Proposition 9.2 Let $H, F \in K[X_0, X_1, X_2]$, F irreducible

(1) Then either $H \in (F)$ and $H(a) = 0 \; \forall a \in V_F$ or $H \notin (F)$ and $|V_F \cap V_H| < \infty$

(2) If $X_j \notin (F)$ for a $j \in \{0,1,2\}$ ($\Leftrightarrow F \notin (X_j)$) $\Rightarrow |\{(a_0 : a_1 : a_2) \in V_F \mid a_j = 0\}| < \infty$

Comment: We recall well-known fact that the intersections of "independent" projective curves is finite. We prove it using "affine" observation 4.4(3)

Proof: Put $d := \deg F$ and denote $\hat{V}_f := \{\hat{b} \in \mathbb{P}^2 \mid b \in V_f\}$ for $f \in K[X_1, X_2]$

(1) Suppose that $H \notin (F) \Rightarrow d \geq 1$

(a) if $F \notin (X_0) \Rightarrow \deg(\pi_0(F)) = d \Rightarrow \widehat{\pi_0(F)} = F$, if $F \in (X_0) \overset{F\text{-irreducible}}{\Longrightarrow} \exists \lambda \in K^* : F = \lambda X_0 \Rightarrow$

$\Rightarrow F \notin (X_1) \cup (X_2) \Rightarrow$ we may switch X_i's to, w.l.o.g suppose $\boxed{F \notin (X_0),\ \hat{f} = F \text{ for suitable } f \in K[X_1, X_2]}$

(b) Put $G := F(0, X_1, X_2) \in K[X_1, X_2] \Rightarrow \deg G = d > 0 \Rightarrow$

$\Rightarrow$ either $\deg_{X_1} G > 0$ or $\deg_{X_2} G > 0 \Rightarrow$ we may suppose w.l.o.g $\boxed{\deg_{X_1} G > 0}$

Then $|\{\lambda \in K \mid G(\lambda : 1) = 0\}| < \infty$ & $(0 : a_1 : 1) \in V_F \Leftrightarrow G(a_1 : 1) = 0$ & $|\{(0 : a_1 : 0)\}| = 1$

$\Rightarrow V_F \setminus \hat{V}_f$ is finite $\Rightarrow$ it remains $\boxed{\text{to prove } |\hat{V}_f \cap V_H| < \infty}$

(c) By (a) $\exists h \in K[X_1, X_2], \exists i \geq 0 \; H = X_0^i \hat{h} \Rightarrow \rho_0(V_H) = V_h$ by Observ. B(5)

$H \notin (F) = (\hat{f}) \Rightarrow h \notin (f) \Rightarrow |\hat{V}_f \cap V_H| = |\rho_0(\hat{V}_f) \cap \rho_0(V_H)| = |V_f \cap V_h| < \infty$ by 4.4(3) as $GCD(h, f) = 1$

(2) follows from (1) applying on $H := X_j$

Corollary 9.3 Let $F, G \in K[X_0, X_1, X_2]$, $V_F = V_G$, $a \in V_F$, F, G irreducible

Then (1) $\exists \lambda \in K^* : F = \lambda G$, (2) F is smooth at $a \Leftrightarrow G$ is smooth at a.

Proposition 9.4 Let $f \in K[X_1, X_2]$ be irreducible and $F = \hat{f}$. Define

$$\left.\begin{array}{l} \varepsilon_f : K(V_f) \to K(V_F) \text{ by } \varepsilon_f\left(\dfrac{g + (f)}{h + (f)}\right) := \dfrac{\hat{g} X_0^{\deg h} + (F)}{\hat{h} X_0^{\deg g} + (F)} \\ \varepsilon : K(X_1) \to K(\mathbb{P}^1) \text{ by } \varepsilon\left(\dfrac{g}{h}\right) := \dfrac{\hat{g} X_0^{\deg h}}{\hat{h} X_0^{\deg g}},\ \varepsilon(0) = \varepsilon_f(0) = 0 \end{array}\right\} \Rightarrow \varepsilon_f \,\&\, \varepsilon \text{ are } K\text{-isomorphisms of fields}$$

Proof: By Observation B(1)&(2) ε_f & ε are K-homomorphisms

Comment: For the proof of correctness of ε_f define $\tilde{\varepsilon}_f$ on $K[X_1, X_2]$ and use 1st Isomorph. Thm

Let $R \in K(V_F) \overset{\text{by defn.}}{\Rightarrow} \exists g, h \in K[X_1, X_2] \; \exists r, s \in \mathbb{N} : r + \deg h = s + \deg g$

and $R = \dfrac{\hat{g} X_0^r + (F)}{\hat{h} X_0^s + (F)} \Rightarrow R = \varepsilon_f\left(\dfrac{g + (f)}{h + (f)}\right) \Rightarrow \varepsilon_f$ is surjective.

The argument for surjectivity of ε is the same omitting (F) and (f)

[Def.N] Let $G \in K[X_0, X_1]$, then $\forall A, B \in K[X_0, X_1] \setminus \{0\}$ define

$V_G(A) := \max\{e \geq 0 \mid G^e \mid A\}$, $V_G\left(\frac{A}{B}\right) = V_G(A) - V_G(B)$, $V_G(0) = \infty$

Comment 9.4 shows that $K(\mathbb{P}^1)$ is an AFF and we describe all its NDV

Lemma 9.5 Let v be normalised discrete valuation (NDV) of the AFF $K(\mathbb{P}^1)$ over K. Then v_F is a NDV $\forall$ irreducible F and
(1) $\exists$ irreducible $G \in K[X_0, X_1]$ such that $v = v_G$
(2) degree of the place $\{u \in K(\mathbb{P}^1) \mid v_G(u) > 0\}$ is $\deg G$.
(3) the mapping $(a_0 : a_1) \to \{u \in K(\mathbb{P}^1) \mid v_{a_1 X_0 - a_0 X_1}(u) > 0\}$ is a bijection $\mathbb{P}^1 \to \mathbb{P}^{(1)}_{K(\mathbb{P}^1)/K}$

Comment! We do not need any special NDV "v_∞" as it corresponds to v_{X_0} for irreducible polynomial X_0!

Proof (1) 9.4 $\Rightarrow$ $v\mathcal{E}$ is a NDV upon $K(X_1)$ $\overset{3.14}{\Rightarrow}$ (a) either $v\mathcal{E} = v_\infty$ or (b) $v\mathcal{E} = v_g$ for $g \in K[X_1]$ irreducible

Then (a) $v(\mathcal{E}(\frac{a}{b})) = v_\infty(\frac{a}{b}) = \deg b - \deg a = v_{X_0}\left(\frac{\hat{a} X_0^{\deg b}}{\hat{b} X_0^{\deg a}}\right)$

(b) $v(\mathcal{E}(\frac{a}{b})) = v_g(\frac{a}{b}) = v_g(a) - v_g(b) = v_{\hat{g}}(\hat{a}) - v_{\hat{g}}(\hat{b}) = v_{\hat{g}}\left(\frac{\hat{a} X_0^{\deg b}}{\hat{b}\, X_0^{\deg a}}\right)$

On the other hand $v_{\hat{g}}\mathcal{E}^{-1} = v_g$ and $v_{X_0}\mathcal{E}^{-1} = v_\infty$ by (a) & (b) $\Rightarrow$

$\Rightarrow$ v_G is a NDV $\forall$ irreducible F.

Comment! $\mathcal{E}$ provides translation of NDV and we need to describe "$\mathcal{E}$-images" of NDV's

(2) Using 9.4 & the proof of (1) we get

$\deg\{u \in K(\mathbb{P}^1) \mid v_{\hat{g}}(u) > 0\} = \deg\{u \in K(X_1) \mid v_g(u) > 0\} = \deg g = \deg \hat{g}$

$\deg\{u \in K(\mathbb{P}^1) \mid v_{X_0}(u) > 0\} = \deg\{u \in K(X_1) \mid v_\infty(u) > 0\} = 1 = \deg X_0$

(3) follows from (1), (2) & 9.3(1).

In the rest of the lecture $F \in K[X_0, X_1, X_2]$ is irreducible

T&N Let $a \in V_F \subseteq \mathbb{P}^2$, define $\mathcal{O}_a := \left\{\frac{G + (F)}{H + (F)} \in K(V_F) \mid H(a) \neq 0\right\}$

$P_a := \left\{\frac{G + (F)}{H + (F)} \in \mathcal{O}_a \mid G(a) = 0\right\}$

Comment: Places P_a corresponds to smooth points of projective curve

Observation (C) Let $a \in V_F$.

(1) if $f \in K[X_1, X_2]$, $F = \hat{f}$ and $\exists \chi \in V_f$ $\boxed{\hat{\chi} = a}$ $\Rightarrow$ for $\frac{g + (f)}{h + (f)} \in K(V_f)$:

$\left(h(\chi) \neq 0 \Leftrightarrow \hat{h} X_0^{\deg g}(a) \neq 0\right)$ & $\left(g(\chi) = 0 \Leftrightarrow \hat{g} X_0^{\deg h}(a) = 0\right)$

$\Rightarrow$ $\mathcal{E}_f(P_\chi) = P_a$

(2) if $F \neq \hat{f}$ $\forall f \in K[X_1, X_2]$ $\Rightarrow$ $\exists \lambda \in K^*$ such that $F = \lambda X_0$

$\Rightarrow$ $K(V_F) = K(V_{\lambda X_0}) = K(V_{X_0}) \cong K(V_{X_2}) \cong K(\mathbb{P}^1)$

$G(X_0, X_1, X_2) \to G(X_2, X_1, X_0)$

~~Observation~~ Comment: Isomorphism of AFF $K(V_F)$ and $K(V_f)$ transfers also set P_χ to $P_{\hat{\chi}}$

Theorem 9.6 Let $P \in \mathbb{P}_{K(V_F)|K}$, $a \in V_F$ (Recall that F is irreducible) $\in K[X_0, X_1, X_2]$

(1) $\exists\, b \in V_F$ such that $P_b \subseteq P$,

(2) if $\deg P = 1$ & $P_a \subseteq P \Rightarrow a \in V_F(K)$,

(3) if F is smooth at $a \in V_F(K) \Rightarrow P_a = P$ & $\deg P_a = 1$.

Proof: If $F = \lambda X_j$ for some $j \in \{0,1,2\}$ & $\lambda \in K^*$ $\overset{\text{Obs } C(3)}{\Longrightarrow} K(V_F) \cong K(\mathbb{P}_1)$

$\Rightarrow$ the assertion follows from 9.5 $\Rightarrow$ $\boxed{\text{we may suppose } F \notin (X_j)\ \forall j = 0,1,2}$

$\Rightarrow \exists\, f \in K[x_1, x_2]$ such that $F = \hat{f}$ & f is irreducible (by Obs B(3))

(1) Put $\xi_j := X_j + (F)$ $\forall j = 0,1,2$ and $m := \max\{v_P(\xi_i/\xi_j) \mid i \neq j\}$

Note that $v_P(\xi_i/\xi_j) = -v_P(\xi_j/\xi_i)$, $v_P(\xi_1/\xi_0) + v_P(\xi_0/\xi_2) + v_P(\xi_2/\xi_1) = v_P(1) = 0$

w.l.o.g. $m = \boxed{v_P(\xi_1/\xi_0) \geq 0}$?? $v_P(\xi_0/\xi_2) > 0 \Rightarrow \underbrace{v_P(\xi_1/\xi_2)}_{> m} = \underbrace{v_P(\xi_1/\xi_0)}_{= m} + \underbrace{v_P(\xi_0/\xi_2)}_{> 0}$

$\Rightarrow \boxed{v_P(\xi_2/\xi_0) \geq 0}$ a contradiction $\Leftarrow$

Applying K-isomorphism ε_f from 9.4 we get:

$Q := \varepsilon_f^{-1}(P) \in \mathbb{P}_{K(V_f)|K}$, $x_1 + (f) = \varepsilon_f^{-1}(\xi_1/\xi_0)$, $x_2 + (f) = \varepsilon_f^{-1}(\xi_2/\xi_0) \in \mathcal{O}_Q$

$\Rightarrow K[V_f] \subseteq \mathcal{O}_Q \overset{5.15}{\Longrightarrow} \tilde{Q} := K[V_f] \cap Q$ is a maximal ideal of $K[V_f]$

$\overset{\text{Hilbert Nullstellensatz}}{\Longrightarrow} \exists\, \gamma \in \mathbb{A}^2 : \omega(I_\gamma) = \tilde{Q} \subseteq K[V_f]$

Comment: See definition of ω for $\boxed{\text{I\&N}}$ between 5.10 and 5.11

Since $(f) \subseteq I_\gamma = \{h \in K[x_1, x_2] \mid h(\gamma) = 0\} \Rightarrow \gamma \in V_f \Rightarrow P_\gamma \subseteq Q$, $\mathcal{O}_\gamma \subseteq \mathcal{O}_Q$

$\Rightarrow P_{\hat{\gamma}} = P_{\hat{\gamma}} = \varepsilon_f(P_\gamma) \subseteq \varepsilon_f(Q) = P$

(2) if $\deg P = 1 \Rightarrow \dim_K K[V_f]/\tilde{Q} \leq \dim_K \mathcal{O}_Q/Q \overset{9.4}{=} \dim_K \underset{\varepsilon_f(\mathcal{O}_Q)/\varepsilon_f(Q)}{\underset{\shortparallel}{\mathcal{O}_P/P}} = 1$

$\Rightarrow \gamma \in V_f(K)$ by Obs. (4) before 5.15 $\Rightarrow a = \hat{\gamma} \in V_F(K)$

(3) follows from 9.1 & 5.13 repeating the argument of the proof of 8.3(4).

Comment: By 9.6 place P_∞ of the WBP corresponds (for a homogeneous variant of WBP) to a projective point of the curve "in infinity" (i.e. of the form $(0:a_1:a_2)$)