Alternative - martingale approach to Optimal investment

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1. Refresher from last year
   - Problem statement
   - Martingale Approach

2. Optimal Wealth Computation
   - Straightforward vs. Dual approaches
   - Power utility: Optimal Wealth and Comparison to HJB Approach

3. Optimal proportion
   - Martingale Representation Theorem and Proportion Computation
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4. Conclusion

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Generalities

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  Further:
- We require a standard bank account with a constant short rate \( r \).
- Of course, we require \( \sigma \) to be positive.
Basic Idea

- We consider an investor with initial capital $x$ and a utility function $U$.
- Any positive DISCOUNTED self–financing portfolio is denoted by $X \in \chi(x)$.
- Our aim is to maximize the expected utility
  \[
  \mathbb{E}^P[U(X_T)] \quad \text{such that} \quad X \in \chi(x).
  \]

- Last year, we viewed such a problem as DYNAMIC and attacked it by the classical Bellman approach.
- But it can be approached from a different side!
Define \( \kappa_T \) as a set of contingent claims at time \( T \), which can be replicated by \( X \in \chi(x) \).

Our basic problem is now different: it is a STATIC problem to maximize

\[
E^P[U(X_T)] \quad \text{such that } X_T \in \kappa_T.
\]

in this formulation, the focus is on the optimal terminal wealth. We have decoupled the problem and the optimal portfolio strategy must (at least in principle) now be computed using more accurate techniques.

We have already seen many times, that a discounted self-financing portfolio is a \( Q \)–martingale, where \( Q \) is the (unique) risk–neutral measure.
It is straightforward (and in one dimension easy) to argue, that we are actually in a situation:

$$\max \mathbb{E}^P[U(X_T)] \quad \text{such that} \quad \mathbb{E}^Q[X_T] = x.$$
Lagrange relaxation

The static problem

$$\max E^P[U(X_T)] \quad \text{such that } E^Q[X_T] = x.$$ 

can of course be easily reformulated as

$$\max E^P[U(X_T)] \quad \text{such that } E^P[X_T Z_T] = x$$

where $Z_T$ is the Doleans(stochastic) exponential known from Girsanov theorem. It is simply the Radon – Nikodym derivative $\frac{dQ}{dP}$ and in our simple setup, we can explicitly compute it:

$$Z_T = \exp(-\mu W_T - \frac{1}{2} \mu^2 T),$$

where $\mu = \frac{\alpha - r}{\sigma}$. We are about to use a standard approach from nonlinear optimization!
We set up the Lagrangian:

\[
L(X_T, \lambda) = E^P[U(X_T)] - \lambda(E^P[X_TZ_T] - x) \\
= E^P[U(X_T) - \lambda X_TZ_T] + \lambda x
\]

- we formally differentiate \( L \) with respect to \( X_T \) and search for extremes.
- heuristically this is an easy step, as we maximize \( L \) for each \( \omega \) separately.
- we do not plunge into functional analysis reasoning why this is really correct.
Hence, after switching $\partial$ and $\int$:

$$E^P[U'(X_T) - \lambda Z_T] = 0$$

and therefore:

$$\hat{X}_T = U'^{-1}(\lambda Z_T) =: I(\lambda Z_T).$$

From the constraint $E^P[X_T Z_T] = \chi$ we compute the optimal $\lambda$. Of course, this is feasible only with quite strict assumptions on the coefficients $\alpha$ and $\sigma$. We have postulated them constant, therefore the computations can be done.
In this approach to Lagrangian optimization we are again not too rigid and assume (requires some stochastic versions of KKT or Slater conditions?), that

$$\sup_{X_T} \inf_{\lambda > 0} L(X_T, \lambda) = \inf_{\lambda > 0} \sup_{X_T} L(X_T, \lambda).$$

We denote our desired maximum as

$$u(x) := \sup_{X_T} \inf_{\lambda > 0} L(X_T, \lambda)$$

and define the conjugate to our utility function:

$$V(y) = \sup_{x \in \mathbb{R}} \{U(x) - yx\} = U(I(y)) - yI(y), \quad y > 0,$$

where of course $I(y) = U'^{-1}(y)$. We define the dual (value) function

$$v(\lambda) := E^P[V(\lambda Z_T)].$$
Under our assumptions (especially we can switch sup and inf), functions $u$ and $v$ are conjugate, which means

$$u(x) = \inf_{\lambda > 0} (v(\lambda) + x\lambda)$$

$$v(\lambda) = \sup_{x > 0} (u(x) - x\lambda)$$

This greatly simplifies the computations and:

$$u(x) = \inf_{\lambda > 0} (v(\lambda) + x\lambda) = v(\hat{\lambda}(x)) + x\hat{\lambda}(x),$$

where clearly $x = -v'(\hat{\lambda})$ and $\hat{\lambda}(x) = u'(x)$. We have computed, combining two slides:

$$\hat{X}_T = U'^{-1} (\lambda Z_T) =: I(\lambda Z_T).$$
To show how explicit the computations can be, we turn to our power utility function from last year:

$$U(x) = \frac{x^{1-p}}{1-p}; \quad p > 0, \ p \neq 1.$$  

Clearly, $U'(x) = x^{-p}$ and $I(y) = y^{-\frac{1}{p}}$. Turning back to our straightforward approach, we search for optimal $\hat{\lambda}$:

$$E^P [\hat{\lambda}^{-\frac{1}{p}} Z_T^{-\frac{1}{p}} Z_T] = x$$

$$\hat{\lambda}^{-\frac{1}{p}} = \frac{x}{k}$$

$$\hat{\lambda} = \left(\frac{x}{k}\right)^{-p}.$$  

where $k = E^P \left[Z_T^{-\frac{1-p}{p}} \right]$
Let's compute $k$: (believing at own risk, however, from the conclusions it seems to be correct)

$$k = E^P[Z_T^{\frac{1-p}{p}}]$$

$$= e^{\left(\frac{1-p}{2p^2} \mu^2 T\right)}$$
HJB approach: We know how a discounted self-financing portfolio evolves:

\[ X_t = x + \int_0^t X_u \theta_u ((\alpha - r) du + \sigma dW_u) \]

that is, easily from Ito formula we confirm, that:

\[ X_t = x \exp \left[ \left( (\alpha - r) \theta - \frac{1}{2} \theta^2 \sigma^2 \right) t + \theta \sigma dW_t \right] . \]

HJB yields the optimal Merton proportion:

\[ \theta = \frac{\alpha - r}{p \sigma^2} \]

When we plug this proportion into the wealth process, we get the optimal wealth process. The terminal wealth is therefore:

\[ \hat{X}_T = x \exp \left[ \left( p - \frac{1}{2} \right) \sigma^2 \theta^2 T + \sigma \theta W_T \right] \]
Martingale approach:

\[ X_T = I(\hat{\lambda} L_T) = (\hat{\lambda} L_T)^{-\frac{1}{p}} \]

\[ = \frac{x}{k} \exp \left( \frac{1}{p} \left( \frac{1}{2} \mu^2 T + \mu W_T \right) \right) \]

\[ = x \exp \left( \frac{1}{p \sigma^2} \left( \mu \sigma^2 W_T + \frac{1}{2p} \mu^2 \sigma^2 p T - \frac{1}{2p} \sigma^2 \mu^2 T \right) \right) \]

\[ = x \exp \left( \theta \sigma W_T + (p - \frac{1}{2}) \sigma^2 \theta^2 T \right) \]

therefore we have obtained the same optimal terminal wealth as in the HJB case.
To match all the results from HJB case we should obtain a whole optimal wealth process and subsequently an optimal control of the process – the optimal proportion of capital invested into the risky asset. We start with a martingale representation theorem:

**Theorem (martingale representation)**

Let $W_t$ be a Brownian motion and let $M_t$ be a martingale adapted to the filtration generated by $W_t$. Then there exists an adapted process $\Gamma_t$ such that:

$$M_t = M_0 + \int_0^t \Gamma_u dW_u.$$
Since we know, that the process of a self–financing portfolio (for any control, any proportion invested into the risky asset) is a $Q$–martingale, we can therefore conclude that there exists an adapted process $\Gamma_t$, such that $X_t$ is a solution to:

$$X_t = x + \int_0^t \Gamma_u dW_u^Q.$$ 

On the other hand,

$$X_t = x + \int_0^t \sigma \theta_u X_u dW_u^Q.$$ 

Hence $\theta_t = \Gamma_t \sigma^{-1} X_t^{-1}$. In general, finding $\Gamma_t$ may unfortunately not be feasible.
However, utilizing a finer approach, assuming constant coefficients, we can compute the optimal proportion (control) explicitly! It of course yields the same proportion as the one obtained by HJB approach. We start with computing the optimal process:

\[
\hat{X}_t = \frac{x}{k} \mathbb{E}^Q \left[ L_T^{-\frac{1}{p}} | \mathcal{F}_t \right] = \frac{x}{k} \mathbb{E}^P \left[ L_T^{-\frac{1}{p}} \frac{L_T}{L_t} | \mathcal{F}_t \right]
\]

\[
= \frac{x}{k} \frac{1}{L_t} \mathbb{E}^P \left[ L_T^{-\frac{1-p}{p}} | \mathcal{F}_t \right].
\]

Clearly,

\[
L_T^{-\beta} = \exp(\mu_\beta W_T + \frac{1}{2} \beta \mu^2 T) = \tilde{L}_T \exp \left( \frac{1}{2} \mu^2 T \beta (\beta + 1) \right),
\]

where \( \tilde{L}_t = \exp(\mu_\beta W_t - \frac{1}{2} \beta^2 \mu^2 t) \) is a martingale.
That is,

\[ L_T^{-\frac{1-p}{p}} = \tilde{L}_T \exp \left( \frac{1}{2} \mu^2 T \frac{1-p}{p^2} \right) \]

\[ \mathbb{E}^P [L_T^{-\frac{1-p}{p}} \mid \mathcal{F}_t] = \tilde{L}_t \exp \left( \frac{1}{2} \mu^2 t \frac{1-p}{p^2} \right) h(t, T) \]

\[ = L_t^{-\frac{1-p}{p}} h(t, T), \]

where \( h(t, T) = \exp \left( \frac{1}{2} \mu^2 (T - t) \frac{1-p}{p^2} \right) \) and we have obtained the optimal wealth process

\[ \hat{X}_t = \frac{x}{k} L_t^{-\frac{1-p}{p}} h(t, T) = \frac{x}{k} L_t^{-\frac{1}{p}} h(t, T). \]
Now only a little work must be done. By Itô formula:

\[ dL_t = -L_t \mu dW_t \]

and plugging this into the optimal wealth process:

\[
\begin{align*}
    d\hat{X}_t &= d \left( \frac{x}{k} h(t, T) L_t^{-\frac{1}{p}} \right) \\
    &= \frac{x}{k} \left( L_t^{-\frac{1}{p}} dh(t, T) + h(t, T) d(L_t^{-\frac{1}{p}}) \right) \\
    &= \frac{x}{k} \left( L_t^{-\frac{1}{p}} h(t, T) \left( -\frac{\mu^2(1-p)}{2p^2} \right) + h(t, T) \left( -\frac{1}{p} \right) L_t^{-\frac{1+p}{p}} dL_t \right) \\
    &= \hat{X}_t \left( -\frac{\mu^2(1-p)}{2p^2} \right) dt + \hat{X}_t \mu \frac{1}{p} dW_t \\
    &= \hat{X}_t \mu \frac{1}{p} dW_t^Q,
\end{align*}
\]

by Girsanov theorem.
As we have seen, it generally holds for a self-financing portfolio that

\[ dX_t = \sigma \theta_t X_t \, dW_t^Q. \]

So, now it only suffices to match the coefficients to see, that

\[ \theta_t = \frac{\mu}{p\sigma} = \frac{\alpha - r}{p\sigma^2}. \]

which matches the result obtained from the HJB approach.
CONCLUSION

- We have developed an alternative way to compute optimal investment.
- Our approach is more computationally tedious, but may offer unrivaled generality.
- We verified for power utility functions, that in the basic Black – Scholes setup both HJB and martingale approaches yield the same results.
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Any questions?
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Any questions?

Thank you for attention.
Bibliography: For this presentation it should suffice to read the following two materials:

- Bjoerk: Arbitrage Theory in Continuous Time
- Janeček: STP185 study material

and compute a lot...

Further interesting literature on this topic, difficult:

- Karatzas, Shreve: Methods of Mathematical Finance
- Pham: Continuous – time Stochastic Control and Optimization with Financial applications

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