In the sequel: L is an AFF over K given by $w(\alpha,\beta)=0$
(and recall that $w = y\,g(x,y) + h(x) + y \in K[x,y]$
for $h \in K[x]\setminus\{0\}$, $g \in K[x,y]$, $m = \operatorname{mult} h \geq 2$, $\operatorname{mult} g \geq 1$)

Lemma 5.3 $\exists\, P \in \mathbb{P}_{L/K}$ such that $v_P(\alpha) > 0$ & $v_P(\beta) > 0$.

Moreover, $v_P(\beta) = m \cdot v_P(\alpha)$.

Proof: Let $\Omega : K[x,y] \to K[\alpha,\beta]$ be the substitution of (α,β).

$$u \mapsto u(\alpha,\beta)$$

$\Rightarrow \ker \Omega = (w) \subseteq (x,y) = I_{(0,0)}$ – maximal in $K[x,y]$ $\Rightarrow$

$$\Rightarrow K[\alpha,\beta]/(\alpha,\beta) \underset{\substack{\uparrow \\ \text{(definition of } L \text{ as AFF given by } w)}}{\cong} \left(K[x,y]/(w) \right) \Big/ \left((x,y)/(w) \right) \underset{\substack{\uparrow \\ \text{2nd Isomorphism Theorem}}}{\cong} K[x,y]/(x,y) \cong K$$

$\Rightarrow (\alpha,\beta)$ is maximal in $K[\alpha,\beta]$ $\underset{2.5}{\Longrightarrow}$ $\exists\, P \in \mathbb{P}_{L/K}$ such that $(\alpha,\beta) \subseteq P$

Then $v_P(\alpha) > 0$, $v_P(\beta) > 0$. As $\deg_x w > 0$, $\deg_y w > 0$ $\Rightarrow \alpha, \beta$ transcendental $\Rightarrow \alpha \neq 0 \neq \beta$.

By 2.17(2) (as $V_p(\alpha)>0$): $V_p(\lambda(\alpha)) = \text{mult}(\lambda)\, V_p(\alpha) = m V_p(\alpha)$

$w(\alpha,\beta)=0 \Rightarrow \lambda(\alpha) = -\beta - \beta g(\alpha,\beta)$

Note: $V_p(\beta g(\alpha,\beta)) \underset{(PV1)}{=} V_p(\beta) + \underbrace{V_p(g(\alpha,\beta))}_{\geq \text{mult } g \geq 1 \ (PV2)} > V(\beta)$

Hence $V_p(\beta) = V_p(-\beta) \overset{2.13}{=} V_p(-\beta-\beta g(\alpha,\beta)) = V_p(\lambda(\alpha)) = m\cdot V_p(\alpha)$ □

Observation: Let $a\in k[x,y]$, $a=\sum a_{ij}\, x^i y^j$.

(1) $\Lambda(a)(\alpha,\beta) = a(\alpha, \underbrace{-\lambda(\alpha)-\beta g(\alpha,\beta)}_{=\beta}) = a(\alpha,\beta)$ in L

(2) Fix $P\in \mathbb{P}_{L/k}$ s.t. $V_p(\beta)=V_p(\alpha)\cdot m>0$ from S.3

if $u = a(\alpha,\beta)\in k[\alpha,\beta]-\{0\} \Rightarrow V_p(u) \geq \min\{V_p(x^iy^j) \mid a_{ij}\neq 0\} =$

$= \min\{i+jm \mid a_{ij}\neq 0\} V_p(\alpha) = \mu(a)\cdot V_p(\alpha) \Rightarrow \boxed{\mu(a)\leq \frac{V_p(u)}{V_p(\alpha)}}$

$\boxed{T\&N}$ Put $\mu(u) := \max\{\mu(a) \mid a \in K[x,y] : a(\alpha,\beta) = u\}$

for each $u \in K[\alpha,\beta] - \{0\}$

(It is correctly defined as $\mu(u) \leq \dfrac{v_p(u)}{v_p(\alpha)}$

<u>Lemma 8.4</u> Let $u \in K[\alpha,\beta] - \{0\}$ and $k = \mu(u)$.

Then $\exists \lambda \in K - \{0\}$, $\exists b \in K[x,y]$ such that

$\mu(b) > k$ and $u = \lambda \alpha^k + b(\alpha,\beta)$

<u>Proof</u>: Let $a \in K[x,y]$ be a polynomial with

$\mu(a) = \mu(u) = k$

Note that $\Lambda(a)(\alpha,\beta) = a(\alpha,\beta) = u \neq 0 \Rightarrow \Lambda(a) \neq 0$

Let $a = \sum a_{ij} x^i y^j = \underbrace{\sum_{(i,j) \in \Delta(a)} a_{ij} x^i y^j}_{S(a)} + \sum_{(i,j) : i e_j m > k} a_{ij} x^i y^j$

As $\Lambda(a)(\alpha,\beta) = u \neq 0 \Rightarrow \mu(\Lambda(a)) \leq \mu(u) \Rightarrow$

$$\Rightarrow \mu(S(\Lambda(a))) \leq \mu(a) \quad (*)$$

Since Λ is a K-homomorphism:

$$\Lambda(a) = \sum_{\underbrace{(i,j)\in\Delta(a)}_{\Rightarrow\, \mu(\Lambda(x^i y^j)) = k}} a_{ij}\cdot\Lambda(x^i y^j) + \sum_{\substack{i,j:\\ i+jm>k}} a_{ij}\cdot\Lambda(x^i y^j)$$

$i+jm>k \Rightarrow \mu(\Lambda(x^i y^j)) > k$

8.1.

$$\Rightarrow S(\Lambda(a)) = \left(\sum a_{ij}\cdot\lambda_{ij}\right) x^k \neq 0 \quad \text{by } (*)$$

$\exists\, \lambda_{ij} \in K^*$

(otherwise $\mu(S(\Lambda(a))) > \mu(a)$)

<u>Observation</u> $\forall u \in K[\alpha,\beta]\ \exists a \in K[x,y]$:

$a = \underset{\neq 0}{\lambda} x^{\mu(u)} + b(x,y)$ where $\mu(b) > \mu(u)$ and $u = a(\alpha,\beta)$

Proposition 5.5 $\exists!\, P \in \mathbb{P}_{L|K}$ such that $v_P(\alpha) > 0$ and $v_P(\beta) > 0$. For such P $v_P(\alpha) = 1$, $v_P(\beta) = m$ and $v_P(u \cdot v^{-1}) = \mu(u) - \mu(v)$ $\forall u, v \in K[\alpha,\beta] \setminus \{0\}$.

Proof: By 5.3 $\exists P \in \mathbb{P}_{L|K}$: $v_P(\beta) = v_P(\alpha) \cdot m > 0$

$$\left.\begin{array}{l}\text{Let } u \in K[\alpha,\beta] \setminus \{0\} \\ \text{and put } r := \mu(u)\end{array}\right\} \overset{5.4}{\Longrightarrow} \exists\, b = \lambda x^r - c(x,y) \in K[x,y]$$

$\ne 0$ with $\mu(c) > r$, $\boxed{u = b(\alpha,\beta)}$

$\forall i,j$ compute: $v_P(\alpha^i \beta^j) \underset{(Dv1)}{=} i \cdot v_P(\alpha) + j \underbrace{v_P(\beta)}_{= m \cdot v_P(\alpha)} = \underbrace{(i + jm) v_P(\alpha)}_{= \mu(x^i y^j) v_P(\alpha)}$

$\Longrightarrow v_P(c(\alpha,\beta)) \geq v_P(\alpha) \cdot \mu(c) > r \cdot v_P(\alpha)$

$\&\ v_P(\lambda \alpha^r) = r \cdot v_P(\alpha) \overset{2.13}{\Longrightarrow} \boxed{v_P(u) = r \cdot v_P(\alpha)}$

As $v_p(\alpha) \mid v_p(u)$ $\forall u \in K[\alpha,\beta] - \{0\}$ $\Rightarrow$

$\forall u, v \in K[\alpha,\beta]$ $\quad v_p(\alpha) \mid v_p(u) - v_p(v) \underset{(DV1)}{=} v_p\left(\frac{u}{v}\right)$

Since v_p is normalised DV, $v_p(\alpha) = 1$ and

$$(+) \quad v_p\left(\frac{u}{v}\right) = \mu(u) - \mu(v)$$

The formula $\oplus$ holds $\forall\, v_Q$ for $Q \in \mathbb{P}_{L|K}$ s.t. $v_Q(\alpha) > 0$, $v_Q(\beta) > 0$ §5.3

$\Rightarrow$ $v_p\left(\frac{u}{v}\right) = \mu(u) - \mu(v) = v_Q\left(\frac{u}{v}\right)$, $v_p(0) = v_Q(0) \Rightarrow$

$\Rightarrow$ $v_p = v_Q$ $\Rightarrow$

$$\underline{P} = \{a \in L \mid v_p(a) \geq 1\} = \{a \in L \mid v_Q(a) \geq 1\} = \underline{Q}$$

<u>Example 5.6</u> Let $w = y\left(x + \frac{y}{2}\right) + \frac{1}{2}(x^2 - x^3) + y$ be ~~[illegible]~~ a polynomial from 5.7.

$L = K(\alpha, \beta)$: $\alpha = x + (w)$, $\beta = y + (w)$

Compute $v_P(\alpha^2 + \beta)$ for $P \in \mathbb{P}_{L/K}$ s.t. $(\alpha, \beta) \subseteq P$

$$v_P(\alpha^2) = 2 \ , \ v_P(\beta) = \mathrm{mult}\left(\tfrac{1}{2}x^2 - x^3\right) = 2$$

$$w(\alpha, \beta) = 0 \Rightarrow \beta = -\beta\left(\alpha + \frac{\beta}{2}\right) + \frac{1}{2}(\alpha^3 - \alpha^2)$$

$$\Rightarrow \alpha^2 + \beta = \underbrace{\frac{1}{2}\alpha^2}_{v_P = 2} + \underbrace{\frac{1}{2}\alpha^3 - \beta\alpha - \frac{\beta^2}{2}}_{v_P > 2}$$

$$\Rightarrow \underline{v_P(\alpha^2 + \beta)} \overset{2.13}{=} v_P\left(\tfrac{1}{2}\alpha^2\right) = \underline{2}$$

$$! \ \underline{v_P(\alpha^2 + 2\beta)} = v_P\Big(\underbrace{\alpha}_{v_P = 1}\underbrace{(\alpha^2 - 2\beta)}_{v_P = 2} - \underbrace{\beta^2}_{v_P = 4}\Big) \overset{2.13}{=} \underline{3}$$