4 Coordinate rings

In the chapter: $K \subseteq \bar{K}$ field, algebraic closure of K; $X = \{x_1, \dots, x_n\}$

Comment: Students are supposed to be familiar with basic properties of the operator I & V. If not, they can find them in Fulton's book

$\boxed{\text{I\&N}}$ Let $U \subseteq \mathbb{A}^n$: $I_U := \{a \in K[X] \mid a(\alpha) = 0\ \forall \alpha \in U\} = \bar{I}_U \cap K[X]$

$\bar{I}_U := \{a \in \bar{K}[X] \mid \text{—"—}\}$

$\alpha \in \mathbb{A}^n$: $I_\alpha := I_{\{\alpha\}}$, $\bar{I}_\alpha := \bar{I}_{\{\alpha\}}$

Observation (1) Let I be an ideal of $K[X]$ s.t. $I \cap K[x_i] = (a_i) \neq 0$, put $d_i := \deg_{x_i} a_i$. Then $K[X]/I = \operatorname{Span}_K(x_1^{i_1} \cdots x_n^{i_n} \mid i_j \le d_j) \Rightarrow \dim_K(K[X]/I) \le \prod(d_j+1)$

(2) If R is a K-algebra, $\dim_K(R) < \infty$ and R is a domain, then $\forall \alpha \in R \setminus \{0\}$: $\alpha^{-1} \in K(\alpha) \underset{\uparrow \text{ algebraic}}{=} K[\alpha] \subseteq R \Rightarrow R$ is a field.

Lemma 4.1 Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{A}^n$. Then

(1) I_α is a maximal ideal

(2) $\alpha \in \mathbb{A}^n(K) \Leftrightarrow I_\alpha$ is maximal & $K + I_\alpha = K[X]$

(3) $\alpha \in \mathbb{A}^n(K) \Rightarrow I_\alpha = (x_1 - \alpha_1, \dots, x_n - \alpha_n)$

Proof: Denote by $\Omega: K[X] \to \bar{K}$ the substitution $\Omega(p) = p(\alpha)$ $\forall p$

$\Rightarrow \Omega$ is a homomorphism of K-algebras & $\Omega(K[X]) = K[\alpha_1, \dots, \alpha_n] \subseteq \bar{K}$ where $K[\alpha_1, \dots, \alpha_n]$ is a domain $\overset{\text{observation (2)}}{\Longrightarrow} \Omega(K[X]) \cong K[X]/\ker\Omega$ is a field

$\Rightarrow \ker\Omega = I_\alpha$ is a maximal ideal $\Leftrightarrow$ (1)

(2) $K + I_\alpha = K[X] \Leftrightarrow \Omega(K) = K[\alpha_1, \dots, \alpha_n] \Leftrightarrow [K[\alpha_1, \dots, \alpha_n] : K] = 1 \Leftrightarrow \alpha_i \in K\ \forall i$

(3) I_α & $(x_1 - \alpha_1, \dots, x_n - \alpha_n)$ are both maximal (by (1) & (2)) and $I_\alpha \supseteq (x_1 - \alpha_1, \dots, x_n - \alpha_n)$

$\Rightarrow I_\alpha = (x_1 - \alpha_1, \dots, x_n - \alpha_n)$

Proposition 4.2 If P is a prime ideal of $K[X]$ s.t. $P \cap K[x_i] \neq 0$ $\forall i$, then $\exists \alpha \in \mathbb{A}^n$ for which $P = I_\alpha$

Proof: Since $\dim_K(K[X]/P) < \infty$ by Observation (1) $\Rightarrow$

$\Rightarrow K[X]/P$ is a field by Observation (2) $\Rightarrow P$ is maximal

Let $\bar{P} := P\bar{K}[X]$, it is an ideal of $\bar{K}[X]$, we will show that $\bar{P}$ is a proper ideal of $\bar{K}[X]$ first.

2.2 To a contradiction assume that $\bar{P} = \bar{K}[X]$ i.e. $1 \in \bar{P}$

$\Rightarrow \exists \alpha_1, \dots, \alpha_r \in \bar{K}$ (algebraic over K) s.t. $1 \in PK(\alpha_1 \dots \alpha_r)[X]$

$[K(\alpha_1, \dots, \alpha_r) : K] < \infty \overset{1.14}{\Longrightarrow} \underline{P = PK(\alpha_1 \dots \alpha_r)[X] \cap K[X] \ni 1}$

a contradiction

(Comment: We have described maximal ideals of $K[X]$ as I_α)

By Zorn's lemma $\exists$ a maximal ideal $M \subsetneq \bar{K}[X]$ such that $P \subseteq \bar{P} \subseteq M$, hence $\exists \alpha \in \mathbb{A}^n : M = I_\alpha$ & 4.1

$\underline{P \subseteq I_\alpha \cap K[X] = I_\alpha = M \cap K[X]} \; \underline{\subsetneq K[X]} \overset{P\text{ maximal}}{\Longrightarrow} \underline{P = I_\alpha}$

Proposition 4.3 If P is a prime ideal of $K[X,Y]$ then either (a) $P = 0$ or (b) $P = (a)$ for irreducible $a \in K[X,Y]$ or (c) $P = I_\alpha$ is maximal (for a suitable $\alpha \in \mathbb{A}^2$.

Proof: 2.8. & 4.2

see: Fulton

Corollary 4.4 Let $0 \neq P \in K[X,Y]$ be a prime ideal

(1) P is maximal $\Leftrightarrow \exists \alpha \in \mathbb{A}^2 : P = I_\alpha \Leftrightarrow V_P$ is finite

(2) $\exists a \in K[X,Y]$ irreducible s.t. $P = (a) \Leftrightarrow V_{(a)}$ is infinite

(3) If $a, b \in K[X,Y]$ are irreducible, $b \notin (a) \Rightarrow V_{(ab)} = V_a \cap V_b$ is finite.

Example 4.5 Let $w = Y^2 - (X^3 - 1) \in \mathbb{K}[X,Y]$ (WEP)

We compute some primes: $0 \subsetneq (w) \subsetneq (Y, X-1) = I_{(1,0)}$

$\subsetneq (Y, X^2+X+1) = I_{(e^{\frac{2\pi i}{3}}, 0)}$

$\subsetneq (Y^2+1, X) = I_{(0,i)}$

$\boxed{\text{T\&N}}$ Let $C = V_a$ be an affine plane curve (Recall: $I_C = (a)$)

$K[C] := K[X,Y]/I_C = K[X,Y]/(a)$ is <u>the coordinate ring of C</u>

C is <u>irreducible</u> if $K[C]$ is a domain.

w WEP: V_w is a <u>Weierstrass curve</u>.

$P(X,Y) + I_C \in K[X]$ are <u>polynomials on C</u>

Observation $C = V_a \subseteq \mathbb{A}^2$

Comment: Polynomials on C can be interpreted as functions on C

(1) C is irreducible $\Leftrightarrow I_C = (a)$ is prime $\Leftrightarrow a$ is irreducible

(2) The mapping $K[C] \to \{C \to K\} : p + (a) \to (\alpha \to p(\alpha))$ is well-defined and injective. ~~Comment: Polynom~~

$\boxed{\text{T\&N}}$ If $C = V_a = V_{(a)}$ is an irreducible curve then the field of fraction of $K[C]$ is called <u>the function field of C</u>: $K(C) = \left\{ \frac{n + (a)}{d + (a)} \;\middle|\; n \in K[X,Y], d \notin (a) \right\}$