proof of Lemma 5.18: (1) $\boxed{\text{Claim: } P_1, \dots, P_n \in \mathbb{P}_{L|K} \Rightarrow \exists s \in L^* \text{ such that } v_1(s) > 0 \ \& \ v_i(s) < 0 \ \forall i = 2, \dots, n \text{ where } v_i := v_{P_i}}$

Comment: $s \in P_1$, $s \notin O_{P_i}\ \forall i \geq 2$

20.4

by induction: step (a) $n=1$: $s \in P_1 \Rightarrow v_1(s) > 0$

step (b) $n=2$: O_1 & O_2 are distinct VR $\Rightarrow \exists a \in O_1 - O_2$, $\exists b \in O_2 - O_1 \Rightarrow$ $v_1(a) \geq 0$, $v_2(b) \geq 0$, $v_2(a) < 0$, $v_1(b) < 0 \Rightarrow v_1(\frac{a}{b}) = v_1(a) - v_1(b) > 0$, $v_2(\frac{a}{b}) < 0$

step (c) $n \geq 2$, the claim holds true for n $\rightsquigarrow$ we will prove it holds for $n+1$:

$\exists \tilde{s} \in L^*: v_1(\tilde{s}) > 0$, $v_i(\tilde{s}) < 0\ \forall i = 2, \dots, n$, if $v_{n+1}(\tilde{s}) < 0$, we are done, hence let $v_{n+1}(\tilde{s}) \geq 0 \overset{(b)}{\Longrightarrow} \exists r: v_1(r) > 0 \ \& \ v_{n+1}(r) < 0 \Rightarrow v_1(s + r^k) \overset{(OV2)}{\geq} \min(\overset{>0}{v_1(s)}, \overset{>0}{k \cdot v_1(r)}) > 0$

by Observ.(3) $v_i(s + r^k) < 0$ for $i \geq 2$ and enough large k as $v_{n+1}(r^k) = k \cdot v_{n+1}(r) < 0$, $v_i(s) < 0\ \forall i = 2, \dots, n$

(2) claim: $\boxed{P_1, \dots, P_n \in \mathbb{P}_{L|K}, v_i := v_{P_i}, h \in \mathbb{Z}, a_1, \dots, a_n \in L \Rightarrow \exists s \in L : v_i(s - a_i) > h \ \forall i = 1, \dots, n}$

Let $k \in \mathbb{Z}$

Put $r_1 := (1 + s^k)^{-1}$ where $s \in L^*$ is transcendental (Observ.(2)): $v_1(s) > 0$, $v_i(s) < 0\ \forall i \geq 2$ which exists by (1)

$\Rightarrow \boxed{v_1(r_1 - 1) = v_1\left(\frac{s^k}{1 + s^k}\right) = k v_1(s) - \underbrace{v_1(\overset{\in O_1^*}{1 + s^k})}_{=0} = k v_1(s) \geq k}$ and

$\boxed{v_i(r_1) = -v_i(1 + s^k) \overset{2.13}{=} -k \cdot \underset{<0}{v_i(s)} \geq k}\ \forall i \geq 2$

By the same way we can define $r_2, \dots, r_n$ such that:

$\forall i = 1, \dots, n\ \forall j \neq i\ v_i(r_j) \geq k \ \& \ v_i(r_i - 1) \geq k$ $\boxed{\text{Put } s := \sum_{i=1}^n a_i r_i}$

we have for $i \neq j$ $v_i(a_j r_j) = v_i(a_j) + \overset{\geq k}{v_i(r_j)} \geq v_i(a_j) + k$

$v_i(a_i(r_i - 1)) = v_i(a_i) + \underbrace{v_i(r_i - 1)}_{\geq k} \geq v_i(a_i) + k$

if $k > h - v_i(a_j)\ \forall i, j \Rightarrow v_i(s - a_i) = v_i\left(\sum_{j \neq i} a_j r_j + a_i(r_i - 1)\right) > h$

T&N If W is a subspace of a K-space V, $B \subset V$, we say that B is LI / a basis modulo W if $\{b + W \mid b \in B\}$ is a LI set / basis of the factor V/W

Comment: We formulate consequences of Theorem 5.19

Corollary 5.20 (1) $\mathbb{P}_{L|K}$ is infinite

(2) If $P, P_1, \dots, P_n \in \mathbb{P}_{L|K}$ are pairwise distinct, and $\ell \geq 0$ then $\exists$ a basis B of the K-algebra O_P modulo P such that $B \subseteq P_i^\ell$ (i.e. $v_{P_i}(b) \geq \ell\ \forall b \in B$)

proof: (1) ?? $\mathbb{P}_{L|K} = \{P_1, \dots, P_n\}$ is finite $\overset{5.19}{\Longrightarrow} \exists s \in L^*$:

$v_{P_i}(s) = 1\ \forall i = 1, \dots, n \underset{s \text{ transcendental}}{\Longrightarrow} (s^{-1}) \subsetneq K[s^{-1}] \subsetneq L \overset{2.5}{\Longrightarrow} \exists Q \in \mathbb{P}_{L|K}: s^{-1} \in Q \Rightarrow v_Q(s) \leq -1$ ↯

(2) Let $d := \deg P = \dim_K(\mathcal{O}_P/P)$ and $\{e_1, \ldots, e_d\}$ be a K-basis of $\mathcal{O}_P$ modulo P.

We apply 5.19 $\forall i = 1 \ldots n$ (i.e. n times) where we put $a_0 = 0$, $a_1 = \ldots = a_n = e_i$ $\Rightarrow \exists s_i \in L : v_{P_j}(s_i - e_i) = \ell \ \forall j$, $v_P(s_i) = 1$

Then $b_i = s_i - e_i \in P_j^\ell \ \forall j \geq 1$ and $s_i \in P \Rightarrow$
$\{b_1 + P, \ldots, b_d + P\} = \{-e_1 + P, \ldots, -e_d + P\}$ is a basis of $\mathcal{O}_P/P$ over K.

Comment: Since $v_P(a) = \ell \Leftrightarrow a \in P^\ell \setminus P^{\ell+1}$ we can describe a LI set modulo a place P using DV v_P:

<u>Observation</u> Let $P \in \mathbb{P}_{L/K}$, $b_1, \ldots, b_n \in \mathcal{O}_P$ be a LI set modulo P over the field K, $s \in P$ such that $v_P(s) = 1$ (i.e. $s \in P \setminus P^2$) and $\lambda_i, \lambda_{ij} \in K$ such that $\exists i, \exists ij : \lambda_i \neq 0, \lambda_{ij} \neq 0$, $e \in \mathbb{N}$. Then:

(1) $v_P(\sum_i \lambda_i b_i) = 0$ (since $\sum \lambda_i b_i \in \mathcal{O}_P \setminus P$)

(2) $v_P(\sum_i \lambda_i b_i s^j) = \overbrace{v_P(\sum \lambda_i b_i)}^{=0} + v_P(s^j) = j$ (by (DV1) & (1)) $\forall j$

(3) $v_P(\sum_{i,j} \lambda_{ij} b_i s^j) = \min\{j \mid \exists i : \lambda_{ij} \neq 0\}$ by (2) & 2.13

(4) the set $\{b_i s^j \mid i = 1, \ldots, n \ \& \ j = 0, \ldots, e-1\}$ is LI modulo P^e over K by (3).

<u>Proposition 5.21</u> Let $P_1, \ldots, P_n \in \mathbb{P}_{L/K}$ be pairwise distinct, $v_i := v_{P_i}$.
If $s \in \bigcap_{i=1}^n P_i$ (i.e. $v_i(s) \geq 1 \ \forall i$) then $[L : K(s)] \geq \sum_{i=1}^n v_i(s) \deg P_i$.

~~Observation~~ Comment: In an AFF $L \ \forall s \in L \setminus \tilde{K}$ $[L : K(s)] < \infty$ and it is easily computable, so we have a strong restriction on number of the places containing s!

<u>Proof</u>: Put $\mathcal{O}_i := \mathcal{O}_{P_i}$, $d_i := \deg P_i$, $e_i := v_i(s)$, $e := \max\{e_i \mid i = 1 \ldots n\}$

By 5.20(2) $\forall i \ \exists B_i = \{b_{i1}, \ldots, b_{id_i}\}$ a K-basis of $\mathcal{O}_i$ modulo P_i such that $\forall i, j, k : k \neq j \ v_k(b_{ij}) \geq e$

By 5.19 $\forall i \ \exists s_i \in P_i : v_i(s_i) = 1 \ \& \ v_k(s_i) = 0 \ \forall k \neq i$

Put $\tilde{B}_i = \{b_{ij} s_i^k \mid j = 1, \ldots, d_i, \ k = 0, \ldots, e_i - 1\} \ \forall i = 1 \ldots n$

$B := \bigcup_{i=1}^n \tilde{B}_i$. RcF B is LI over $(K(s)) \Rightarrow \sum_{i=1}^n \underbrace{v_i(s)}_{=e_i} \underbrace{\deg P_i}_{=d_i} = |B| \geq [L : K(s)]$

?? B is LD over $K(s) \overset{1.7}{\Longrightarrow} \exists \lambda_{ijk} \in K, \exists a_{ijk} \in K[s] : \exists i, j, k \ \lambda_{ijk} \neq 0$ &

$\sum_{i=1}^n \boxed{\sum_{j,k} (s a_{ijk} + \lambda_{ijk}) b_{ij} s_i^k} = 0 \Rightarrow \sum_{i=1}^n c_i = 0 \Rightarrow v_i(c_i) = v_i(-\sum_{k \neq i} c_k) \geq e \geq e_i \ \forall i$

put c_i

$\Rightarrow \sum b_{ij} s^k (\underbrace{s a_{ijk}}_{\in P_i^{e_i}} + \lambda_{ijk}) = c_i \in P_i^{e_i} \overset{s \in P_i^{e_i}}{\Longrightarrow} \sum \lambda_{ijk} b_{ij} s_i^k \in P_i^{e_i} \underset{\text{Observ.(4)}}{\Longrightarrow} \lambda_{ijk} = 0 \ \forall ijk$

a contradiction