Functional Linear Models II.
applications

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Introduction

Functional Data Analysis with R and Matlab connected with FDA r package which includes a scripts www.functionaldata.org

- basis systems
- data smoothing techniques
- functional principal component analysis
- functional linear models (ch. 9, 10)
A Scalar Response Model for Log Annual Precipitation

The aim is to predict the logarithm of annual precipitation for 35 Canadian weather stations from their temperature profiles.

\[ y_i = \alpha_0 + \int x_i(t)\beta(t)dt + \epsilon_i \]

- Low-Dimensional Regression Coefficient Function \( \beta \)
- Coefficient \( \beta \) Estimate Using a Roughness Penalty
- Scalar Response Models by Functional Principal Components

\[ \beta(t) = \sum_{k=1}^{K} b_k \phi_k(t) \]
Functional predictor

65 fourier basis functions without roughness penalty are used to create a functional data from daily temperature averages.
Non-sinusoidal character of the data

\[ x_i(t) = c_{i1} + c_{i2} \sin\left(\frac{2\pi t}{T}\right) + c_{i3} \cos\left(\frac{2\pi t}{T}\right) \]

define the linear (acceleration) operator \( L = \left(\frac{2\pi}{T}\right)^2 D + D^3 \)
Low-Dimensional Regression Coefficient Function $\beta$

using small dimension $K$ of $\beta$ relatively to number of observation
if $K = 5$

\[
\begin{align*}
\phi_1(t) &= 1 \\
\phi_2(t) &= \sin(2\pi/T \ t) \\
\phi_3(t) &= \cos(2\pi/T \ t) \\
\phi_4(t) &= \sin(4\pi/T \ t) \\
\phi_5(t) &= \cos(4\pi/T \ t)
\end{align*}
\]

r function \texttt{fRegress}
with parameters: response, predictor, basis for beta
gives: compute, $\beta$ function, predicted values, ...
squared multiple correlation: \( \frac{SSE_0 - SSE_1}{SSE_0} = 0.8 \)

F statistic: \( \frac{SSE_0 - SSE_1}{p_1 - p_0} / \frac{SSE_1}{n - p_1} = 22.6 \text{ with 5 and 29 df} \)
Coefficient $\beta$ Estimate Using a Roughness Penalty $\lambda$

\[ PENSSE_\lambda(\beta) = \sum \left[ y_i - \int x_i(t)\beta(t)dt \right]^2 + \lambda \int [L\beta(t)]^2 dt \]

using acceleration operator $L$ and increasing $\lambda$ will force $\beta(t)$ looks more like sine wave

\[ CV(\lambda) = \sum_{i=1}^{N} \left[ y_i - \int x_i(t)\beta_{\lambda}^{(-i)} dt \right]^2 \]

\[ CV(\lambda) = \sum_{i=1}^{N} \left( \frac{y_i - \hat{y}_i}{1 - H_{ii}} \right)^2 \]
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Functional Linear Regression with a Scalar Response
Coefficient $\beta$ Estimate Using a Roughness Penalty $\lambda$

CV minimize $\lambda = 10^{12.5}$

Squared multiple correlation: 0.75
F statistic: 25.1 with 3.7 and 30.3 df
**Confidence interval for \( \beta(t) \)**

\[
\text{Var}[\hat{b}] = (Z'Z)^{-1} Z'\Sigma Z (Z'Z)^{-1}
\]
Scalar Response Models by FPCA

- derive FPC scores for $X$
- regress $y_i$ on principal component scores $c_{ij}$

\[
    x_i(t) = \bar{x}(t) + \sum_{j \geq 0} c_{ij} \xi_j(t)
\]

\[
    y_i = \sum c_{ij} \beta_j + \epsilon_i
\]

\[
    c_{ij} = \int \xi_j(t)(x_i(t) - \bar{x}(t))dt
\]

\[
    y_i = \sum \int \beta_j \xi_j(t)(x_i(t) - \bar{x}(t))dt + \epsilon_i
\]

\[
    \beta(t) = \sum \beta_j \xi_j(t)
\]
Confidence interval for $\beta(t)$

$$\text{var}[\hat{\beta}(t)] = [\xi_1(t) \ldots \xi_k(t)] \text{var}[\beta][\xi_1(t) \ldots \xi_k(t)]'$$
Statistical test of significance

calculation statistic: \( \frac{\text{var}[\hat{y}]}{1/n \sum (y_i - \hat{y}_i)^2} \) for permutation of response
Functional Responses with Functional Predictors

\[ y_i(t) = \sum_{j=1}^{q-1} x_{ij}(t) \beta_j(t) + \epsilon_i(t) \]

this model relates \(y\) and \(x\) at same time \(t\) (concurrent model)

\[ y_i(t) = \beta_0(t) + \int_{\Omega_t} \beta_1(s, t)x_i(t)ds + \epsilon_i(t) \]

”Mortality models”
How much control does the hip angle have over the knee angle?
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Functional Responses with Functional Predictors

[Graph showing the relationship between Knee angle (degrees) and Hip angle (degrees)]
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Functional Responses with Functional Predictors
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Functional Responses with Functional Predictors
Estimation procedure

\[ y(t) = Z(t)\beta(t) + \epsilon(t) \]

\[ r(t) = y(t) - Z(t)\beta(t) \]

\[ \text{LMSSE}(\beta) = \int r(t)'r(t)\,dt + \sum_{j} \lambda_j \int [L_j\beta_j(t)]^2\,dt \]

\[ \beta_j(t) = \sum_{k} b_{kj}\theta_{kj}(t) = \theta_j(t)'b_j \]

\[ r(t) = y(t) - Z(t)\Theta(t)b \]
\[ \mathbf{b} = (b'_1, \ldots, b'_q)' \]

\[ \Theta(t) = \begin{bmatrix}
\theta_1(t)' & 0 & \ldots & 0 \\
0 & \theta_2(t)' & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \theta_q(t)' \end{bmatrix} \]

\( R(\lambda) \) block diagonal matrix with \( j \)th block

\[ \lambda_j \int [L_j \theta_j(t)'][L_j \theta_j(t)'] dt \]

differentiating \( LMSSE \) wrt. \( \mathbf{b} \) setting equal to zero gives penalized normal equations:

\[ \left[ \int \Theta'(t)Z'(t)Z(t)\Theta(t) dt + R(\lambda) \right] \hat{\mathbf{b}} = \int \Theta'(t)Z'(t)y(t) dt \]
Confidence intervals

The use of a basis expansion for $y = C\phi(t)$ allows the flexibility of modelling variation in $y$ by itself.

$$\hat{b} = A^{-1} \left[ \int \phi(t)' \otimes (\Theta'(t)Z'(t)) dt \right] \text{vec}(C)$$

$$= c2b\text{Map} \ \text{vec}(C)$$

$$\Sigma_e^* = \frac{1}{N}rr'$$

where $r$ is a matrix of residuals

$$r_{ij} = y_{ij} - Z_j(t_i)\beta(t_i)$$

$$\text{var}[\hat{b}] = c2b\text{Map} \ y2c\text{Map} \ \Sigma_e^* \ y2c\text{Map}' \ c2b\text{Map}'$$
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Functional Responses with Functional Predictors

![Graphs showing intercept and hip coefficient](image-url)
Model for acceleration
Ramsay J.O., Hooker G., Graves S.  
*Functional Data Analysis with R and MATLAB.*  

Ramsay J.O., Silverman B.W.  
*Functional Data Analysis.*  

"R/library/fda/scripts/fdarm-ch**.r"