Recall that every WEP is absolutely irreducible § 4.9 (i.e irreducible in $\tilde{K}[x,y]$) and if $w \in K[x,y]$ is absolutely irreducible, then $K = \tilde{K}$ in $K(V_w)$ § 4.10, so we get a consequence:

<u>Corollary 4.11</u> If C is a Weierstrass curve, then each $g \in K(C) \setminus K$ is transcendental over K.

<u>Example 4.12</u> $w = y^2 + yx + x^3 + 1 \in \mathbb{F}_2[x,y]$ is WEP and $\mathbb{F}_2(V_w) = L :=$ the fraction field of $\mathbb{F}_2[x,y]/(w) \cong \underbrace{\mathbb{F}_2[x,y]_{(w)}}_{\text{the localization of } \mathbb{F}_2[x,y] \text{ in } (w)}/(w)$

Then $\tilde{\mathbb{F}}_2 = \mathbb{F}_2$ are the set of all (! only 2) algebraic elements of L over $\mathbb{F}_2$.

$\Rightarrow$ x^2+x+1, x^3+x+1 has no roots in L $\Rightarrow$ they are irreducible in L.

If we consider $w \in \mathbb{F}_{2^n}[x,y]$ $\underset{\text{again}}{\Rightarrow}$ $\tilde{\mathbb{F}}_2 = \mathbb{F}_{2^n}$ in $\mathbb{F}_{2^n}(V_w)$

$\boxed{\text{T\&N}}$ Let $w \in K[x,y]$, L be an AFF over K, $\alpha, \beta \in L$.
We say that an AFF L is given by (the equation) $w(\alpha,\beta) = 0$ if (1) $L = K(\alpha,\beta)$, (2) w is irreducible, (3) $w(\alpha,\beta) = 0$ in L

<u>Observation</u> If $K(V_w)$ is the function field of V_w for irreducible $w \in K[x,y]$ and $\alpha := x + (w)$, $\beta := y + (w)$. Then $K(V_w)$ is an AFF given by $w(\alpha,\beta) = 0$.

<u>Comment</u>: w is a polynomial determining the affine planar curve V_w, so we have two views into AFF's: either as function fields of a curve or an AFF given by an equation, both the description ~~are~~ are based on the polynomial „w".

<u>5. Places</u>

In the whole section K will be a field, $w = y\,g(x,y) + h(x) + y \in K[x,y]$ where $h \in K[x] \setminus \{0\}$, $g \in K[x,y]$, $\underline{m} := \text{mult}(h) \geq 2$, $\text{mult}(g) \geq 1$.

<u>Comment</u> Our aim is to say more about places in an AFF given by ~~w(α,β)=0~~ $w(\alpha,\beta) = 0$ where is (and could be everything) of positive multiplicity, i.e without constants (in fact, the condition $m \geq 2$ & $\text{mult}\, g \geq 1$ are stronger but satisfactable)

$\boxed{\text{T\&N}}$ Let $a = \sum a_{ij} x^i y^j \in K[x,y] \setminus \{0\}$, then define:
$\mu(a) := \text{mult}(a(x, y^m)) = \min\{i + jm \mid a_{ij} \neq 0\}$ „m-weighted multiplicity"

$s(a) := \{(i,j) \mid i+jm = \mu(a),\ i\geq 0, j\geq 0\}$

$S(a) := \sum_{(i,j)\in s(a)} a_{ij} x^i y^j \in K[x,y]-\{0\}$ – „m-socle"

Comment: We need to determine a „lower part" of a polynomial (weighted by m) as a part defining/restricting multiplicities.

Observation Let $a, b \in K[x,y]-\{0\}$, $a_{ij}, b_{kl} \in \mathbb{N}\cup\{0\}$

(1) $mult(a\cdot b) = mult(a) + mult(b)$, if $mult(a) < mult(b) \Rightarrow mult(a+b) = mult(a)$

(cf. it with the corresponding claims about degree: the proof use minimality due lexicographic order on (i,j) a_{ij})

(2) $\mu(a\cdot b) = mult(a(x,y^m)\, b(x,y^m)) \overset{(1)}{=} mult(a(x,y^m)) + mult(b(x,y^m)) = \mu(a)+\mu(b)$

if $\mu(a)<\mu(b) \Leftrightarrow mult(a(x,y^m)) < mult(b(x,y^m)) \overset{(1)}{\Rightarrow} \mu(a+b) = \mu(a)$

(3) $(i+jm)+(k+lm) = \mu(a)+\mu(b) \overset{(2)}{=} \mu(a\cdot b)$ & $i+jm > \mu(a) \Rightarrow k+lm < \mu(b) \Rightarrow b_{kl}=0$

$\Rightarrow S(a)\cdot S(b) = \sum_{\substack{(i,j)\in s(a)\\(k,l)\in s(b)}} a_{ij} b_{kl} x^{i+k} y^{j+l} \overset{(*)}{=} \sum_{\substack{i,j,k,l\\ i+jm+k+lm=\mu(ab)}} a_{ij} b_{kl} x^{i+k} y^{j+l} = S(a\cdot b)$

(4) $\mu(a) = \mu(S(a))$ & $\mu(a) \geq mult(a)$

if $\mu(a) < \mu(b) \overset{(2)}{\Rightarrow} S(a+b) = S(a)$

$\boxed{T\&N}$ Let $\Lambda: K[x,y]\to K[x,y]$ be the K-endomorphism defined by the rule $\Lambda(u(x,y)) = u(x, -f(x)-y\,g(x,y))$ – it is substitution (to the second coordinate)

Lemma 5.1 For every $i,j\geq 0$ $\mu(\Lambda(x^iy^j)) = i+jm$ and $\exists \lambda\in K^*$ such that $S(\Lambda(x^iy^j)) = \lambda\cdot x^{i+jm}$

Comment: Note that $\mu(x^iy^j) = i+jm = \mu(\Lambda(x^iy^j))$, i.e. x^iy^j and $\Lambda(x^iy^j)$ has the same m-weighted multiplicity. Furthermore Λ „shifts" the m-socle to the variable x.

Proof: Note: $\mu(-f) = \mu(f) = mult(f) = m$

$\mu(-yg) = \mu(yg) \overset{(2)}{=} \mu(y)+\mu(g) > \mu(y) = mult(y^m) = m$ as $\mu(g) \geq mult(g)$

$\Rightarrow \mu(-f-yg) \overset{(2)}{=} \mu(-f) = m$ & $S(-f-yg) \overset{(4)}{=} S(-f) = (\underset{\neq 0}{-f_m})\, x^m$ $\boxed{f = \sum f_i x^i}$ ≥ 1

To sum up: $S(\Lambda(x^iy^j)) = S(x^i(-f-yg)^j) \overset{(3)}{=} S(x)^i \cdot S(-f-yg)^j = x^i(-f_m x^m)^j = \underset{(-f_m)^j \in K^*}{\lambda}\, x^{i+jm}$

Example 5.2 Let $\tilde{w} = (y+x+1)^2 - (x^3+2x+1) \in \mathbb{R}[x,y]$

$GCD((x^3+2x+1), \underset{3x^2+2}{(x^3+2x+1)'}) = 1 \Rightarrow \tilde{w}$ is a smooth WSP

Put $w := \frac{1}{2}\tilde{w} = \frac{1}{2}(y^2+x^2+2yx+2y-x^3) = y\underbrace{(x+\tfrac{y}{2})}_{g} + \underbrace{\tfrac{1}{2}(x^2-x^3)}_{f} + y$

Since $mult\, g = 1$, $m := mult(f) = 2 \Rightarrow w$ is of required type

Compute: $\mu(g) = mult(x+\frac{y^2}{2}) = 1$, $S(g) = x$, $\mu(f) = 2$, $S(f) = \frac{1}{2}x^2$

$\mu(x^3y^2) = 3+2\cdot 2 = 7$, $\mu(x^2y^3) = 2+3\cdot 2 = 8 \Rightarrow \mu(x^3y^2+x^2y^3) = 7$

$S(\Lambda(x^3y^2+x^2y^3)) = S(\Lambda(x^3y^2)) = \frac{1}{4}x^7$ (by the proof of 5.1)