2. CFF Homework, series 2, to be submitted till 13th April

All steps should be explained in detail (preferably by reference to the class assertions).

Let $f = y^2 - (x^3 + 2x^2 + 1) \in \mathbb{Q}[x, y]$ and L be an algebraic function field given by $f(\alpha, \beta) = 0$ (hence $\alpha = x + (f), \beta = y + (f) \in \mathbb{Q}[\alpha, \beta] \subset \mathbb{Q}(\alpha, \beta)$).

2.1. Consider L as a vector space over fields $\mathbb{Q}(\alpha)$ and over $\mathbb{Q}(\beta)$.
(a) Determine a base A of L over $\mathbb{Q}(\alpha)$,
(b) determine a base B of L over $\mathbb{Q}(\beta)$,
(c) compute coordinates $[\alpha^3\beta^3]_A$ $[\alpha^3\beta^3]_B$ of $\alpha^3\beta^3$ with respect to both the bases A, B.
Hint: apply Proposition 4.7 and the proof of Lemma 4.6.

5 points

2.2. Prove that f is smooth at the point $(1, 2) \in \mathbb{A}^2(\mathbb{Q})$ and find
(a) an affine mapping σ, and polynomials $h \in \mathbb{Q}[x]$ and $g \in \mathbb{Q}[x, y]$ such that $\sigma(1, 2) = (0, 0)$, $\sigma^*(h(x) + yg(x, y) + y) = f$, $\operatorname{mult}(h) \geq 2$, and $\operatorname{mult}(g) \geq 1$,
(b) all points $\mathbf{a} \in \mathbb{A}^2(\mathbb{Q})$ for which there exists σ satisfying conditions of (a) and, moreover, $\sigma(0, 0) = \mathbf{a}$.
Hint: use Lemma 5.7.

8 points

2.3. Suppose ν is a normalized discrete valuation of L such that $\nu(\mathbb{Q}\setminus 0) = 0$, $\nu(\alpha - 1) > 0$ and $\nu(\beta - 2) > 0$. Determine all $(l_0, l_1, l_2) \in \mathbb{Q}^3$ such that
(a) $\nu(l_0 + l_1\alpha + l_2\beta) = 1$,
(b) $\nu(l_0 + l_1\alpha + l_2\beta) = 2$,
(c) $\nu(l_0 + l_1\alpha + l_2\beta) = 3$.
Hint: apply the proof of Theorem 5.8 and Theorem 2.15.

7 points