In the sequel: L is an AFF over k given by $w(\alpha,\beta)=0$

(Recall: $w = y\,g(x,y) + \ell(x) + y \in k[x,y]$ where $\ell \in k[x]\setminus\{0\}$, $g \in k[x,y]$ | $m = \mathrm{mult}\,\ell \geq 2$, $\mathrm{mult}\,g \geq 1$)

Comment: w has to be irreducible by the definition

Comment: $v_P(\beta) = m\,v_P(\alpha)$ holds for each such P!

Lemma 5.3: $\exists P \in \mathbb{P}_{L/k}$ such that $v_P(\alpha)>0$ & $v_P(\beta)>0$. Moreover, $v_P(\beta) = m \cdot v_P(\alpha)$

Proof: Let $\Omega : k[x,y] \to k[\alpha,\beta]$ be the substitution of (α,β)

$\Rightarrow \ker\Omega = (w) \subseteq (x,y) \subseteq I_{0,0}$ is a maximal ideal of $k[x,y]$ $\Rightarrow$

$\Rightarrow k[\alpha,\beta]/(\alpha,\beta) \underset{\uparrow}{\cong} k[x,y]/(w) \Big/ (x,y)/(w) \underset{\uparrow}{\cong} k[x,y]/(x,y) \cong k$ $\Rightarrow$

by the definition of AFF — 2nd Isomorphism Theorem

$\Rightarrow (\alpha,\beta)$ is maximal in $k[\alpha,\beta] \overset{2.5}{\Rightarrow} \exists P \in \mathbb{P}_{L/k} : (\alpha,\beta) \subseteq P$

Clearly: $v_P(\alpha)>0$, $v_P(\beta)>0$, $\deg_x w > 0$, $\deg_y w > 0 \Rightarrow \alpha,\beta$ - transcendental $\Rightarrow \alpha \neq 0 \neq \beta$

$v_P(\alpha)>0 \overset{2.12(2)}{\Longrightarrow} v_P(\ell(\alpha)) = \mathrm{mult}(\ell)\,v_P(\alpha) = m \cdot v_P(\alpha)$

$w(\alpha,\beta)=0 \Rightarrow \ell(\alpha) = -\beta - \beta g(\alpha,\beta)$, $v_P(\beta g(\alpha,\beta)) \underset{DV1}{=} v_P(\beta) + \overbrace{v_P(g(\alpha,\beta))}^{\geq \mathrm{mult}\,g \geq 1} > v(\beta)$

$\Rightarrow \underline{v_P(\beta)} = v_P(-\beta) \overset{2.13}{=} v_P(-\beta-\beta g(\alpha,\beta)) = v_P(\ell(\alpha)) = \underline{m \cdot v_P(\alpha)}$

Observation: Let $a \in k[x,y]$, $a = \sum a_{ij} x^i y^j$

(1) $\Lambda(a)(\alpha,\beta) = a(\alpha, -\ell(\alpha) - \beta g(\alpha,\beta)) = a(\alpha,\beta)$ in L

(2) Fix $P \in \mathbb{P}_{L/k}$ s.t. $v_P(\beta) = v_P(\alpha)\cdot m > 0$ from 5.3 and $u = a(\alpha,\beta) \in k[\alpha,\beta]\setminus\{0\}$

$\Rightarrow v_P(u) \geq \min\{v_P(x^i y^j) \mid a_{ij} \neq 0\} = \min\{i+jm \mid a_{ij}\neq 0\}\cdot v_P(\alpha) = \mu(a)\cdot v_P(\alpha)$

$\Rightarrow \mu(a) \leq \dfrac{v_P(u)}{v_P(\alpha)} \Rightarrow$ **T&N** (it is correct) $\quad \mu(u) := \max\{\mu(a) \mid a \in k[x,y], a(\alpha,\beta)=u\}$ $\left(\leq \dfrac{v_P(u)}{v_P(\alpha)}\right)$

Comment: We can find $\forall u \in k[\alpha,\beta]$ a polynomial with the maximal μ; the next assertion shows that a could be chosen with μ determined by $x^{\mu(u)}$.

Lemma 5.4 Let $u \in k[\alpha,\beta]\setminus\{0\}$ and $r := \mu(u)$. Then $\exists \lambda \in k\setminus\{0\}$, $\exists b \in k[x,y]$ such that $\mu(b) > r$ and

$$u = \lambda \alpha^r + b(\alpha,\beta)$$

Proof: Let $a \in k[x,y]$ be a polynomial with $\mu(a) = \mu(u) = r$

Then $\Lambda(a)(\alpha,\beta) = a(\alpha,\beta) = u \neq 0 \Rightarrow \Lambda(a) \neq 0$, Let $a = \sum a_{ij} x^i y^j$

$\Rightarrow a = \underbrace{\sum_{(i,j)\in E(a)} a_{ij} x^i y^j}_{S(a)} + \sum_{a_{ij}:\, i+jm > r} a_{ij} x^i y^j$ | Note that $\mu(S(\Lambda(a))) \leq \mu(a)$ since $(*)$ $\Lambda(a)(\alpha,\beta) = u$ & $\mu(\Lambda(a)) \leq \mu(u)$

Λ is a K-homomorphism $\Rightarrow \Lambda(a) = \sum_{(i,j)\in\sigma(a)} a_{ij}\Lambda(x^i y^j) + \sum_{i,j} a_{ij}\Lambda(x^i y^j)$

$\rightarrow \mu(\Lambda(x^i y^j)) = k$; $i e_j m > k \Rightarrow \mu(\Lambda(\ldots)) > k$

$\overset{S.1}{\Longrightarrow} \exists\, \lambda_{ij} \in K^* : S(\Lambda(a)) = \underbrace{\left(\sum a_{ij}\lambda_{ij}\right)}_{\neq 0} x^k \neq 0$ $\lightning$ (*)

<u>Proposition 5.5</u> $\exists!\, P \in \mathbb{P}_{L|K}$ such that $v_P(\alpha) > 0$ and $v_P(\beta) > 0$.

For such P: $v_P(\alpha) = 1$, $v_P(\beta) = m$, $v_P(u \cdot v^{-1}) = \mu(u) - \mu(v)$ $\forall u, v \in K[\alpha,\beta]$

<u>Comment</u>: α,β uniquely determine $P \in \mathbb{P}_{L|K}$ s.t. $v_P(\alpha) > 0$ and $v_P(\beta) > 0$,
v_P could be computed using μ!

<u>proof</u>: By 5.3. $\exists P \in \mathbb{P}_{L|K}$ s.t. $v_P(\beta) = v_P(\alpha) \cdot m > 0$.

Let $u \in K[\alpha,\beta] \setminus \{0\}$, put $k := \mu(u)$ $\overset{5.4}{\Longrightarrow} \exists\, h = \underset{\neq 0}{\lambda} x^k - c(x,y) \in K[x,y]$
s.t. $\mu(c) > k$ & $u = h(\alpha,\beta)$

we can $\forall i,j$ compute $v_P(\alpha^i\beta^j) \underset{(DV1)}{=} i \cdot v_P(\alpha) + j\, \underbrace{v_P(\beta)}_{\overset{5.3}{=} m \cdot v_P(\alpha)} = \underbrace{(i + jm)\, v_P(\alpha)}_{= \mu(x^i y^j)\, v_P(\alpha)}$

$\Rightarrow v_P(c(\alpha,\beta)) \geq$ ~~...~~ $\mu(c) \cdot v_P(\alpha) > k \cdot v_P(\alpha)$
$v_P(\lambda\alpha^k) = k \cdot v_P(\alpha)$ $\underset{2.13}{\Longrightarrow} v_P(u) = k \cdot v_P(\alpha)$

$\Rightarrow v_P(\alpha) \mid v_P(u)$ $\forall u \in K[\alpha,\beta] \setminus \{0\}$

Comment: $\forall\, \delta \in K(\alpha,\beta) \setminus \{0\}$ $\exists u, v \in K[\alpha,\beta] : \delta = \frac{u}{v}$

$\Rightarrow \forall u, v \in K[\alpha,\beta] \setminus \{0\}$ $v_P(\alpha) \mid v_P(u) - v_P(v) = v_P\left(\frac{u}{v}\right)$

As v_P is normalised DV: $v_P(\alpha) = 1$ & $v_P\left(\frac{u}{v}\right) = \mu(u) - \mu(v)$

Let $Q \in \mathbb{P}_{L|K}$ $\Rightarrow$ using the same proof as for P: $v_Q\left(\frac{u}{v}\right) = \mu(u) - \mu(v)$

$\Longrightarrow v_P = v_Q \Rightarrow$ <u>$P = \{a \in L \mid v_P(a) \geq 1\} = \{a \in L \mid v_Q(a) \geq 1\} = Q$</u>
(& $v_P(0) = \infty = v_Q(0)$)

<u>Example 5.6</u> Let $w = y\left(x + \frac{y}{2}\right) + \frac{1}{2}(x^2 - x^3)$ from 5.2
$L = K(\alpha,\beta)$ $\alpha = x + (w)$, $\beta = y + (w)$ | Let $P \in \mathbb{P}_{L|K} : (\alpha,\beta) \subseteq P$ $\Rightarrow v_P(\alpha^2) = 2$, $v_P(\beta) = 2$

?. $v_P(\alpha^2 + \beta) = ?$ $\beta = -\beta\left(\alpha + \frac{\beta}{2}\right) + \frac{1}{2}(\alpha^3 - \alpha^2)$

$\Rightarrow \alpha^2 + \beta = \underbrace{\frac{1}{2}\alpha^2}_{v_P = 2} + \underbrace{\frac{1}{2}\alpha^3 - \beta\alpha - \frac{\beta^2}{2}}_{v_P > 2}$ $\Rightarrow v_P(\alpha^2 + \beta) \overset{2.13}{=} v_P\left(\frac{1}{2}\alpha^2\right) = 2$

?. $v_P(\alpha^2 + 2\beta) = v_P\left(\underbrace{\alpha}_{v_P = 1}\underbrace{(\alpha^2 - 2\beta)}_{v_P = 2} - \underbrace{\beta^2}_{v_P = 4}\right) \overset{2.13}{=} \underline{3}$