<u>Observation</u>: Let $a \in K[x,y]$ be irreducible, $f \in K[x,y]$ and $C = V_a$.

(1) $f(x+(a), y+(a)) = 0$ in $K(C) \Leftrightarrow f \in (a)$

(2) $K(C) = K(x+(a), y+(a))$ (Recall: $K(C)$ is the fraction field of $K[C] = K[x,y]/(a)$)

(3) $x+(a)$ is algebraic over $K \Leftrightarrow$

$\Leftrightarrow \exists\, p \in K[x] \setminus \{0\}$ s.t. $\underbrace{p(x) \in (a)}_{\Leftrightarrow\, a/p}$ (by (1)) $\Leftrightarrow \deg_y a(x,y) = 0$.

$\boxed{\text{Again, } K \text{ is a field in the sequel}}$

<u>Lemma 4.6</u> Let $K \subseteq L$ be a field extension, $w \in K[x,y]$ be irreducible and $\alpha, \beta \in L$ such that α is transcendental over K, $L = K(\alpha, \beta)$ and $w(\alpha, \beta) = 0$.

Then $[L : K(\alpha)] = \deg_y(w(x,y))$.

<u>Proof</u>: α-transcendental & $w \neq 0 \Rightarrow w(\alpha, y) \neq 0$

$\Rightarrow m(y) := w(\alpha, y) \in K(\alpha)[y] \setminus \{0\}$ & $m(\beta) = w(\alpha,\beta) = 0$. i.e.

β is a root of $m(y)$ $(\neq 0)$

w irreducible $\underset{\text{(cf. Observation before 2.7)}}{\overset{\text{Gauss lemma}}{\Longrightarrow}}$ w is irreducible as a polynomial $K(x)[y]$ (in variable y)

$\overset{\alpha\text{-transcendental}}{\Longrightarrow}$ $m(y)$ is irreducible $(\in K(\alpha)[y])$ $\overset{\text{(Galois theory)}}{\Longrightarrow}$

$$[L : K(\alpha)] = [K(\alpha)(\beta) : K(\alpha)] = \deg_y m(y) = \deg_y w$$

Proposition 4.7 Let $w \in K[x,y]$ be irreducible, $C = V_w$, $\alpha := x + (w)$, $\beta := y + (w) \in K[C] \subseteq K(C) = K(\alpha,\beta)$.

Then: (1) α is transcendental over $K \Leftrightarrow \deg_y(w) > 0$,

(2) —— " —— $\Rightarrow [K(C) : K(\alpha)] = \deg_y w$,

(3) $K(C)$ is an AFF over K.

Proof: (1) follows from the Observation (3)

(2) as $m(\alpha,\beta)=0$ by Observation (1) the claim follows from Lemma 4.6.

(3) m irreducible $\Rightarrow m \in K[x,y]\setminus K \Rightarrow$

$\Rightarrow$ either $\deg_x m > 0 \overset{(1)}{\Rightarrow} \beta$ is transcendental over K

or $\deg_y m > 0 \overset{(1)}{\Rightarrow} \alpha$ —— " ——

w.l.o.g. α-transcendental $\overset{(2)}{\Rightarrow} [K(C):K(\alpha)] < \infty$

$\Rightarrow K(C)$ is an AFF

<u>Corollary 4.8</u> Let $K \subseteq L$ be a field extension. Then

$\exists \alpha,\beta \in L : L = K(\alpha,\beta)$ is an AFF over K $\iff$ $\exists$ an irreducible affine curve $C \subseteq \mathbb{A}^2$ such that L is K-isomorphic to $K(C)$.

proof: ($\Leftarrow$) by 4.7, ($\Rightarrow$) Let $\Omega : K[x,y] \to K[\alpha,\beta]$, $m \mapsto m(\alpha,\beta)$

where $L = K(\alpha,\beta)$, α-transcendental, $[L:K(\alpha)] < \infty$

not isomorph. then

$\Rightarrow K[x,y]/\ker\Omega \cong K[\alpha,\beta]$ is a domain $\Rightarrow \ker\Omega$ is prime

Since $\ker\Omega$ is non-maximal $\overset{4.4}{\Longrightarrow}$ $\exists\, w \in k[x,y]$ prime s.t. $\ker\Omega = (w)$

$\Rightarrow$ for $C := V_w$ $k[C] \overset{\bar\Omega}{\cong}_k k[\alpha,\beta] \Rightarrow k(C) \cong_k k(\alpha,\beta)$
(is an irreducible affine curve)

$\boxed{\text{T\&N}}$ $f \in k[x,y]$ is called <u>absolutely irreducible</u> if it is irreducible in $\bar{k}[x,y]$ ($\bar{k}$ - the algebraic closure of k).

<u>Lemma 4.9</u> Let $f, g \in k[x]$ such that $\deg g \leq 1$ and $\deg f \geq 3$ odd. Then $w = y^2 + yg - f \in k[x,y]$ is absolutely irreducible. In particular, every WEP
——— " ———.

<u>proof</u>: Let $w = u \cdot v \in \bar{k}[x,y]$

(a) Assume that $u, v \notin \bar{k}[x] \Rightarrow \deg_y u, \deg_y v > 0$

$2 = \deg_y w = \deg_y u + \deg_y w \Rightarrow$ both ——— " ——— $= 1$

we may assume w.l.o.g that u & v are monic i.e. (in y)

$\exists\, \Delta_1, \Delta_2 \in \overline{K}[x]$: as $1 = lc_y(w) = lc_y(u) \cdot lc_y(v)$

Recall lc_y = leading coefficient of the polynomial in the variable y

$$u = y - \Delta_1$$
$$v = y - \Delta_2$$

$$\Rightarrow \quad w = y^2 \underbrace{- (\Delta_1 + \Delta_2)}_{= g} y + \underbrace{\Delta_1 \Delta_2}_{= -f}$$

$\deg f = \deg \Delta_1 \Delta_2 = \deg \Delta_1 + \deg \Delta_2$ is odd $\Rightarrow \deg \Delta_1 \neq \deg \Delta_2$

w.l.o.g. $\deg \Delta_1 < \deg \Delta_2 \Rightarrow \deg(\underbrace{-(\Delta_1 + \Delta_2)}_{g}) = \deg \Delta_2 \leq 1$

$\deg g \leq 1$ & the problem

$\Rightarrow \deg f = \deg \Delta_1 \Delta_2 \leq 2 \deg \Delta_2 \leq 2$, a contradiction

(b) either $u \in K[x]$ or $v \in K[x]$ w.l.o.g $u \in K[x]$

$\Rightarrow 1 = lc_y(w) = \underbrace{lc_y(u)}_{= u} \cdot lc_y(v) \Rightarrow u \in \overline{K}^* \Rightarrow$

$\deg_y(u) = 0$

$\Rightarrow w$ is irreducible over $\overline{K}$

Lemma 4.10 Let $w \in K[x,y]$ be irreducible, $C = V_w$ and $\tilde{K}$ ~~be~~ the field of constants of AFF $K(C)$.

Then: (1) $K = \tilde{K} \iff w$ is irreducible over $\tilde{K}$

(2) w is absolute irreducible $\Rightarrow K = \tilde{K}$

proof: (1) ($\Rightarrow$) clear by the hypothesis

($\Leftarrow$) Recall that $\tilde{K} = \{ u \in K(C) \mid u \text{ algebraic over } K \}$

$K(C)$ is an AFF $\xRightarrow{\S 4.7,\, 4.8}$ $\exists \alpha \in K(C)$ transcendental over $K \Rightarrow$ over $\tilde{K}$

& $\exists \beta \in K(C)$ s.t. $K(C) = K(\alpha,\beta)$, $w(\alpha,\beta) = 0$

$\tilde{K} \subseteq K(\alpha,\beta) \Rightarrow \tilde{K}(\alpha,\beta) = K(\alpha,\beta)$

w irreducible over $\tilde{K}$ & $w(\alpha,\beta) = 0$

$$\xRightarrow{4.6} [\underbrace{\tilde{K}(\alpha,\beta)}_{=K(\alpha,\beta)=K(C)} : \tilde{K}(\alpha)] \overset{4.6}{=} \deg_y w \overset{4.6}{=} [\overbrace{K(\alpha,\beta)}^{=K(C)} : K(\alpha)]$$

$\Rightarrow \dim_{K(\alpha)} K(C) = \dim_{\tilde{K}(\alpha)} K(C) \Rightarrow K(\alpha) = \tilde{K}(\alpha) \xRightarrow{1.8(2)} K = \tilde{K}$

(2) w irreducible over $\bar{K} \supseteq \tilde{K} \xRightarrow{(1)} K = \tilde{K}$