Lemma 5.11 If f is singular at $x \in V_K(k) \Rightarrow \mathcal{O}_x$ is not a VR.

Proof: B 3.10 f is singular at $x = \tau_x(\underline{0}) \Leftrightarrow \tau_x^*(f)$ is singular at $\underline{0}$

$\boxed{\underline{0} := (0,0)}$ & Observation (5) ${}_f\mathcal{O}_x = {}_{\tau_x^*(f)}\mathcal{O}_{\underline{0}} \Rightarrow$

we may assume w.l.o.g. that f is singular at $\underline{0}$

Then & 3.8 $L(f) = A_0(f) = 0 \Rightarrow \operatorname{mult} f \geq 2$

Comment: This w.l.o.g. is a standard trick using [illegible] etc.

?? Assume to contrary that $\mathcal{O}_{\underline{0}}$ is VR

Hence either $\frac{\alpha}{\beta} \in \mathcal{O}_{\underline{0}}$ or $\frac{\beta}{\alpha} \in \mathcal{O}_{\underline{0}}$, w.l.o.g. let $\frac{\alpha}{\beta} \in \mathcal{O}_{\underline{0}}$

$\Rightarrow \exists\, a, b \in K[x,y]$ with $\operatorname{mult} a \geq 1$, $\operatorname{mult} b \geq 1$ & $\exists\, \nu \in K\ \exists\, \lambda \in K^*$:

$$\frac{\alpha}{\beta} = \frac{a(\alpha,\beta)+\nu}{b(\alpha,\beta)+\lambda} \Rightarrow \alpha(b(\alpha,\beta)+\lambda) - \beta(a(\alpha,\beta)+\nu) = 0 \Rightarrow f \mid \underbrace{xb + \overset{\neq 0}{\lambda} x - ya - \nu y}_{\operatorname{mult} = 1}$$

$\operatorname{mult} f \geq 2$ divides a polynomial of $\operatorname{mult} = 1 \Rightarrow$ a contradiction

(the argument for $\frac{\beta}{\alpha}$ is symmetric).

Comment: $\mathcal{O}_x \neq \mathcal{O}_P$ for any $P \in \mathbb{P}_{L|K}$ and any singular element x

Lemma 5.12

Let L be an AFF given by $w(u,v) = 0$ (with the standard "w"-properties: $w = h(x) + y\, g(x,y) + y$, $\operatorname{mult} g \geq 1$, $m := \operatorname{mult} h \geq 2$, $\deg h \geq 2$). Suppose $P \in \mathbb{P}_{L|K}$ such that $u, v \in P$, $v_P(u) = 1$.

If $r \in K[u,v] - \{0\}$, then $\exists\, a, b \in K[x,y]$ with $a(\underline{0}) \neq 0 \neq b(\underline{0})$ (i.e. $\operatorname{mult}(a) = \operatorname{mult}(b) = 0$) and

$$\frac{r}{u^{v_P(r)}} = \frac{a(u,v)}{b(u,v)} \in {}_w\mathcal{O}_{\underline{0}}^* = {}_w\mathcal{O}_{\underline{0}} - {}_wP_{\underline{0}}.$$

Comment: Recall the mapping $\mu(c) = \operatorname{mult}(c(x, y^m))$.

Proof: Put $k := v_P(r) = \mu(r)$ & 5.5

& 5.4 $\exists\, c \in K[x,y]\ \exists\, \lambda \in K^*$ such that $r = \lambda u^k + c(u,v)$, $\mu(c) > k$

Denote $c = \sum c_{ij} x^i y^j$; if $c_{ij} \neq 0 \xRightarrow[\mu(c) > k]{} i + jm > k \Rightarrow \boxed{i + jm - k \geq 0}$

$\operatorname{mult} h = m \Rightarrow \exists\, \tilde{h} \in K[x]$: $h(x) = -\tilde{h}(x) \cdot x^m$

Since ~~...~~ $0 = w(u,v) = \underbrace{h(u)}_{= -\tilde{h}(u) \cdot u^m} + v(1 + g(u,v)) \Rightarrow \boxed{\frac{v}{u^m} = \frac{\tilde{h}(u)}{1 + g(u,v)}}$

If $c_{ij} \neq 0 \Rightarrow \frac{u^i v^j}{u^k} = \left(\frac{v}{u^m}\right)^j \cdot u^{\overbrace{i+jm-k}^{}}_{\geq 0} \Rightarrow \left(\frac{c(x,y)}{x^k}\right)(u,v) =$

$$= \sum c_{ij} \frac{u^i v^j}{u^k} = \sum c_{ij} \left(\frac{v}{u^m}\right)^j u^{i+jm-k} = \sum c_{ij} \frac{\tilde{h}(u)^j\, u^{i+jm-k}}{(1 + g(u,v))^j}$$

put $l := \max\{j \mid c_{ij} \neq 0\}$ and $b := (1 + \underbrace{g(x,y)}_{\text{mult} \geq 1})^l \Rightarrow \text{mult}\, b = 0$

$\exists d \in K[x,y]$ mult $d \geq 1$: $\boxed{\text{Construct } d := \sum c_{ij}\, \tilde{x}(x)^j x^{i} \cdot (1+g(x,y))^{l-j}}$

such that $\left(\frac{c(x,y)}{x^r}\right)(u,w) = \frac{d}{b}(u,w)$; put $a := \lambda b + d$
$\Rightarrow$ mult $a = 0$

$$\Rightarrow \frac{r}{u^r} = \lambda + \frac{c(u,w)}{u^r} = \frac{\lambda b(u,w) + d(u,w)}{b(u,w)} = \frac{a(u,w)}{b(u,w)} = \frac{a}{b}(u,w)$$

Proposition 5.13 Let f be smooth at $x \in V_f(K)$ and $P \in \mathbb{P}_{L|K}$ satisfies $v_P(\alpha - x_1) > 0$, $v_P(\beta - x_2) > 0$. Then

(1) $\exists u \in P_x$: $v_P(u) = 1$ and $\forall r \in K[\alpha,\beta]$: $\frac{r}{u^{v_P(r)}} \in \mathcal{O}_x^*$

(2) $P = P_x$

Comment: So P_x are the only places satisfying $\alpha - x_1, \beta - x_2 \in P_x$ for x smooth, by 5.8

Proof: (1) We use transformation from 5.2 & 5.8 again
i.e. $f \mapsto w_\sigma$ $(\ell_\sigma(\sigma^{-1})^*(f) = w_\sigma)$, $(u,w) = \sigma(\alpha,\beta)$, $v_P(u) = 1$ by 5.5
Since $\text{Span}(u,w) = \text{Span}(\alpha - x_1, \beta - x_2)$ (cf. the proof of 5.8) we can extend the Observation (S) to (S'):

$$\boxed{{}_f\mathcal{O}_x = {}_{w_\sigma}\mathcal{O}_0 \qquad {}_fP_x = {}_{w_\sigma}P_0}$$

As $u = \lambda(u) \Rightarrow u \in {}_{w_\sigma}P_0 = {}_fP_x$

By 5.12: $\frac{r}{u^{v_P(r)}} \in {}_{w_\sigma}\mathcal{O}_0^* = {}_f\mathcal{O}_x^*$

(2) By 25 $\exists Q \in \mathbb{P}_{L|K}$: $P_x \subseteq Q$, $\mathcal{O}_x \subseteq \mathcal{O}_Q$; $\alpha - x_1, \beta - x_2 \in P$

$\underset{5.8}{\Rightarrow} P = Q \Rightarrow P_x \subseteq P$

we have to prove $P \subseteq P_x$ (using (1))

Let $r \in P \Rightarrow \exists r_1, r_2 \in K[\alpha,\beta]$ s.t. $r = \frac{r_1}{r_2}$

Then by (1) for $i = 1,2$ $\exists o_i \in \mathcal{O}_x^*$: $r_i = u^{v_P(r_i)} \cdot o_i$ $\Rightarrow r = \frac{r_1}{r_2} = \underbrace{\left(\frac{o_1}{o_2}\right)}_{\in \mathcal{O}_x^*} \cdot u^{v_P(r_1) - v_P(r_2)}$

$$\Rightarrow \underset{r \in P}{0 <} v_P(r) = \underbrace{v_P\left(\frac{o_1}{o_2}\right)}_{=0} + v_P(r_1) - v_P(r_2) \Rightarrow r = \underbrace{\frac{o_1}{o_2}}_{\in \mathcal{O}_x} \cdot \underbrace{u^{\overbrace{v_P(r)}^{>0}}}_{\in P_x} \in P_x$$

Example 5.14 Repeating 5.10: $f = y^2 + xy + x^5 + 32 \in \mathbb{R}[x,y]$

we have $(-2,2) \in V_f(\mathbb{R})$, $t_{(-2,2)}(f) = 82x + 2y + 160$

$\Rightarrow P = P_{(-2,2)} \in \mathbb{P}_{\mathbb{R}(\alpha,\beta)} = (\alpha + 2) = \{(\alpha+2) \cdot r(\alpha,\beta) \mid r \in R_{(-2,2)}\}$

$v_P(\alpha + 2) = 1$, $v_P(\beta - 2) \geq 0$ introduced by 210/215 $= \{(\alpha+2) \cdot \frac{h(\alpha,\beta)}{g(\alpha,\beta)} \mid h, g \in \mathbb{R}[x]\ g(-2,2) \neq 0\}$