Stochastic modeling in economics and finance

Test of hypotheses with panel data

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Introduction I

Restricted model: represents a behavioral equation with the same parameters over time and across individuals.

Unrestricted model: is the same behavioral equation but with different parameters across time or individuals.

Note: we restrict to test poolability of the data for the case of pooling across individuals (pooling over time can be obtained in a similar fasion).

The restricted panel data regression model:

$$y_{it} = \alpha + X_{it}^T \beta + u_{it} \qquad i = 1, \ldots, N, t = 1, \ldots, T,$$
$$u_{it} = \mu_i + \nu_{it} \qquad i = 1, \ldots, N, t = 1, \ldots, T.$$

The unrestricted panel data regression model:

$$y_{it} = \alpha_i + X_{it}^T \beta_i + u_{it} \qquad i = 1, \ldots, N, t = 1, \ldots, T,$$
$$u_{it} = \mu_i + \nu_{it} \qquad i = 1, \ldots, N, t = 1, \ldots, T.$$

Introduction II

Why do we pool data?

- Pooling lead to widen database, and therefore we can obtain better and more reliable estimates of the parameters.
- Panel data allows to study individual and time effects.
- Panel data models are popular in applied economics (they allow to control for individual heterogeneity).

Vector formulation of restricted and unrestricted model I

The unrestricted panel data regression model:

$$y_i = \alpha_i + X_i\beta_i + u_i = Z_i\delta_i + u_i, \qquad i = 1, \ldots, N,$$

where $y_i^T = (y_{i1}, \ldots, y_{iT})$, $Z_i = (\iota_T, X_i)$ and $\delta_i = (\alpha_i, \beta_i)$.

$y_i : T \times 1, X_i : T \times K, Z_i : T \times (K + 1), \delta_i : (K + 1) \times 1, u_i : T \times 1.$

The restricted panel data regression model:

$$y = \alpha\iota_{NT} + X\beta + u = Z\delta + u,$$

where $Z^T = (Z_1^T, \ldots, Z_N^T)$ and $u^T = (u_1^T, \ldots, u_N^T)$.

$y : NT \times 1, X : NT \times K, Z : NT \times (K + 1), \delta : (K + 1) \times 1, u : NT \times 1.$

We want to test the hypothesis $H_0 : \delta_i = \delta$ for all $i = 1, \ldots, N$.

Vector formulation of restricted and unrestricted model II

The unrestricted model can be reformulated as:

$$
y = \begin{pmatrix} Z_1 & 0 & \cdots & 0 \\ 0 & Z_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Z_N \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_N \end{pmatrix} + u = Z^* \delta^* + u.
$$

In this case $Z = Z^* I^*$ with $I^* = (\iota_N \otimes I_{K'})$, $K' = K + 1$.

$y : NT \times 1, Z^* : NT \times N(K+1), \delta^* : N(K+1) \times 1, u : NT \times 1.$

We aim to compare restricted and unrestricted in the forms derived above:

$$
y = Z\delta + u
$$
$$
y = Z^* \delta^* + u.
$$

Test for poolability under assumption $u \sim N(0, \sigma^2 I_{NT})$ I

For the restricted model under $u \sim N(0, \sigma^2 I_{NT})$ the minimum variance unbiased (MVU) estimator for δ is:

$$\hat{\delta}_{ols} = \hat{\delta}_{mle} = (Z^T Z)^{-1} Z^T y$$

and therefore

$$y = Z\hat{\delta}_{ols} + e$$
$$e = y - Z\hat{\delta}_{ols} = (I_{NT} - Z(Z^T Z)^{-1} Z^T)y = My = M(Z\delta + u) = Mu.$$

For the unrestricted model under $u \sim N(0, \sigma^2 I_{NT})$ MVU estimator for δ_i is:

$$\hat{\delta}_{i,ols} = \hat{\delta}_{i,mle} = (Z_i^T Z_i)^{-1} Z_i^T y_i$$

and therefore

$$y_i = Z_i\hat{\delta}_{i,ols} + e_i$$
$$e_i = y_i - Z_i\hat{\delta}_{i,ols} = (I_T - Z_i(Z_i^T Z_i)^{-1} Z_i^T)y_i = M_i y_i = M_i(Z_i\delta_i + u_i) = M_i u_i.$$

For the unrestricted model given as $y = Z^* \delta^* + u$ under $u \sim N(0, \sigma^2 I_{NT})$ MVU estimator for δ_i is:
$$\hat{\delta}^*_{ols} = \hat{\delta}^*_{mle} = (Z^{*T} Z^*)^{-1} Z^{*T} y$$

and therefore

$$y = Z^* \hat{\delta}^*_{ols} + e^*$$
$$e^* = y - Z^* \hat{\delta}^*_{ols} = (I_{NT} - Z^*(Z^{*T} Z^*)^{-1} Z^{*T}) y = M^* y = M^*(Z^* \delta^* + u) = M^* u.$$

It can be shown that

$$M^* = \begin{pmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_N \end{pmatrix},$$

M and M^* are idempotent and symmetric matrices with $M M^* = M^*$.

Test for poolability under assumption $u \sim N(0, \sigma^2 I_{NT})$ III

Chow test extended to the case of *N* linear regressions (Baltagi (2005))

Under $H_0 : \delta_i = \delta$ for $i = 1, \ldots, N$ and $u \sim N(0, \sigma^2 I_{NT})$, the statistic F_{obs} given as

$$F_{obs} = \frac{(e^T e - e^{*T} e^*)/(\text{tr}(M) - \text{tr}(M^*))}{e^{*T} e^*/\text{tr}(M^*)} = \frac{(e^T e - e_1^T e_1 - \ldots - e_N^T e_n)/(K^{'}(N-1))}{(e_1^T e_1 + \ldots + e_N^t e_n)/N(T - K^{'})}$$

is distributed as an $F((N-1)K^{'}, N(T - K^{'})$. Hence the critical region for this test is defined as $\{F_{obs} > F((N-1)K^{'}, N(T - K^{'}; \alpha_0))\}$ where α_0 denotes the level if significance of the test.

Proof. Using properties of matrices M and M^* one can easily derive:

$$e^T e - e^{*T} e^* = (Mu)^T(Mu) - (M^* u)^T(M^* u) = u^T M u - u^T M^* u = u^T(M - M^*)u,$$
$$e^{*T} e^* = u^T M^* u.$$

Since M^* and $(M - M^*)$ are idempotent and $u \sim N(0, \sigma^2 I_{NT})$, $u^T(M - M^*)u/\sigma^2$ follows χ^2 distribution with $\text{tr}(M - M^*)$ degrees of freedom, and similarly, $u^T M^* u/\sigma^2$ follows χ^2 distribution with $\text{tr}(M^*)$ degrees of freedom.

Matrices M and M^* are idempotent and therefore:

$$\text{tr}(M) = \text{r}(M) = NT - K^{'},$$
$$\text{tr}(M^*) = \text{r}(M^*) = NT - K^{'} N = N(T - K^{'}),$$
$$\text{tr}(M - M^*) = \text{tr}(M) - \text{tr}(M^*) = K^{'}(N - 1).$$

To finish the proof it remains to note that $u^T(M - M^*)u$ and $u^T M^* u$ are independent variable, $(M - M^*)M^* = 0$. Statistic F_{obs}, as the ration of two independent random variables with χ^2 distribution both divided by their degrees of freedom, has to follow F distribution.

Test for poolability under assumption $u \sim N(0, \Omega)$ I

In the general case $u \sim N(0, \Omega)$ one seeks for a suitable transformation of the model so as the Chow test can be applied. Namely, consider that $\Omega = \sigma^2 \Sigma$ and multiply restricted as well as unrestricted models by $\Sigma^{-1/2}$, then we get:

$$\bar{y} = \bar{Z}\delta + \bar{u},$$
$$\bar{y} = \bar{Z}^*\delta^* + \bar{u},$$

where $\bar{y} = \Sigma^{-1/2}y$, $\bar{Z} = \Sigma^{-1/2}Z$, $\bar{u} = \Sigma^{-1/2}u$ and $\bar{Z}^* = \Sigma^{-1/2}Z^*$. In this case

$$\mathbb{E}(\bar{u}\bar{u}^T) = \mathbb{E}(\Sigma^{-1/2}uu^T\Sigma^{-1/2T}) =' \text{Sigma}^{-1/2}\mathbb{E}(uu^T)\Sigma^{-1/2T} = \sigma^2 I_{NT}.$$

For the restricted and unrestricted reformulated models we gain have:

$$\hat{\bar{\delta}}_{ols} = (\bar{Z}^T\bar{Z})^{-1}\bar{Z}^T\bar{y}, \quad \bar{e} = \bar{y} - \bar{Z}\hat{\bar{\delta}}_{ols}, \quad \bar{e} = \bar{M}\bar{y} = \bar{M}\bar{u},$$
$$\hat{\bar{\delta}}^*_{ols} = (\bar{Z}^{*T}\bar{Z}^*)^{-1}\bar{Z}^{*T}\bar{y}, \quad \bar{e}^* = \bar{y} - \bar{Z}^*\hat{\bar{\delta}}^*_{ols}, \quad \bar{e}^* = \bar{M}^*\bar{y} = \bar{M}^*\bar{u}.$$

Test for poolability under assumption $u \sim N(0, \Omega)$ II

In order to apply Chow test, we need to verify:

- $\bar{Z} = \bar{Z}^* I^*$,
- $\bar{M}$, $\bar{M}^*$ are symmetric and idempotent and $\bar{M}\bar{M}^* = \bar{M}^*$, where $\bar{M} = I_{NT} - \bar{Z}(\bar{Z}^T \bar{Z})^{-1}\bar{Z}^T$ and $\bar{M}^* = I_{NT} - \bar{Z}^*(\bar{Z}^{*T}\bar{Z}^*)^{-1}\bar{Z}^{*T}$.

Roy-Zellner test for poolability (Baltagi (2005))

Under $H_0 : \delta_i = \delta$ for $i = 1, \ldots, N$ and $u \sim N(0, \Omega)$ the statistic F_{obs} given as

$$F_{obs} = \frac{(\bar{e}^T \bar{e} - \bar{e}^{*T} \bar{e}^*)/(\mathrm{tr}(\bar{M}) - \mathrm{tr}(\bar{M}^*)}{\bar{e}^{*T} \bar{e}^*/\mathrm{tr}(\bar{M}^*)}$$

is distributed as an $F((N - 1)K^{'}, N(T - K^{'})$. Hence the critical region for this test is defined as $\{F_{obs} > F((N - 1)K^{'}, N(T - K^{'}; \alpha_0)\}$ where α_0 denotes the level if significance of the test.

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Introduction I

All presented tests are dedicated for two-way error component model given as:

$$y_{it} = \alpha + X_{it}^T \beta + u_{it} \qquad\qquad i = 1, \ldots, N, t = 1, \ldots, T,$$
$$u_{it} = \mu_i + \lambda_t + \nu_{it} \qquad\qquad i = 1, \ldots, N, t = 1, \ldots, T,$$

for which $\mu_i \sim IID(0, \sigma_\mu^2)$, $\lambda \sim IID(0, \sigma_\lambda^2)$ and $\nu_{it} \sim IID(0, \sigma_\nu^2)$.

We want to test the hypotheses:

$$H_0^a : \sigma_\mu^2 = 0,$$
$$H_0^b : \sigma_\lambda^2 = 0,$$
$$H_0^c : \sigma_\mu^2 = \sigma_\lambda^2 = 0$$
$$H_0^d : \sigma_\mu^2 = 0 | \sigma_\lambda^2 > 0$$
$$H_0^e : \sigma_\lambda^2 = 0 | \sigma_\mu^2 > 0.$$

Introduction II

All presented test statistics stem from the likelihood function, which under assumption of normality has the following form:

$$L(\delta, \theta) = \text{constant} - \frac{1}{2} \log |\Omega| - \frac{1}{2} u^T \Omega^{-1} u,$$

where $\theta^T = (\sigma_\mu^2, \sigma_\lambda^2, \sigma_\nu^2)$ and Ω is given as

$$\Omega = \sigma_\mu^2 (I_N \otimes J_T) + \sigma_\lambda^2 (J_N \otimes I_T) + \sigma_\nu^2 I_{NT}.$$

Breusch and Pagan (1980) derived a Lagrange multiplier (*LM*) statistic to test $H_0^c : \sigma^\mu = \sigma_\lambda^2 = 0$ based on the Fisher score and the Fisher information matrix.

In the following text we denote as $\tilde{\theta}_{mle}$ MLE of θ under H_0^c, similarly $\tilde{\Omega}$ MLE of Ω under H_0^c. Note that $\tilde{\Omega} = \tilde{\sigma}_\nu^2 I_{NT}$ where $\tilde{\sigma}_\nu^2 = \tilde{u}^T \tilde{u} / NT$ and $\tilde{u}$ are the OLS residuals.

Introduction III

Derivation of the Fisher score and the Fisher information matrix. It has been shown that

$$\frac{\partial L}{\partial \theta_r} = -\frac{1}{2}\text{tr}(\Omega^{-1}(\partial\Omega/\partial\theta_r)) + \frac{1}{2}(u^T\Omega^{-1}(\partial\Omega/\partial\theta_r)\Omega^{-1}u).$$

In our specific case:

$$\partial\Omega/\partial\theta_1 = (I_N \otimes J_T), \qquad \partial\Omega/\partial\theta_2 = (J_N \otimes I_T), \qquad \partial\Omega/\partial\theta_3 = I_{NT}.$$

Using $\text{tr}(I_N \otimes J_T) = \text{tr}(J_N \otimes I_T) = \text{tr}(I_{NT}) = NT$, one gets:

$$D(\tilde{\theta}) = \left[\frac{\partial L}{\partial \theta}\right]_{\tilde{\theta}_{mle}} = -\frac{NT}{2\tilde{\sigma}_\nu^2}\left(\begin{array}{c} 1 - \frac{\tilde{u}^T(I_N \otimes J_T)\tilde{u}}{\tilde{u}^T\tilde{u}} \\ 1 - \frac{\tilde{u}^T(J_N \otimes I_T)\tilde{u}}{\tilde{u}^T\tilde{u}} \\ 0 \end{array}\right).$$

Introduction IV

In order to calculate the information matrix for this model we need

$$\mathbb{E}\left(\frac{\partial^2 L}{\partial\theta_s\partial\theta_r}\right) = -\frac{1}{2}\mathrm{tr}\left(\Omega^{-1}(\partial\Omega/\partial\theta_r)\Omega^{-1}(\partial\Omega/\partial\theta_s)\right)$$

Using $\mathrm{tr}((I_N \otimes J_T)(J_N \otimes I_T)) = \mathrm{tr}(J_{NT}) = NT$, $\mathrm{tr}(I_N \otimes J_T)^2 = NT^2$ and $\mathrm{tr}(J_N \otimes I_T)^2 = N^2T$, the information matrix for this model is:

$$J(\tilde{\theta}) = \mathbb{E}\left[-\frac{\partial^2 L}{\partial\theta_r\partial\theta_s}\right]_{\tilde{\theta}_{mle}} = \frac{NT}{2\tilde{\sigma}_\nu^4}\left(\begin{array}{ccc} T & 1 & 1 \\ 1 & N & 1 \\ 1 & 1 & 1 \end{array}\right)$$

with

$$J^{-1}(\tilde{\theta}) = \frac{\tilde{\sigma}_\nu^4}{NT(N-1)(T-1)}\left(\begin{array}{ccc} (N-1) & 0 & (1-N) \\ 0 & (T-1) & (1-T) \\ (1-N) & (1-T) & (NT-1) \end{array}\right).$$

The Breusch-Pagan test I

1. The Breusch-Pagan test (Breusch and Pagan (1980))

Under $H_0^c : \sigma_\mu^2 = \sigma_\lambda^2 = 0$, the Breusch-Pagan test statistic LM given as:

$$LM = \frac{NT}{2(T-1)} \left(1 - \frac{\tilde{u}^T (I_N \otimes J_T) \tilde{u}}{\tilde{u}^T \tilde{u}} \right)^2 + \frac{NT}{2(N-1)} \left(1 - \frac{\tilde{u}^T (J_N \otimes I_T) \tilde{u}}{\tilde{u}^T \tilde{u}} \right)^2$$

is asymptotically distributed as χ_2^2.

Proof. The statistic can be obtain as: $LM = \tilde{D}^T \tilde{J}^{-1} \tilde{D}$.

The Breusch-Pagan test II

Notes on the Breusch-Pagan test:

- The test is very popular (it requires only calculation of OLS residuals $\tilde{u}$.
- The test statistic LM_1,

$$
LM_1 = \frac{NT}{2(T-1)} \left(1 - \frac{\tilde{u}^T(I_N \otimes J_T)\tilde{u}}{\tilde{u}^T \tilde{u}} \right)^2
$$

  is asymptotically distributed as χ_1^2 and can be used to test $H_0^a : \sigma_\mu^2 = 0$.

- The test statistic LM_2,

$$
LM_2 = \frac{NT}{2(N-1)} \left(1 - \frac{\tilde{u}^T(J_N \otimes I_T)\tilde{u}}{\tilde{u}^T \tilde{u}} \right)^2
$$

  is asymptotically distributed as χ_1^2 and can be used to test $H_0^b : \sigma_\lambda^2 = 0$.

- Both test statistic LM_1 and LM_2 can be applied on condition $\sigma_\lambda^2 = 0$ and $\sigma_\mu^2 = 0$.

Other test for individual and time effects I

The problem with the Breusch-Pagan test is that it assumes that the alternative hypothesis is two-sided, but variance are nonnegative, thus the alternative hypothesis should be one-sided.

The Honda tests (Honda (1985))

Under hypothesis $H_0^a : \sigma_\mu^2 = 0$, the Honda test statistic HO given as:

$$HO \equiv A = \sqrt{\frac{NT}{2(T-1)}} \left(1 - \frac{\tilde{u}^T (I_N \otimes J_T) \tilde{u}}{\tilde{u}^T \tilde{u}} \right)$$

has the asymptotic normal distribution $N(0, 1)$.

Other test for individual and time effects II

Under hypothesis $H_0^b : \sigma_\lambda^2$, the Honda test statistic HO given as:

$$HO \equiv B = \sqrt{\frac{NT}{2(N-1)}} \left(1 - \frac{\tilde{u}^T (J_N \otimes I_T) \tilde{u}}{\tilde{u}^T \tilde{u}} \right)$$

has the asymptotic normal distribution $N(0, 1)$.

Under hypothesis $H_0^c : \sigma_\mu^2 = \sigma_\lambda^2 = 0$, the Honda test statistic HO given as:

$$HO = (A + B)/\sqrt{2}$$

has the asymptotic normal distribution $N(0, 1)$.

Other test for individual and time effects III

King and Wu (1997) suggested the alternative test statistic to testing $H_0^c : \sigma_\mu^2 = \sigma_\lambda^2 = 0$.

The King and Wu test (King and Wu (1997))

Under hypothesis $H_0^c : \sigma_\mu^2 = \sigma_\lambda^2 = 0$, the King and Wu test statistic KW given as:

$$KW = \frac{\sqrt{T-1}}{\sqrt{N+T-2}}A + \frac{\sqrt{N-1}}{\sqrt{N+T-2}}B$$

has the asymptotic normal distribution $N(0,1)$.

Other test for individual and time effects IV

Moulton and Randolph (1989) suggested an alternative standardized Lagrange multiplier test, because they revealed poor performance of Honda tests (especially if the number of regressors is high).

The standardized Lagrange multiplier test (Moulton and Randolph (1989))

Under hypothesis $H_0^a : \sigma_\mu^2 = 0$ or $H_0^b : \sigma_\lambda^2 = 0$, the standardized Lagrange multiplier test statistic SLM given as:

$$SLM = \frac{HO - \mathbb{E}(HO)}{\sqrt{\text{var}(HO)}}$$

has the asymptotic normal distribution $N(0, 1)$.

Other test for individual and time effects V

Gourieroux, Holly, and Monfort (1982) note that A and B can be negative for a specific application and suggest corrected test for testing $H_0^c : \sigma_\mu^2 = \sigma_\lambda^2 = 0$.

The Gourieroux, Holly and Monfort test (Gourieroux, Holly, and Monfort (1982))

Under hypothesis $H_0^c : \sigma_\mu^2 = \sigma_\lambda^2 = 0$, the Gourieroux, Holly and Monfort test statistic *GLM* is given as:

$$\chi_m^2 = \begin{cases} A^2 + B^2 & \text{if } A > 0, B > 0 \\ A^2 & \text{if } A > 0, B \leq 0 \\ B^2 & \text{if } A \leq 0, B > 0 \\ 0 & \text{if } A \leq 0, B \leq 0 \end{cases}$$

χ_m^2 denotes the mixed χ^2 distribution. Under the null hypothesis,

$$\chi_m^2 \sim \frac{1}{4}\chi^2(0) + \frac{1}{2}\chi^2(1) + \frac{1}{4}\chi^2(2),$$

where $\chi^2(0)$ equals 0 with probability one.

Other test for individual and time effects VI

When using above tests for $H_0^a : \sigma_\mu^2 = 0$, one implicitly assumes that $\sigma_\lambda^2 = 0$. This may lead to incorrect decisions especially when the variance σ_λ^2 is large.

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the conditional *LM* tests (Baltagi and Li (1992))

Under hypothesis $H_0^d : \sigma_\mu^2 = 0$ (allowing $\sigma_\lambda^2 > 0$), the conditional *LM* test given as:

$$LM_\mu = \frac{\sqrt{2}\tilde{\sigma}_2^2\tilde{\sigma}_\nu^2}{\sqrt{T(T-1)(\tilde{\sigma}_\nu^4 + (N-1)\tilde{\sigma}_2^4)}}\tilde{D}_\mu,$$

where

$$\tilde{D}_\mu = \frac{T}{2\tilde{\sigma}_2^2}\left(\frac{\tilde{u}^T(\bar{J}_N \otimes \bar{J}_T)\tilde{u}}{\tilde{\sigma}_2^2} - 1\right) + \frac{T(N-1)}{2\tilde{\sigma}_\nu^2}\left(\frac{\tilde{u}^T(E_N \otimes \bar{J}_T)\tilde{u}}{(N-1)\tilde{\sigma}_\nu^2} - 1\right)$$

with $\tilde{\sigma}_2^2 = \tilde{u}^T(\bar{J}_N \otimes I_T)\tilde{u}/T$ and $\tilde{\sigma}_\nu^2 = \tilde{u}^T(E_N \otimes I_T)\tilde{u}/T(N-1)$, is asymptotically distributed as $N(0,1)$.

Other test for individual and time effects VII

Under hypothesis $H_0^e : \sigma_\lambda^2 = 0$ (allowing $\sigma_\mu^2 > 0$), the conditional *LM* test given as:

$$LM_\lambda = \frac{\sqrt{2}\tilde{\sigma}_1^2\tilde{\sigma}_\nu^2}{\sqrt{N(N-1)(\tilde{\sigma}_\nu^4 + (T-1)\tilde{\sigma}_1^4)}}\tilde{D}_\lambda,$$

where

$$\tilde{D}_\lambda = \frac{N}{2\tilde{\sigma}_1^2}\left(\frac{\tilde{u}^T(\bar{J}_N \otimes \bar{J}_T)\tilde{u}}{\tilde{\sigma}_1^2} - 1\right) + \frac{N(T-1)}{2\tilde{\sigma}_\nu^2}\left(\frac{\tilde{u}^T(\bar{J}_N \otimes E_T)\tilde{u}}{(T-1)\tilde{\sigma}_\nu^2} - 1\right)$$

with $\tilde{\sigma}_1^2 = \tilde{u}^T(I_N \otimes \bar{J}_T)\tilde{u}/N$ and $\tilde{\sigma}_\nu^2 = \tilde{u}^T(I_N \otimes E_T)\tilde{u}/N(T-1)$, is asymptotically distributed as $N(0, 1)$.

Other test for individual and time effects VIII

ANOVA F tests can be used as universal tests for testing each of the considered hypotheses.

The ANOVA F tests

The ANOVA F test statistics have the following form:

$$F = \frac{y^T MD(D^T MD)^{-1}D^T My/(p-r)}{y^T Gy/(NT-(\tilde{k}+p-r))},$$

where $M = Z(Z^T Z)^{-1}Z^T$ and $G, D, \tilde{k}, p, r$ are chosen depending on hypothesis tested. Under the null hypothesis, this statistic has a F distribution with $p - r$ and $NT - ((k + \tilde{p} - r)$ degrees of freedom.

Other test for individual and time effects IX

The likelihood ratio tests can be used as universal tests for testing each of the considered hypotheses.

The likelihood ratio tests

The one-sided likelihood ratio LR tests have the following form:

$$LR = -2 \log \left(\frac{l(res)}{l(unres)} \right)$$

where $l(res)$ denotes the restricted maximum likelihood value (under the null hypothesis), while $l(unres)$ denotes the unrestricted maximum likelihood value. For H_0^a, H_0^b, H_0^d and H_0^e, LR is distributed as $\frac{1}{2}\chi^2(0) + \frac{1}{2}\chi^2(1)$ and for H_0^c as $\frac{1}{4}\chi^2(0) + \frac{1}{2}\chi^2(1) + \frac{1}{4}\chi^2(2)$.

Comparison of the tests I

Baltagi and Li (1992) carried out a Monte Carlo simulation in order to compare performance of presented tests on two-way error component model.

They obtained the following results:

- When $H_0^a : \sigma_\mu^2 = 0$ is true and σ_λ^2 is large, all usual tests (*BP*, *HO*, *KW*, *SLM*, *GMH*) preformed badly (they ignore the fact that $\sigma_\lambda^2 > 0$).

- When $\sigma_\mu^2 >> 0$ all tests performed well in rejecting H_0^a, but for small σ_μ^2 the power of the tests decreases as σ_λ^2 increases.

- when testing $H_0^d : \sigma_\mu^2 = 0 | \sigma_\lambda^2 > 0$, LM_μ, *LR* and *F* performed well. Moreover, the power of the tests increases as σ_λ^2 increases.

- Overspecifying the model, i. e. assuming the model to be two-way error component when it is one-way, does not hurt the power of tests LM_μ, *LR* or *F*. Therefore, one should not ignore $\sigma_\lambda^2 > 0$ when testing $\sigma_\mu^2 = 0$.

- When testing $H_0^c : \sigma_\mu^2 = \sigma_\lambda^2 = 0$ all tests are possible, but *GHM* and *F* are recommended.

Comparison of the tests II

	H_0^a $\sigma_\mu^2 = 0$	H_0^b $\sigma_\lambda^2 = 0$	H_0^c $\sigma_\mu^2 = \sigma_\lambda^2 = 0$	H_0^d $\sigma_\mu^2 = 0 \mid \sigma_\lambda^2 > 0$	H_0^e $\sigma_\lambda^2 = 0 \mid \sigma_\mu^2 > 0$
BP	✓	✓	✓	-	-
HO	✓	✓	✓	-	-
KW	✓	✓	✓	-	-
SLM	✓	✓	-	-	-
GHM	-	-	✓	-	-
F	✓	✓	✓	✓	✓
LR	✓	✓	✓	✓	✓
LM_μ	-	-	-	✓	-
LM_λ	-	-	-	-	✓

Table: Suitability of the tests

How to proceed when seeking for the most proper model:

STEP 1 Testing $H_0^c : \sigma_\mu^2 = \sigma_\lambda^2 = 0$ by *GHM*. If H_0^c is not rejected, then use OLS. If H_0^c is rejected, continue to STEP 2.

STEP 2 Calculate LM_μ and LM_λ to test H_0^d and H_0^e. Depending on the results apply one-way or two-way error component model to data.

Contents

Introduction I

Firstly, we will consider random one-way error component model whose critical assumption is that $\mathbb{E}(u_{it}|X_{it}) = 0$.

We intend to test the hypothesis $H_0 : \mathbb{E}(u_{it}|X_{it}) = 0$.

All test presented in this section are based on comparing estimates $\hat{\beta}_{GLS}$, $\hat{\beta}_{within}$ and $\hat{\beta}_{between}$.

$\hat{\beta}_{GLS}$ is derived from the following model:

$$
\begin{aligned}
y_{it} &= \alpha + X_{it}^T \beta + u_{it} \qquad i = 1, \ldots, N, \ t = 1, \ldots, T, \\
\Omega &= \mathbb{E}(u^T u), \\
\Omega^{-1/2} y &= \Omega^{-1/2} \alpha \iota_{NT} + \Omega^{-1/2} X \beta + \Omega^{-1/2} Z_\mu \mu + \Omega^{-1/2} \nu \\
\hat{\beta}_{GLS} &= (X^T \Omega^{-1} X)^{-1} X^T \Omega^{-1} y.
\end{aligned}
$$

$\hat{\beta}_{within}$ is derived from the following model:

$$
\begin{aligned}
y_{it} - \bar{y}_{i\cdot} &= (X_{it} - \bar{X}_{i\cdot})^T \beta + (\nu_{it} - \bar{\nu}_{i\cdot}) \qquad i = 1, \ldots, N, \ t = 1, \ldots, T, \\
Q &= (I_N \otimes E_T), \\
Qy &= QX\beta + Q\nu \\
\hat{\beta}_{within} &= (X^T Q X)^{-1} X^T Q y.
\end{aligned}
$$

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Introduction II

$\hat{\beta}_{between}$ is derived from the following model:

$$\bar{y}_{i.} = \alpha + \bar{X}_{i.}^T \beta + \bar{u}_{i.} \qquad i = 1, \ldots, N,$$
$$P = (I_N \otimes \bar{J}_T),$$
$$Py = P\alpha\iota_{NT} + PX\beta + PZ_\mu\mu + P\nu,$$
$$\hat{\beta}_{between} = (X^T PX)^{-1} X^T Py.$$

In the case $\mathbb{E}(u_{it}|X_{it}) \neq 0$, the GLM estimator $\hat{\beta}_{GLM}$ becomes biased and inconsistent for β, whereas the Within transformation leaves the Within estimator $\hat{\beta}_{within}$ unbiased and consistent for β.

The test is based on the difference between $\hat{\beta}_{GLS}$ and $\hat{\beta}_{within}$:

$$\begin{aligned}
\hat{q}_1 &= \hat{\beta}_{GLS} - \hat{\beta}_{within} \\
&= (\hat{\beta}_{GLS} - \beta) - (\hat{\beta}_{within} - \beta) \\
&= (X^T\Omega^{-1}X)^{-1}X^T\Omega^{-1}u - (X^TQX)^{-1}X^TQu.
\end{aligned}$$

To derive the test statistic we need to calculated the mean and variance of $\hat{q}_1$. Obviously, $\mathbb{E}(\hat{q}_1) = 0$. In order to calculate the variance, we proceed as follows:

$$\begin{aligned}
\text{cov}(\hat{\beta}_{GLS}, \hat{q}_1) &= \text{cov}(\hat{\beta}_{GLS}, \hat{\beta}_{GLS} - \hat{\beta}_{within}) \\
&= \text{var}(\hat{\beta}_{GLS}) + \text{cov}(\hat{\beta}_{GLS}, \hat{\beta}_{within}) \\
&= (X^T\Omega^{-1}X)^{-1} - (X^T\Omega^{-1}X)^{-1}X^T\Omega^{-1}\mathbb{E}(uu^T)QX(X^TQX)^{-1} \\
&= (X^T\Omega^{-1}X)^{-1} - (X^T\Omega^{-1}X)^{-1} = 0
\end{aligned}$$

Hausman's specification test II

$$\hat{\beta}_{within} = \hat{\beta}_{GLS} - \hat{q}_1$$
$$\mathrm{var}(\hat{\beta}_{within}) = \mathrm{var}(\hat{\beta}_{GLS}) + \mathrm{var}(\hat{q}_1)$$
$$\mathrm{var}(\hat{q}_1) = \mathrm{var}(\hat{\beta}_{within}) - \mathrm{var}(\hat{\beta}_{GLS})$$
$$= \sigma_\nu^2 (X^T Q X)^{-1} - (X^T \Omega^{-1} X)^{-1}.$$

Hausman's specification test (Hausman (1978))

Under $H_0 : \mathbb{E}(u_{it}|X_{it}) = 0$, the Hausman's specification test statistic given as:

$$m_1 = \hat{q}_1^T (\mathrm{var}(\hat{q}_1))^{-1} \hat{q}_1$$

is asymptotically distributed as χ_K^2, where K denoted the dimension of slope vector β.

An alternative asymptotically equivalent test I

Consider the following regression:

$$\sigma_\nu \Omega^{-1/2} y = \sigma_\nu \Omega^{-1/2} X\beta + QX\gamma + \omega$$
$$y^* = X^*\beta + \tilde{X}\gamma + \omega.$$

Then the Hausman's test ($\hat{\beta}_{within} = \hat{\beta}_{GLS}$) is equivalent to test whether $\gamma = 0$. To the later one can apply standard Wald test for omission of variables $\tilde{X}$.

Performing OLS on the above stated mode, one gets the estimates:

$$\begin{pmatrix} \hat{\beta} \\ \hat{\gamma} \end{pmatrix} = \begin{pmatrix} X^T(Q + \phi^2 P)X & X^T QX \\ X^T QX & X^T QX \end{pmatrix}^{-1} \begin{pmatrix} X^T(Q + \phi^2 P)y \\ X^T Qy \end{pmatrix},$$

$$\hat{\beta} = \hat{\beta}_{between} = (X^T PX)^{-1} X^T Py$$

$$\hat{\gamma} = (X^T QX)^{-1} X^T Q\nu - (X^T PX)^{-1} XPu$$

$$= \hat{\beta}_{within} - \hat{\beta}_{between}.$$

where $\sigma_\nu \Omega^{-1/2} = Q + \phi P$ and $\phi = \sigma_\nu / \sigma_1$.

An alternative asymptotically equivalent test II

The alternative statistic is based on $\hat{q}_3 = \hat{\gamma} = \hat{\beta}_{within} - \hat{\beta}_{between}$ for which

$$\mathbb{E}(\hat{q}_3) = 0$$
$$\text{var}(\hat{q}_3) = \mathbb{E}(\hat{q}_3 \hat{q}_3^T) = \sigma_\nu^2 (X^T Q X)^{-1} + \sigma_1^2 (X^T P X)^{-1}$$
$$= \text{var}(\hat{\beta}_{within}) + \text{var}(\hat{\beta}_{between})$$

The alternative specification test

Under $H_0 : \mathbb{E}(u_{it}|X_{it}) = 0$, the alternative specification test statistic given as:

$$m_3 = \hat{q}_3^T (\text{var}(\hat{q}_3))^{-1} \hat{q}_3$$

is asymptotically distributed as χ_K^2, where K denoted the dimension of slope vector β.

The test m_1 and m_3 are numerically exactly identical and are also identical with the statistic $m_2 = \hat{q}_2^T (\text{var}(\hat{q}_2))^{-1} \hat{q}_2$, where $\hat{q}_2 = \hat{\beta}_{GLS} - \hat{\beta}_{between}$. This follows from the relationship between the estimators:

$$\hat{\beta}_{GLS} = W_1 \hat{\beta}_{within} + W_2 \hat{\beta}_{between}.$$

Hausman's test for the two-way error component model

The Hausman's test for the two-way error component model is based on difference between the fixed effects estimator (with both time and individual dummies) and the two-way random effects GLS estimator, i. e. the Within and GLS estimators.

The equivalent tests cannot be executed anymore, since there are two Between estimators. However there are other type of equivalences. Kang (1985) classifies five testable hypothesis, which consider between time periods estimator $\hat{\beta}_T$ and between cross section estimator $\hat{\beta}_C$:

- Assume μ_i fixed and test $\mathbb{E}(\lambda_t | X_{it}) = 0$ based upon $\hat{\beta}_{within} - \hat{\beta}_T$.
- Assume μ_i random and test $\mathbb{E}(\lambda_t | X_{it}) = 0$ based upon $\hat{\beta}_T - \hat{\beta}_{GLS}$.
- Assume λ_t fixed and test $\mathbb{E}(\mu_i | X_{it}) = 0$ based upon $\hat{\beta}_{within} - \hat{\beta}_C$.
- Assume λ_t random and test $\mathbb{E}(\mu_i | X_{it}) = 0$ based upon $\hat{\beta}_C - \hat{\beta}_{GLS}$.
- Test $\mathbb{E}(\mu_i | X_{it}) = \mathbb{E}(\lambda_t | X_{it}) = 0$ upon $\hat{\beta}_{GLS} - \hat{\beta}_{within}$, where $\hat{\beta}_{GLS}$ is the estimates assuming both μ_i and λ_t random and $\hat{\beta}_{within}$ is the estimates assuming both μ_i and λ_t fixed.

Contents

Grunfeld investment equation I

Grunfeld (1958) considered the following investment equation for 10 large US manufacturing firms over 20 years, 1935 - 1954:

$$I_{it} = \alpha + \beta_1 F_{it} + \beta_2 C_{it} + u_{it}, \qquad i = 1, \ldots, 10, \quad t = 1, \ldots, 20,$$

I_{it}: real gross investment for firm i in year t,
F_{it}: the real value of the firm (shares outstanding),
C_{it}: the real value of the capital stock

In order to find a proper model we need to answer the following questions:

- Can we use the restricted model, can we pool the data across firms or/and time?

- If we choose the restricted model, are there any individual and time effects and should we use one-way or two-way error component model?

- If we choose the restricted model with random individual or/and time effects, do disturbances u_{it} contain invariant effect which are unobservable and uncorrelated with explanatory variables?

Tests for poolability of the data

	RRSS	URSS	F_{obs}	Distribution	Quantile
Across firms	1 755 850.48	324 728.57	27.75	F(27, 170)	1.55
Across firms*	523 478.15	324 728.57	5.78	F(18, 170)	1.62
Across time	1 755 850.48	1 205 817.97	1.12	F(57, 140)	1.42
Across time*	523 478.15	1 205 817.97	-2.08	F(38, 140)	1.49

Table: Poolability of Grunfeld investment data across firms and time under assumption $u \sim N(0, \sigma^2 I_{NT})$, (* denotes poolbility allowing varying intercept).

	F_{obs}	Distribution	Quantile
Across firms	4.35	F(27, 170)	1.55
Across time	2.72	F(57, 140)	1.42

Table: Poolability of Grunfeld investment data across firms and time under assumption $u \sim N(0, \Omega)$.

Tests for individual and time effects

	H_0^a $\sigma_\mu^2 = 0$	H_0^b $\sigma_\lambda^2 = 0$	H_0^c $\sigma_\mu^2 = \sigma_\lambda^2 = 0$	H_0^d $\sigma_\mu^2 = 0 \mid \sigma_\lambda^2 > 0$	H_0^e $\sigma_\lambda^2 = 0 \mid \sigma_\mu^2 > 0$
BP	798.162	6.454	804.615	-	-
	(3.841)	(3.841)	(5.991)		
HO	28.252	-2.540	18.181	-	-
	(1.645)	(1.645)	(1.645)		
KW	28.252	-2.540	21.832	-	-
	(1.645)	(1.645)	(1.645)		
SLM	32.661	-2.433	-	-	-
	(1.645)	(1.645)			
GHM	-	-	798.162	-	-
			(4.231)		
F	49.177	0.235	17.403	52.672	1.142
	(1.930)	(1.645)	(1.543)	(1.648)	(1.935)
LR	193.091	0	193.108	193.108	0.017
	(2.706)	(2.706)	(4.231)	(2.706)	(2.706)
LM_μ	-	-	-	28.252	-
				(2.706)	
LM_λ	-	-	-	-	0.110
					(2.706)

Table: Tests for individual and time effects for Grunfeld investment data

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Hausman's specification test

m_1	m_2	m_3
2.3304	2.1314	2.1725
(5.9915)	(5.9915)	(5.9915)

Table: Hausman's test Grunfeld investment data modelled as one-way error component model (with individual effects).

Conclusion

Based on the tests executed we arrive at the following conclusions:

- The tests reject poolability across firms as well as poolability across time.
- *GHM* test rejects hypothesis $\sigma_\mu^2 = \sigma_\lambda^2 = 0$. *LM*$_\mu$ and *LM*$_\lambda$ tests revealed that $\sigma_\lambda^2 = 0$ and $\sigma_\mu^2 \neq 0$. Therefore if we decide to model Grunfeld data using panel data model, the model should be formulated as:

$$I_{it} = \alpha + \beta_1 F_{it} + \beta_2 C_{it} + \mu_i + \nu_{it}, \qquad i = 1, \ldots, 10, \quad t = 1, \ldots, 20,$$

  where μ_i are random effects.

- Hausman's test and its alternative do not reject the hypothesis $\mathbb{E}(u_{it}|X_{it})$ and for estimating the regression parameters we can use GLS estimator. Thus the coefficients of the model are:

$$\hat{\alpha} = -57.83$$
$$\hat{\beta}_1 = 0.11$$
$$\hat{\beta}_2 = 0.31$$

Thank you for your attention

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