Summary of 1. risk theory

Cramer-Lundberg model
- inter arrival times have exponential distribution with parameter λ
- iid. claims X_i with mean μ, df. F and positive support
- initial capital u and income rate c

+ risk proces

$$U(t) = u + c\ t - \sum_{i=1}^{N(t)} X_i$$

+ ruin probability

$$\psi(u) = P(U(t) < 0,\ \text{for some } t)$$

$$1 - \psi(u) = \left(1 - \frac{\mu\lambda}{c}\right) \sum_{n=0}^{\infty} \left(\frac{\mu\lambda}{c}\right)^n F_I^{n*}(u) \tag{1.11}$$

where $F_I^{n*}(x) = \int_0^x 1 - F^{n*}(y)dy/(n\ \mu)$
and F^{n*} denotes $n-$th convolution of df. F

Theorem 1.2.2 (Craner-Lundberg theorem)
for sufficiently light tails exists $\nu > 0$ such that:

$$\int_0^\infty e^{\nu x}(1 - F(x))dx = \frac{c}{\lambda}$$

ruin probability is exponentialy bounded:

$$\psi(u) \le e^{-\nu u}$$

and condition $\int_0^\infty x\ e^{\nu x}\overline{F}(x)dx < \infty$ yields approximation:

$$\lim_{u\to\infty} e^{\nu u}\psi(u) = \left[\frac{\nu}{c/\lambda - \mu} \int_0^\infty x\ e^{\nu x}(1 - F(x))dx\right]$$

small claim distribution: exponential, gamma, truncated normal
for exponential distribution:

$$\psi(u) = \frac{\mu\lambda}{c} \exp\left[-u\left(\frac{1}{\mu} - \frac{\lambda}{c}\right)\right]$$

proof of (1.11)
multiplying (1.20) by e^{-su} and integrating over $(0,\infty)$ yields on rhs:

$$\int_0^\infty \delta'(u)e^{-su}du = -\delta(0) + s\underbrace{\int_0^\infty \delta(u)e^{-su}du}_{\tilde{\delta}(s)}$$

and on lhs

$$\frac{\lambda}{c}\left(\int_0^\infty \delta(u)e^{-su}du - \int_0^\infty \int_0^u \delta(u-x)dF(x)e^{-su}du\right) =$$

$$= \frac{\lambda}{c}\left(\tilde{\delta}(s) - \int_0^\infty \int_x^\infty \delta(u-x)e^{-su}du\ dF(x)\right) =$$

$$= \frac{\lambda}{c}\left(\tilde{\delta}(s) - \int_0^\infty \int_0^\infty \delta(u)e^{-s(u+x)}du\ dF(x)\right) =$$

$$= \frac{\lambda}{c}\tilde{\delta}(s)\left(1 - \tilde{f}(s)\right)$$

equating rhs = lhs gives

$$-\delta(0) + s\tilde{\delta}(s) = \frac{\lambda}{c}\tilde{\delta}(s)\left(1 - \tilde{f}(s)\right)$$

which together with

$$(1-F\tilde{)}(s) = \frac{1-\tilde{f}(s)}{s}$$

and

$$f_I(x) = \frac{1 - F_X(x)}{\mu}$$

yields:

$$\tilde{\delta}(s) = \frac{1}{s}\underbrace{\frac{1-\mu\lambda/c}{1-\mu\lambda/c\ \tilde{f}_I(s)}}_{M_S(-s)}$$

where $M_S(s)$ stands for moment generating function of compound geometric process, with probability of success $1-\mu\lambda/c$ and claim size distribution F_I

uniqueness of laplace tranformation gives:

$$\delta(u) = \left(1 - \frac{\mu\lambda}{c}\right)\sum_{n=0}^{\infty}\left(\frac{\mu\lambda}{c}\right)^n F_I^{n*}(u)$$

detailed explanation in Risk Modelling in General Insurance: From Principles to Practice; Gray, Pitts

heavy-tailed distributions
+ reularly varying function $h \in \mathcal{R}_\alpha$

$$\lim_{x\to\infty} \frac{h(tx)}{h(x)} = t^\alpha$$

+ slowly varying function: $\alpha = 0$

Corollary 1.3.2
if $\overline{F}(x) = x^{-\alpha}L(x)$ for $\alpha \geq 0$ and $L \in \mathcal{R}_0$ then

$$\overline{F^{n*}}(x) \sim n\overline{F}(x), \qquad x \to \infty$$

which for maximum M_n from n iid. X_i and sum S_n:

$$P(M_n > x) \sim n\overline{F}(x), \qquad x \to \infty$$
$$P(S_n > x) = \overline{F^{n*}}(x)$$

says, that the tail of maximum determines tail of sum
and from the (1.11) follows

$$\frac{\psi(u)}{\overline{F_I}(u)} = \left(1 - \frac{\mu\lambda}{c}\right) \sum_{n=0}^{\infty} \left(\frac{\mu\lambda}{c}\right)^n \frac{\overline{F_I^{n*}}(u)}{\overline{F_I}(u)}$$

$$\to \left(1 - \frac{\mu\lambda}{c}\right) \sum_{n=0}^{\infty} \left(\frac{\mu\lambda}{c}\right)^n n = \frac{\mu\lambda}{c - \mu\lambda}, \qquad u \to \infty$$

which is natural ruin estimate whenever $\overline{F_I}$ is regularly varying
- **pareto**, burr, loggamma, truncated stable

Definition 1.3.3
subexponential df. $F \in \mathcal{S}$ if

$$\lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{\overline{F}} = n$$

Definition 1.3.3 Cramer-Lundberg theorem for large claims

$$\psi(u) \sim \frac{\mu\lambda}{c - \mu\lambda} \overline{F}_I(u), \qquad u \to \infty$$

for subexponential integrated tail distribution ($\overline{F}_I \in \mathcal{S}$)
- **lognormal**, benktander I and II, weibull ($\tau < 1$)