> In the sequel: K is a field, $\bar{K}$ is an algebraic closure of K; $\mathbb{X} = \{x_1, \dots, x_n\}$ a set of variables

T&N Let $K \subseteq L \subseteq \bar{K}$ be field extensions ($\Rightarrow L$ is algebraic over K)

Denote by $\mathbb{A}^n := \bar{K}^n$ = set of points of algebraic closure (an affine space over $\bar{K}$)

$\mathbb{A}^n(L) := L^n$ = set of L-rational points (an affine space over L)

$M \subset K[\mathbb{X}]$ $\quad V_M := \{\underline{\alpha} \in \mathbb{A}^n \mid a(\underline{\alpha}) = 0 \ \forall a \in M\}$ – common points of M

$V_M(L) := V_M \cap \mathbb{A}^n(L)$ – L-rational points of M

$V_a := V_{\{a\}}$, $V_a(L) := V_{\{a\}}(L)$

<u>Observation</u>: Let $M \subset K[\mathbb{X}]$, $a \in K[\mathbb{X}]$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{A}^n$

(1) $V_M = V_{(M)}$ (i.e. M could be chosen finite as $K[\mathbb{X}]$ is noetherian)

(2) $\text{mult}\, \tau_\beta^*(a) \geq 1 \Leftrightarrow \text{mult}(a(x+\beta_1, \dots, x+\beta_n)) \geq 1 \Leftrightarrow [a(\beta_1, \dots, \beta_n) = 0 \Leftrightarrow \beta \in V_a$

T&N Let $a = \sum \underbrace{a_{i_1 \dots i_n}}_{\text{coefficient corresponds to } x_1^{i_1} \cdots x_n^{i_n}} x_1^{i_1} \cdots x_n^{i_n} \in K[\mathbb{X}]$ & $a = \sum b_j x_i^j \in (K[\mathbb{X} \setminus \{x_i\}])[x_i]$, i.e. coeff. $b_j \in K[\mathbb{X} \setminus \{x_i\}]$

Define: $L(a) = \sum_{i_1 \dots i_n : \sum i_j = 1} a_{i_1 \dots i_n} x_1^{i_1} \cdots x_n^{i_n} = \sum a_{\delta_{1j} \cdots \delta_{nj}} x_j$ – coefficient of x_j only! (δ_{ij} = Kronecker δ) – the linear part of a

$\forall i$: $\frac{\partial a}{\partial x_i} := \sum_{j \geq 0} (j+1)\, b_{j+1} x^j$ – (partial) derivative (in variable x_i) (= usual derivative of a is considered as a polynomial in x_i!)

(Exercise: define general $\frac{\partial a}{\partial^{i_1} x_1 \cdots \partial^{i_n} x_n}$ and proof the correctness!)

Let $\underline{\alpha} \in V_a$ (i.e. $a(\underline{\alpha}) = 0$) and put $c_i = \frac{\partial a}{\partial x_i}(\underline{\alpha})$ ($\in \bar{K}$, if $\underline{\alpha} \in K \Rightarrow c_i \in K$)

a is <u>smooth at $\underline{\alpha}$</u> ("nonsingular") if $\exists i: c_i \neq 0$, a is <u>singular at $\underline{\alpha}$</u> if $\forall i\ c_i = 0$.

$\Delta_{\underline{\alpha}}(a) := \sum_{i=1}^n c_i x_i - \sum_{i=1}^n c_i \alpha_i = \sum c_i (x_i - \alpha_i)$ ($\in \bar{K}[\mathbb{X}]$, if $\alpha_i \in L \Rightarrow \Delta_{\underline{\alpha}}(a) \in L[\mathbb{X}]$)
– <u>tangent</u> of a at $\underline{\alpha}$

<u>Observation</u>: Let $a \in K[\mathbb{X}]$, $\underline{\alpha} \in \mathbb{A}^n$. Then

(1) a is smooth at $\underline{\alpha} \Leftrightarrow \Delta_{\underline{\alpha}}(a) \neq 0$ (i.e. $\exists i\ c_i \neq 0$) (2) $\underline{\alpha} \in V_{\Delta_{\underline{\alpha}}(a)}$

<u>Example 3.7.</u> Let $w = y^2 - (x^3 + x - 2)$ $\in \mathbb{R}[x,y]$ (be a short WEP), we can easily compute: $L(w) = -x$, $\frac{\partial w}{\partial x} = -3x^2 - 1$, $\frac{\partial w}{\partial y} = 2y$, $(1,0) \in V_w$ as $w(1,0) = 0$

$\Rightarrow$ for $\underline{\alpha} = (1,0)$ $c_1 = -4$, $c_2 = 0 \Rightarrow \Delta_{(1,0)}(w) = -4(x-1) = -4x + 4$

(Draw a picture! Is $\Delta_{(1,0)}$ really a equation of tangent of the curve?)

<u>Lemma 3.8</u> If $a \in K[\mathbb{X}]$ and $\underline{\alpha} \in V_a$, then $\Delta_{\underline{\alpha}}(a) = \tau_{-\underline{\alpha}}^*(L(\tau_{\underline{\alpha}}^*(a)))$

proof: Put $c = \sum c_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n} := \tau_{\underline{\alpha}}^*(a) \in \overline{K}[\underline{x}] \Rightarrow a = \tau_{-\underline{\alpha}}^*(c) = c(x_1-\alpha_1, \dots, x_n-\alpha_n)$

$\Rightarrow \frac{\partial a}{\partial x_i}(\underline{\alpha}) = c_{\delta_{1i} \dots \delta_{ni}}$, $\sum c_{\delta_{1i} \dots \delta_{ni}} x_i = L(c) = L(\tau_{\underline{\alpha}}^*(a))$

substitute $x_i \leftarrow x_i - \alpha_i$: $\underline{A_{\underline{\alpha}}(a)} = \sum c_{\delta_{1i} \dots \delta_{ni}}(x_i - \alpha_i) = \tau_{-\underline{\alpha}}^*(L(c)) = \tau_{\underline{\alpha}}^*(L(\tau_{\underline{\alpha}}^*(a)))$ $\square$

Comment: tangent is „linear part the polynomial shifted to the point"

$\boxed{\text{T\&N}}$ Let $a \in K[x,y]$, $\deg a \geq 1$ (where $\deg a = \deg \sum a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n} = \max\{\sum_{j=1}^{n} i_j \mid a_{i_1 \dots i_n} \neq 0\}$)

$C = V_a$ - an affine plane curve

$\underline{\alpha} \in C$ is smooth (resp. singular) point of C if a is smooth (resp. singular) at $\underline{\alpha}$

a singularity of C = a singular point of C (Comment: singularity means that tangent has no geometrical sense ~ cannot be geometrically defined)

a is smooth if a is smooth at α $\forall \alpha \in V_a$

a is singular at $\exists \alpha \in V_a$ such that a is singular at $\underline{\alpha}$

C is smooth (resp. singular) if a is smooth (resp. singular) where $C = V_a$

Lemma 3.9 Let $a \in \overline{K}[\underline{x}]$, $\underline{\alpha} \in \mathbb{A}^n$, $\sigma \in \mathrm{Aff}_n(\overline{K})$.
Then $A_{\underline{\alpha}}(\sigma^*(a)) = \sigma^*(A_{\sigma(\underline{\alpha})}(a))$

Comment: all following assertions could be treated/proof for $n=2$ & curves

proof: we have from that $\exists \underline{\gamma} \in \mathbb{A}^n(\overline{K})$, $A \in GL_n(\overline{K})$ s.t. $\sigma = \tau_{\underline{\gamma}} \vartheta_A$

Hence: $\boxed{\sigma \tau_{\underline{\alpha}} = \tau_{\gamma} \vartheta_A \tau_{\alpha} = \tau_{\gamma} \tau_{\vartheta_A(\alpha)} \vartheta_A = \tau_{\gamma + \vartheta_A(\alpha)} \vartheta_A = \tau_{\sigma(\underline{\alpha})} \vartheta_A}$ $(*)$

and: $\boxed{\tau_{-\sigma(\underline{\alpha})} \sigma = \tau_{-\gamma - A(\alpha)} \tau_{\gamma} \vartheta_A = \tau_{-A(\alpha)} \vartheta_A = \vartheta_A \tau_{A(-\alpha)} \tau_{A(\alpha)} \vartheta_A = \vartheta_A \tau_{-\underline{\alpha}}}$ $(**)$

compute using $(*)$, $(**)$ & 3.8:

$L\vartheta_A^*(x) = L(A(x)) = AL(x)$

$\underline{A_{\underline{\alpha}}(\sigma^*(a))} \overset{3.8}{=} \tau_{-\underline{\alpha}}^* L(\underbrace{\tau_{\underline{\alpha}}^*(\sigma^*(a))}_{(\sigma \tau_{\underline{\alpha}})^*}) \overset{(*)}{=} \tau_{-\alpha}^* L \vartheta_A^* \tau_{\sigma(\underline{\alpha})}^*(a) =$

$= \underbrace{\tau_{-\alpha}^* \vartheta_A^*}_{(\vartheta_A \tau_{-\alpha})^*} L \tau_{\sigma(\underline{\alpha})}^*(a) \overset{(**)}{=} \sigma^* \tau_{-\sigma(\alpha)}^* L(\tau_{\sigma(\underline{\alpha})}^*(a)) \overset{3.8}{=} \sigma^*(A_{\sigma(\underline{\alpha})}(a))$

Comment: Tangents are naturally shifted by affine automorphisms!

Lemma 3.10 Let $a \in \overline{K}[\underline{x}]$, $\sigma \in \mathrm{Aff}_n(K)$

$\boxed{\text{T\&N}}$ $\sigma = \tau_{\underline{\gamma}} \vartheta_A \in \mathrm{Aff}_n(K)$ then $\overline{\sigma}: \tau_{\underline{\gamma}} \vartheta_A \in \mathrm{Aff}_n(\overline{K})$ (given $\underline{\gamma} \in K^n \subseteq \overline{K}^n$ and $A \in GL_n(K) \subseteq GL_n(\overline{K})$)

(1) $\overline{\sigma}(V_{\sigma^*(a)}) = V_a$ (2) $\sigma^*(a)$ is singular at $\underline{\alpha} \in V_{\sigma^*(a)} \Leftrightarrow a$ is singular at $\overline{\sigma}(\alpha)$

proof: (1) $\underline{\alpha} \in V_{\sigma^*(a)} \Leftrightarrow \sigma^*(a)(\underline{\alpha}) = 0 \Leftrightarrow a(\overline{\sigma}(\underline{\alpha})) = 0 \Leftrightarrow \overline{\sigma}(\alpha) \in V_a$

(2) follows by (1) and 3.9 since $\sigma^*(A_{\alpha}(a)) = 0$ $A_{\beta}(a) = 0$ $\forall \sigma$

Corollary 3.11 Let w & $\tilde{w}$ be K-equivalent WEP's. Then

w is smooth $\Leftrightarrow$ $\tilde{w}$ is smooth

proof: $\exists \sigma$ $\exists \lambda \in K^*$ s.t. $\sigma^*(w) = \lambda \tilde{w}$ $\Rightarrow$ w is smooth $\Leftrightarrow \sigma^*(w)$ is smooth by 3.10(2)

$\boxed{\text{T\&N}}$ WE $w = 0$ is smooth (resp. singular) if WEP w is smooth (resp. singular)

Recall: $f \in K[x]$ is separable if f has no multiple roots in $\overline{K}$

K is perfect if $\forall$ irreducible polynomial is separable

Fresh: all fields of char 0 are perfect!

Proposition 3.12 Let char $K \neq 2$ and $w = y^2 - f(x) \in K[x,y]$ be a short WEP

Then: (1) w has at most 1 singularity

(2) if K is perfect, then any singularity is K-rational (i.e. $\in K^2$)

(3) w is smooth $\Leftrightarrow$ f is separable.

Comment: for K of char $\neq 2$ $\forall \tilde{w} \in$ WEP $\exists$ K-equivalent WEP w of SS
So by 3.11 we can recognize smoothness of w by checking short WEP $\tilde{w}$ using 3.12(3)

Proof of 3.12: compute $\frac{\partial w}{\partial x} = f'(x)$ and $\frac{\partial w}{\partial y} = 2y$ ($\neq 0$ as char $K \neq 2$)

(1) Let w be singular at $\alpha = (\alpha_1, \alpha_2) \in \mathbb{A}^2 = \bar{K}^2$, $\alpha \in V_w \Rightarrow w(\alpha) = 0$ & $f'(\alpha_1) = 0$ $2\alpha_2 = 0$

char $K \neq 0$ $\Rightarrow \alpha_2 = 0 \Rightarrow f(\alpha_1) = 0 = f'(\alpha_1) \Rightarrow \alpha_1$ is a multiple root of $f \Rightarrow (x-\alpha_1)^2 \mid f$

$\Rightarrow f$ is not separable (and we have proved ($\Leftarrow$) of (3))

Since $\deg f = 3$ & α_1 is a multiple root; $\exists!$ such $\alpha_1 \Rightarrow (\alpha_1, 0)$ is unique (if exists)

(2) Let K be perfect and $f = (x-\alpha_1)^2(x-\beta)$ / if $\alpha_1 \neq \beta \Rightarrow (x-\alpha_1) = GCD(f, f') \in K[x] \Rightarrow \alpha_1 \in K$

if $\alpha_1 = \beta \Rightarrow f = (x-\alpha_1)^3$ is reducible $\Rightarrow (x-\alpha_1) \in K[x] \Rightarrow \alpha_1 \in K$

$\Rightarrow \alpha = (\alpha_1, 0) \in K^2 = \mathbb{A}_2(K) = V \cap V_w = V_w(K)$

(3) we have proved $\Leftarrow$ (indirectly)

($\Rightarrow$) Let f be non-separable i.e. $\exists \alpha_1 \in \bar{K}$: $(x-\alpha_1)^2 \mid f \Rightarrow f(\alpha_1) = 0 = f'(\alpha_1) \Rightarrow$

$\Rightarrow (\alpha_1, 0)$ is a singularity of $w \Rightarrow f$ is not smooth.

Example 3.13 (1) $y^2 - (x^3+1) \in \mathbb{R}[x,y]$ is smooth since (x^3+1) is separable

(2) $y^2 - (x^3 - x^2 - x + 1) = y^2 - (x-1)^2(x+1)$ is singular with singularity $(1,0)$

Comment: singularity of short WEP $w = y^2 - f(x)$ can be computed as $(\alpha_1, 0)$ where α_1 is the unique multiple root of $f(x)$, in general case (over K of char $\neq 2$) we can compute $\sigma \in \mathrm{Aff}_2(K)$ s.t. $\sigma^*(w)$ is short WEP (then if β is a singular point of $\sigma^*(w)$ then $\sigma^{-1}(\beta)$ is the singular point of w by 3.10