad Labem, Czech Republic
E-mail address: maly@karlin.mff.cuni.cz