$\boxed{L \text{ is an AFF over } K \text{ and } \tilde{K} \text{ is a field of constants}}$

Errata: $A \in \mathrm{Div}(L|K)$ we define $\boxed{A_- := -\min(A, \underline{0})} = (-A)_+ = A_+ - A$

Comment: both A_+ and A_- are positive divisors such that $A = A_+ - A_-$, the original definition has the opposite sign.

Lemma 6.2 If $A, B \in \mathrm{Div}(L|K)$ such that $A \leq B$, then $\mathcal{L}(A)$ is a K-subspace of the K-space $\mathcal{L}(B)$ and $\dim_K(\mathcal{L}(B)/\mathcal{L}(A)) \leq \deg_K(B-A)$

Comment: $\dim(\mathcal{L}(B)/\mathcal{L}(A)) = \dim(\mathcal{L}(B)) - \dim(\mathcal{L}(A))$ and $\deg(B-A) = \deg B - \deg A$

Proof: Let $A = \sum_{P \in \mathbb{P}_{L|K}} a_P P$, $B = \sum_{P \in \mathbb{P}_{L|K}} b_P P$

$r \in \mathcal{L}(A) \Rightarrow (r) + B \underbrace{\geq}_{\text{Obser. B(4)}} (r) + A \geq 0 \Rightarrow r \in \mathcal{L}(B) \Rightarrow \mathcal{L}(A) \subseteq \mathcal{L}(B)$

Remind: $r \in \mathcal{L}(B) \Leftrightarrow (r) \geq -B \Leftrightarrow v_P(r) \geq -b_P \ \forall P \in \mathbb{P}_{L|K}$ & $r \in \mathcal{L}(A) \Leftrightarrow v_P(r) \geq -a_P \ \forall P$

Define $\varphi: \mathcal{L}(B) \to \prod_{P \in \mathbb{P}_{L|K}} P^{-b_P} := \prod p^{-b_P} \mathcal{O}_P$ where $(p) = P$

Comment: $P^x := \{a \in L^* \mid v_P(a) \geq x\} = (p^x) = \{p^x \cdot u \mid u \in \mathcal{O}_P\}$ - it is a cyclic module over $\mathcal{O}_P$

by $\varphi(r) = \prod_P r$ - the constant in all coordinates (i.e. $\varphi(r)_P = r$)

Then $\varphi(r) \in \prod_P P^{-a_P} \Leftrightarrow v_P(r) \geq -a_P \ \forall P \Leftrightarrow r \in \mathcal{L}(A)$ & φ is K-linear

$1.2 \Rightarrow \exists$ injective K-linear $\mathcal{L}(B)/\mathcal{L}(A) \hookrightarrow \prod P^{-b_P} / \prod P^{-a_P} \cong \prod_P P^{-b_P}/P^{-a_P}$

Observation C(3) $\Rightarrow \dim_K P^{-b_P}/P^{-a_P} = (b_P - a_P)\deg P \geq 0 \Rightarrow \dim \mathcal{L}(B)/\mathcal{L}(A) \leq \underbrace{\sum (b_P - a_P)\deg P}_{= \deg(B-A)}$

Proposition 6.3 Let $K = \tilde{K}$ (i.e. L is a full constant AFF) and $A, B \in \mathrm{Div}(L|K)$

Comment: We formulate "basic" properties of the ordering $\leq$ on $\mathrm{Div}(L|K)$!

(D1) if $A \geq 0 \Rightarrow 1 \leq \ell(A) \leq \deg A + 1$

(D2) if $A < 0 \Rightarrow \ell(a) = 0$

Comment: Riemann-Roch space of any "negative" divisor $= 0$

(D3) $\ell(A) \leq \ell(A_+) < \infty$

Comment: Hence each Riemann-Roch space $\mathcal{L}(A)$ is finitely generated!

(D4) if $A \leq B \Rightarrow \deg A - \ell(A) \leq \deg B - \ell(B)$

Proof: (D1) $0 \leq \dim(\mathcal{L}(A)/\mathcal{L}(\underline{0})) = \ell(A) - \overbrace{\ell(\underline{0})}^{=1} = \ell(A) - 1 \Rightarrow \boxed{\ell(A) \geq 1}$

$6.2 \Rightarrow \ell(A) - 1 = \dim(\quad " \quad) \leq \deg A - \deg \underline{0} = \deg A$, $\Rightarrow \boxed{\ell(A) \leq \deg A + 1}$

(D2) $A < 0 \Rightarrow \mathcal{L}(A) \subseteq \mathcal{L}(\underline{0}) = \tilde{K} = K$, if $s \in K^* \Rightarrow v_P(s) = 0 \ \forall P \in \mathbb{P}_{L|K} \Rightarrow (s) = \underline{0}$

$\Rightarrow A + (s) < 0 \Rightarrow \mathcal{L}(A) = \{\underline{0}\}$

(D3) $\mathcal{L}(A) \subseteq \mathcal{L}(A^+) \Rightarrow \ell(A) \leq \ell(A_+) \overset{(D1)}{\leq} \deg A_+ + 1 < \infty$ as $\deg P < \infty \ \forall P$

(D4) Lemma 6.2 & (D3) $\Rightarrow \overbrace{\ell(B)}^{\text{finite}} - \ell(A) = \dim(\mathcal{L}(B)/\mathcal{L}(A)) \leq \deg B - \deg A$

$\Rightarrow \deg B - \ell(B) \leq \deg A - \ell(A)$

Lemma 6.4 If $s \in L - \tilde{K}$, then $\exists B \in \mathrm{Div}(L|K)$ such that $B \geq \underline{0}$ & $\forall k \geq 0$:

(1) $(k+1)[L:K(s)] \leq \ell(k \cdot (s)_- + B)$

(2) $\quad " \quad \leq k \cdot \deg_K((s)_-) + \deg_K B + 1$

(3) $k \cdot [L:K(s)] - \ell(k \cdot (s)_-) \leq \deg_K B - [L:K(s)]$

Comment: "Ugly" technical lemma. Moving that $[L:K(s)] \approx \ell(k(s)_-)$ for $k \to \infty$.

Proof of 6.4: (1) Let $l_1,\dots,l_n \in L$ be a $K(s)$-basis of L $\overset{1.9.}{\Longrightarrow}$
$\Rightarrow E_k := \{l_i s^j \mid i=1,\dots,n,\ j=0,\dots,k\}$ is LI over K (since $K(s)\cong K(x)$) and we find $B=\sum_{P\in\mathbb{P}_{L|K}} b_P P \in \mathrm{Div}(L|K)$ such that $E_k \subseteq \mathcal{L}(k(s)_- + B)$ $\forall k\ge 0$

Define $b_P := \max\{0, -v_P(l_1),\dots,-v_P(l_n)\}$ $\forall P \Rightarrow b_P + v_P(l_i) \ge 0$ $(*)$ & $b_P\ge 0$ $\forall P$

$\{P\in\mathbb{P}_{L|K} \mid b_P\ne 0\} \subseteq \bigcup_{i=1}^{n} \{P\in \mathbb{P}_{L|K} \mid v_P(l_i)\ne 0\}$ - is finite by 5.22. $\Rightarrow B=\sum b_P P \in \mathrm{Div}(L|K)$

Note that $l_i s^j \in \mathcal{L}(k(s)_-+B) \overset{\text{by definition}}{\iff} D=\sum d_P P = B + (l_i s^j) + k(s)_- \ge 0$ $(**)$

- we will check coefficients d_P of D, let $P\in\mathbb{P}_{L|K}$:

if $v_P(s)<0 \Rightarrow d_P = \underbrace{b_P+v_P(l_i)}_{\ge 0 \text{ by } (*)} + j v_P(s) - k v_P(s) \ge \overbrace{(j-k)}^{\le 0}\overbrace{v_P(s)}^{<0} \ge 0$

if $v_P(s)\ge 0 \Rightarrow d_P = b_P + v_P(l_i) + j v_P(s) \ge j v_P(s) \ge 0$ $(j\le k)$ $\quad \ell(k(s)_-+B)\ge |E_k|$

(2) $(k+1)[L:K(s)] \overset{(1)}{\le} \ell(\underbrace{k(s)_-+B}_{\ge 0}) \le \deg(k(s)_-+B)+1 = k\cdot\deg((s)_-)+\deg B+1$ (by 6.3 (D1)) $\quad (k+1)[L:K(s)]$

(3) $B\ge 0 \overset{\text{Obs. B(4)}}{\Longrightarrow} k(s)_- \le k(s)_-+B \overset{6.2}{\Longrightarrow} \ell(k(s)_-+B) - \ell(k(s)_-) \le$
$\le \deg(\text{— " —}) - \deg(\text{— " —})$

$\Rightarrow (k+1)[L:K(s)] - \ell(k(s)_-) \le \deg(k(s)_-+B) - \deg(k(s)_-) = \deg B \quad \Big|\, -[L:K(s)]$

$\Rightarrow \quad k[L:K(s)] - \ell(k(s)_-) \le \deg B - [L:K(s)]$

Comment: 6.5 shows that the approximation of 5.21 is the best possible. It allows to compute number of places containing a transcendental element (for an algebraic it is clear: all places contain it)

Theorem 6.5 Let $K=\tilde{K}$ and $s\in L\setminus\tilde{K}$. Then $\deg((s)_-)=\deg((s)_+)=[L:K(s)]$ and $\deg((s))=0$

Proof (a) $\deg((s)_+) = \sum_{P:\, s\in P} v_P(s)\deg P \le [L:K(s)]$ $\boxed{5.21}$

(b) $K(s)=K(s^{-1})$ & $(s)_- = (s^{-1})_+ \Rightarrow \boxed{\deg((s)_-) = \deg((s^{-1})_+) \le [L:K(s^{-1})]=[L:K(s)]}$

(c) 6.4(2) $\Rightarrow \exists B\in\mathrm{Div}(L|K)$ such that $\forall k\ge 0$: $(k+1)[L:K(s)] \le k\deg((s)_-)+\deg B+1 \Rightarrow$

$\Rightarrow [L:K(s)] \le \deg((s)_-) + \frac{\deg B+1-[L:K(s)]}{k} \to \deg((s)_-)$ for $k\to\infty$

$\overset{(b)}{\Rightarrow} [L:K(s)] = \deg((s)_-)$ using also (a)

Comment: Here it is used the limit argument to show that $[L:K(s)]\le\deg((s)_-)$

(d) (b)&(c) $\Rightarrow \deg((s)_+) = \deg((s^{-1})_-) = [L:K(s^{-1})] = [L:K(s)]$

Finally: $\deg((s)) = \deg((s)_+ - (s)_-) = \deg((s)_+) - \deg((s)_-) = 0$

Corollary 6.6 If $A\sim B$, then (1) $\deg A = \deg B$ (2) $\dim_{L|K} A = \dim_{L|K} B$

Proof: $A\sim B \iff \exists s\in L^*: A=B+(s)$ and

(1) $\deg A = \deg(B+(s)) = \deg B + \deg((s)) = \deg B$ as $\deg((s))=0$ by 6.5

(2) define mapping $\mathcal{L}(A)\to\mathcal{L}(B)$ by $r\mapsto s\cdot r$; as $r\in\mathcal{L}(A) \iff (r)+A\ge 0 \iff$
$\iff (r)+(s)+B\ge 0 \iff (r\cdot s)+B\ge 0$ it is bijective; clearly, it is linear $\Rightarrow$ it is an isomorphism

Comment 6.6 shows that deg, dim will work on $Cl(L|K) = \mathrm{Div}(L|K)/\mathrm{Princ}(L|K)$

Example 6.7 Consider the AFF L over $\mathbb{F}_2$ from S.24 which is given by $f(\alpha,\beta)=0$ for $f = y^2+y-(x^3+1) \in \mathbb{F}_2[x]$, we will compute principal divisors $(\alpha+1)$ and (α)!

(a) $\deg((\alpha+1)_\varepsilon) = \sum_{P:\alpha+1\in P} \underbrace{v_P(\alpha+1)}_{\geq 1} \underbrace{\deg P}_{\geq 1} \overset{6.5}{=} [L : \overbrace{\mathbb{F}_2(\alpha+1)}^{=\mathbb{F}_2(\alpha)}] = \underline{2}$,

we know that $\alpha+1 \in P_{(1,0)}, P_{(1,1)} \in \mathbb{P}_{L|\mathbb{F}_2} \Longrightarrow \deg P_{(1,0)} = \deg P_{(1,1)} = 1$

$\Rightarrow \mathbb{P}^{(1)}_{L|\mathbb{F}_2} := \{P_{(1,0)}, P_{(1,1)}, P_\infty\}$ is the complete set of places of degree 1

$\Rightarrow \boxed{(\alpha+1) = 1\cdot P_{(1,0)} + 1\cdot P_{(1,1)} - 2\cdot P_\infty} \in \mathrm{Div}(L|\mathbb{F}_2)$

(b) $\deg((\alpha)_\varepsilon) = \sum_{P:\alpha\in P} v_P(\alpha+1) \deg P \overset{6.5}{=} [L : \mathbb{F}_2(\alpha)] = 2$ & $\alpha \notin P_{(1,0)}, P_{(1,1)}, P_\infty$

$\Longrightarrow \exists!\, P_{\alpha} \in \mathbb{P}_{L|\mathbb{F}_2}$ of degree $=2$ containing $\alpha \Rightarrow (\alpha) = 1\cdot P_{\alpha} - 2P_\infty$

Proposition 6.8 Let $k=\tilde{k}$ and $A, B \in \mathrm{Div}(L|k)$, then

(D5) $\ell(B-A) \geq 1 \Leftrightarrow \exists A' \in \mathrm{Div}(L|k) : A \sim A' \leq B$

(D6) if $\ell(B-A) \geq 1 \Rightarrow \deg A - \ell(A) \leq \deg B - \ell(B)$

(D7) $\ell(A) \geq 1 \Leftrightarrow \exists s \in L^* : A + (s) \geq 0$

(D8) if $\deg A < 0 \Rightarrow \ell(a) = 0$

(D9) $\mathcal{L}(s) = k s^{-1} = \{r s^{-1} \mid r \in k\}\ \forall s \in L^*$

Comment: We complete properties of 6.3 by those on dimensions of Riemann-Roch spaces over $\mathrm{Div}(L|k)$

Proof: (D5) $\ell(B-A) \geq 1 \Leftrightarrow \exists s \in L^* : (s) + B - A \geq 0 \Leftrightarrow \exists s \in L^* : B \geq A - (s)$
$\Leftrightarrow \exists A' \sim A : A' \leq B$

(D6) $(D5) \Rightarrow \exists A' \sim A : A' \leq B \overset{6.3(D4)}{\Longrightarrow}$

$\Rightarrow \underline{\deg A - \ell(A)} \overset{6.6}{=} \deg A' - \ell(A') \underline{\leq \deg B - \ell(B)}$

(D7) $\ell(A) \geq 1 \Leftrightarrow \exists s \in \mathcal{L}(A) - \{0\} \Leftrightarrow \exists s \in L^* : A + (s) \geq 0$

(D8) $\deg A < 0 \Rightarrow \deg(A + (s)) \overset{6.5}{=} \deg A < 0 \ \forall s \in L^* \Rightarrow$
$\Rightarrow \mathcal{L}(A) = 0$

(D9) $r \in \mathcal{L}((s)) \Leftrightarrow (r\cdot s) = (r) + (s) \geq 0 \overset{6.3\, B(6)}{\Longleftrightarrow} rs \in k \Leftrightarrow r \in k s^{-1}$