Recall: $\deg : Div(L|k) \to \mathbb{Z}$ is a group homomorphism and $Princ(L|k) \overset{(8.5)}{\leq} \ker(\deg) \leq Div(L|k)$ (normal subgroups)

T&N $Pic^0(L|k) := \ker(\deg)/Princ(L|k)$ is called the Picard group

Denote $[A] := A + Princ(L|k) \in Pic^0(L|k)$ (the coset given by A)

Comment: We will "copy" the structure of $Pic^0(L|k)$ to the curve $V_w(k) \cup \{\infty\}$ using the sets $\mathbb{P}^{(1)}_{L|k}$ for L EFF given by smooth WEP w

Lemma 8.6 Let L be an EFF over k, $P_1, P_2, Q \in \mathbb{P}^{(1)}_{L|k}$ and $A \in Div(L|k)$.

Comment: (1) proves injectivity and (2) surjectivity of ψ_Q which allows us to translate group structure of $Pic^0(L|k)$ to $\mathbb{P}^{(1)}_{L|k}$

(1) if $P_1 - P_2 \in Princ(L|k) \Rightarrow P_1 = P_2$,

(2) if $\deg A = 1 \Rightarrow \exists! P \in \mathbb{P}^{(1)}_{L|k} : P - A \in Princ(L|k)$

(3) the mapping $\psi_Q : \mathbb{P}^{(1)}_{L|k} \to Pic^0(L|k)$ defined by $\psi_Q(P) := [P-Q]$ is a bijection

Proof: Note that $\deg A \geq 1 \overset{3.6(2)}{\Rightarrow} \ell(A) = \deg(A)$

(1) $\exists z \in L\setminus k : P_2 + (z) = P_1 \geq 0 \overset{\text{by def.}}{\Rightarrow} 1, z \in \mathcal{L}(1P_2)$ are $L|k$ $\Rightarrow \ell(1P_2) = \dim_k(\mathcal{L}(1P_2)) \geq 2$ $\Rightarrow$ a contradiction with $\ell(1P_2) = 1$ $\Rightarrow P_1 = P_2$

(2) ∃ (a) if $A \geq 0, \deg A = 1 \Rightarrow \exists P \in \mathbb{P}^{(1)}_{L|k} : A = 1P$ ✓

(b) Let A is general: $\deg A = 1 \Rightarrow \exists z \in L^* : z \in \mathcal{L}(A) \overset{\text{by def.}}{\Rightarrow} A + (z) \geq 0$
$\overset{(a)}{\Rightarrow} \exists P \in \mathbb{P}^{(1)}_{L|k} : A + (z) = P \Rightarrow P - A = (z) \in Princ(L|k)$

Uniquely: if $A - P_1, A - P_2 \in Princ(L|k) \Rightarrow P_1 - P_2 \in Princ(L|k) \overset{\text{def}}{\Rightarrow} P_1 = P_2$ ✓

(3) Injectivity: if $\psi_Q(P_1) = \psi_Q(P_2) \Rightarrow [P_1 - P_2] = [0] \overset{(1)}{\Rightarrow} P_1 = P_2$

Surjectivity: if $B \in Div(L|k) : \deg B = 0 \Rightarrow \deg(Q + B) = 1 \overset{(2)}{\Rightarrow} \exists! P \in \mathbb{P}^{(1)}_{L|k}$ such that $P - (B+Q) \in Princ(L|k) \Rightarrow \psi_Q(P) = [P-Q] = [B]$.

T&N Let L be an EFF, $Q \in \mathbb{P}^{(1)}_{L|k}$, we define an operation $\oplus$ on $\mathbb{P}^{(1)}_{L|k}$:

$P_1 \oplus P_2 := \psi_Q^{-1}(\psi_Q(P_1) + \psi_Q(P_2))$ for $P_1, P_2 \in \mathbb{P}^{(1)}_{L|k}$ and ψ_Q from 8.6(3)

Observation Let L be an EFF over k; $Q, P_0, P_1, \dots, P_n \in \mathbb{P}^{(1)}_{L|k}$, ψ_Q from 8.6(3)

(1) $\mathbb{P}^{(1)}_{L|k}$ forms an abelian group with the operation $\oplus$ and neutral element Q

(2) ψ_Q is a group isomorphism,

(3) $P_1 \oplus P_2 = P_3 \Leftrightarrow [P_1 + P_2] = [P_3 + Q]$.

(4) $P_1 \oplus \dots \oplus P_n = P_0 \Leftrightarrow P_0 + (1-n)Q + \sum_{i=1}^{n} P_i \in Princ(L|k)$.

Comment: The group of $\mathbb{P}^{(1)}_{L|k}$ is defined in order for ψ_Q to be a homomorphism. $\ominus P_1 = P_2$ such that $P_1 \oplus P_2 = [0] \Leftrightarrow [P_2] = [Q - P_1]$

T&N Let $\hat{\ell} = cx + dy + e \in k[x,y]$ for $c, d, e \in k$ such that $(c,d) \neq (0,0)$. Then $\ell = \hat{\ell} + (w) = \hat{\ell}(\alpha, \beta) \in k[V_w] = k[\alpha, \beta]$ where $\alpha := x + (w)$, $\beta := y + (w)$ is called a line on V_w represented by $\hat{\ell}$, we say that ℓ passes through $\gamma \in V_w$ if $\gamma \in V_{\hat{\ell}}$.

Lemma 8.7 Let $w = y^2 + a_1xy + a_3y - \overbrace{(x^3 + a_2x^2 + a_4x + a_6)}^{r(x)}$ is a smooth WEP at $V_w(k)$, $\gamma = (\gamma_1, \gamma_2) \in V_w(k)$ and $\hat{\ell} \in k[x,y]$ represents a line $\ell = \hat{\ell} + (w) \in k[V_w]$. Then

(1) if $\hat{\ell} = x - \gamma_1 \Rightarrow \exists! \delta = (\gamma_1, \delta_1) \in V_w(k) : (\ell) = P_\gamma + P_\delta - 2P_\infty$ and $\boxed{\delta_2 = -a_1\gamma_1 - a_3 - \gamma_2}$

(2) if $\hat{\ell} = y - \lambda x - \mu$ for $\lambda, \mu \in k$ and ℓ passes through ∞, then $(\ell)_- = 3P_\infty$ and either

Comment: We describe possible intersections of lines and the curve

(a) $\exists P \in \mathbb{P}^1_{/k}$: $\deg P = 2$, $(\ell)_+ = P_\chi + P$, $\hat{\ell} \notin (A_\chi(w))$, and $V_w(k) \cap V_{\hat{\ell}} = \{\chi\}$

or

(b) $\exists \delta = (\delta_1, \delta_2), \eta = (\eta_1, \eta_2) \in V_w(k)$: $(\ell)_+ = P_\chi + P_\delta + P_\eta$, $V_w \cap V_{\hat{\ell}} = \{\chi, \delta, \eta\}$, $\eta_i = \chi_1 - \delta_1 - a_2 + \lambda^2 + a_1\lambda$, and $\hat{\ell} \in (A_\chi(w)) \iff \chi \in \{\delta, \eta\}$.

Proof: Recall that we denote $\alpha := x_\epsilon(w)$, $\beta := y_\epsilon(w)$, $\ell = \hat{\ell}(\alpha, \beta) \in k[V_w] = k[\alpha, \beta] \subset k(\alpha, \beta)$

(1) By 8.3: $(\ell)_- = (\alpha - \chi_1)_- \overset{\text{as } v_P(\chi_1) = 0 \ \forall P \neq \infty}{=\!=} (\alpha)_- \overset{8.3(3)}{=\!=} 2P_\infty$

by 5.8: $P_\chi \le (\ell)_+$ as $\hat{\ell}(\chi) = 0$; by 8.3(e): $\deg P_\chi = 1$

$\overset{8.3(e)}{\Longrightarrow}$ $\exists! \delta = (\delta_1, \delta_2) \in V_w(k)$: $P = P_\delta \Rightarrow (\ell)_+ = P_\chi + P_\delta$

$\overset{G5}{\Rightarrow} \exists! P \in (\mathbb{P}^1_{/k})^{(1)}$: $(\ell)_+ = P_\chi + P$

Comment: Note that $P_\chi \le (\ell)_+$ exactly means that $\chi \in V_{\hat{\ell}}$!

Since $w(\chi) = 0 = w(\delta)$: $\exists \lambda \in k$ such that δ_2, χ_2 are roots of $\underbrace{\lambda^2 + (a_1\chi_1 + a_3)\lambda + \lambda}_{g(\lambda)} \in k[\lambda]$

$\Rightarrow \boxed{\delta_2 + \chi_2 = -a_1\chi_1 - a_3}$ (as $g(\lambda) = (\lambda - \delta_2)(\lambda - \chi_2)$ comparing coefficients of λ)

(2) Again by 8.3(4) & 5.8 we have $(\ell)_- = (3P_\infty)$ and $P_\chi \le (\ell)_+$, then by G5

- either (a) $(\ell)_+ = P_\chi + P$ for $P \in \mathbb{P}^1_{/k}$ of degree 2 $\Rightarrow P_\delta \notin (\ell)_+ \ \forall \delta \in V_w(k) - \{\chi\}$

$\Rightarrow V_w(k) \cap V_{\hat{\ell}} = \{\chi\}$

Comment: The case (a) describes lines with only one rational intersection with the curve V_w!

- or (b) $(\ell)_+ = P_\chi + P_\delta + P_\eta$ for $\delta, \eta \in V_w(k)$

Note that $\hat{\ell} \in (A_\chi(w)) \overset{5.8}{\iff} 2P_\chi \le (\ell)_+ \iff \chi \in \{\delta, \eta\}$

Put $\delta = (\delta_1, \delta_2)$, $\eta = (\eta_1, \eta_2)$

↖ this is impossible in (a) $\Rightarrow$ in (a) $\hat{\ell} \notin (A_\chi(w))$

Let $w = y^2 + a_1xy + a_3y - f(x)$ where $f(x) = x^3 + a_2x^2 + a_4x + a_6$

Suppose (b): $(\ell)_+ = P_\chi + P_\delta + P_\eta$ and put $g(\lambda) = -w(\lambda_1, \lambda\lambda + \mu) = f(\lambda) - (\lambda\lambda + \mu)^2 - a_1(\lambda\lambda + \mu) - a_3(\lambda\lambda + \mu) \in k[\lambda]$

if $\rho \in \underbrace{\{\chi_1, \delta_1, \eta_1\}}_{R:=}$ or $(\rho, \sigma) \in V_w \cap V_{\hat{\ell}} \Rightarrow g(\rho) = 0$, i.e. ρ is a root of g & $\deg g = 3$

on the other hand by 8.9 $\Rightarrow$ the multiplicity of the root $\rho \in \{\chi_1, \delta_1, \eta_1\}$ is

or since $|R| = 3$

(if $|R| < 3 \Rightarrow \exists \rho \in R$: $\hat{\ell} \in (A_\rho(w)) \Rightarrow \frac{\partial w}{\partial \sigma}(\rho) \neq 0$) equal to $v_P(\ell)$ for the corresponding $P \in \{P_\chi, P_\delta, P_\eta\}$

$\Rightarrow \{\chi_1, \delta_1, \eta_1\}$ are exactly all roots of the monic polynomial $g(\lambda)$ (with multiplicities)

$\Rightarrow$ comparing the coefficients of λ^2 of g: $\boxed{-(\alpha_1 + \beta_1 + \chi_1) = a_2 - \lambda^2 - a_1\lambda}$

Definition Let w be a smooth WEP and L be an EFF.

Consider the group structure on $(\mathbb{P}^1_{/k})^{(1)}$ determined by Ψ_{P_∞} from 8.6(3).

Put $E(k) := V_w(k) \cup \{\infty\}$ and define operations $\oplus$ and $\ominus$ on $E(k)$:

$\boxed{\chi \oplus \delta = \eta \overset{\text{def}}{\iff} P_\chi \oplus P_\delta = P_\eta \overset{8.6(3)}{\iff} [P_\chi + P_\delta] = [P_\eta + P_\infty]}$, $\boxed{\ominus \chi = \delta \overset{\text{def}}{\iff} P_\chi \oplus P_\delta = P_\infty}$