As usual: $\boxed{L \text{ is an AFF over } K \text{ of genus } g, K \text{ is the field of constants}}$ 1

<u>Theorem 7.3</u>: If $K = \tilde{K}$, then

(1) $\dim_L(\Omega_{L|K}) = 1$

(2) if $\omega \in \Omega_{L|K} \setminus \{0\}$, $A \in \mathrm{Div}(L|K)$, then $\psi_{\omega A}: \mathcal{L}((\omega) - A) \to \Omega_{L|K}(A)$ given by $\psi_{\omega A}(s) = s\omega$ is a $K$-isomorphism.

<u>Proof</u>: (1) Consider $\psi_\omega$ from 7.2, let $\omega, \tilde{\omega} \in \Omega_{L|K} \setminus \{0\}$

$\overset{7.2(3)}{\Longrightarrow} \exists B \in \mathrm{Div}(L|K): \psi_\omega(\mathcal{L}((\omega)-B)) \cap \psi_{\tilde{\omega}}(\mathcal{L}((\tilde{\omega})-B)) \subseteq \Omega_{L|K}(B)$

$\Rightarrow \exists a, \tilde{a} \in L^*: \underbrace{\psi_{\tilde{\omega}}(\overset{\mathcal{L}((\tilde{\omega})-B)}{\tilde{a}})}_{\tilde{a}\tilde{\omega}} \overset{0^*}{=} \underbrace{\psi_\omega(\overset{\mathcal{L}((\omega)-B)}{a})}_{a\omega} \Rightarrow \tilde{\omega} = \frac{a}{\tilde{a}}\omega$ where $\frac{a}{\tilde{a}} \in L^*$

$\Rightarrow \Omega_{L|K} = \mathrm{Span}_L(\omega) \Rightarrow \dim_L \Omega_{L|K} = 1$

(2) Note that $\psi_{\omega A} = \psi_\omega \restriction \mathcal{L}((\omega)-A)$ and it is injective $K$-linear into $\Omega_{L|K}(A)$ by 7.2(2)

Suppose $\tilde{\omega} \in \Omega_{L|K}(A)$, we search $\lambda \in \mathcal{L}((\omega)-A) : \varphi_{\omega A}(\lambda) = \tilde{\omega}$

if $\tilde{\omega} = 0$, then $\lambda := 0$.

let $\tilde{\omega} \neq 0$ then $\exists \lambda \in L^* : \tilde{\omega} = \lambda\omega \in \Omega_{L|K}(A)$ by (1)

$\xRightarrow{7.1} (\lambda\omega) - A \overset{7.2}{=} (\lambda) + (\omega) - A \geq 0 \Rightarrow \lambda \in \mathcal{L}((\omega)-A)$

<u>Corollary 7.4</u> Let $K = \bar{K}$. The canonical divisors form exactly one coset modulo $\mathrm{Princ}(L|K)$.

(i.e. if $W$ is canonical, then $A \sim W \iff A$ is canonical).

<u>Proof</u>: Let $\omega \in \Omega_{L|K} \setminus \{0\}$, then

$(\omega) \sim A \iff \exists a \in L^* : A = (a) + (\omega) \overset{7.2(1)}{=} (a\omega) \overset{7.3(1)}{\iff} \exists \tilde{\omega} \in \Omega_{L|K} : A = (\tilde{\omega})$

<u>Theorem 7.5</u> (Riemann-Roch) If $K = \bar{K}$ and $W$ is a canonical divisor, then $\ell(A) = \deg(A) + \ell(W-A) + 1 - g \quad \forall A \in \mathrm{Div}(L|K)$

Proof: Let $W = (\omega)$ for $\omega \in \Omega_{4K} \setminus \{0\} \overset{2.3}{\Longrightarrow} \mathcal{L}(W-A) \cong \Omega_{4K}(A)$

$\Longrightarrow \ell(W-A) = \dim_K(\mathcal{L}(W-A)) \overset{\text{Obs.(1)}}{=} i(A) = \underline{g-1-\deg A + \ell(A)}$

## Corollary 7.6

Let $A, W \in \mathrm{Div}(4K)$, $K = \tilde{K}$.

(1) If $W$ is canonical $\Rightarrow \ell(W) = g$, $\deg(W) = 2g-2$, $i(W) = 1$,

(2) (Main consequence of the Riemann-Roch theorem)

If $\deg A \geq 2g-1 \Longrightarrow \ell(A) = \deg(A) + 1 - g$

Proof: (1) follows from 7.5 for $A = 0$ & $A = W$

(2) $(1) \Rightarrow \deg(W-A) \leq 2g-2-(2g-1) = -1 \overset{(D8)}{\Longrightarrow}$

$\ell(W-A) = 0 \overset{2.5}{\Longrightarrow} \ell(A) = \deg A + 1 - g$

**Lemma 7.7** Let $K = \bar{K}$, $A \in \mathrm{Div}(L|K)$, then

(1) if $\deg A = 2g-2$, $\ell(A) \geq g \Rightarrow A$ is canonical,

(2) if $g = 1 \Rightarrow A$ is canonical if and only if $A$ is principal.

**Proof**: (1) $i(A) = \underbrace{\ell(A)}_{\geq g} - \underbrace{\deg A}_{= 2g-2} + g - 1 \geq 1 \xrightarrow{\text{Obs. (1)}}$

$\Rightarrow \exists\, \omega \in \Omega_{L|K}(A) - \{0\} \xRightarrow{7.1} A \leq (\omega)$

$7.6(1) \Rightarrow \deg A = \deg((\omega))$

$\Bigg\} \Rightarrow A = (\omega)$

(2) $7.68(1) \Rightarrow A$ is canonical $\Leftrightarrow \deg A = 0,\ \ell(A) = 1 \overset{6.9}{\Longleftrightarrow} A$ is principal

**Proposition 7.8** Let $K = \bar{K}$, $A, B \in \mathrm{Div}(L|K)$, $g = 0$.

(1) $A$ is principal $\Leftrightarrow \deg A = 0$,

(2) $A \sim B \Leftrightarrow \deg A = \deg B$,

(3) $A$ is canonical $\Leftrightarrow \deg A = -2$.

proof of 7.8: (1) ($\Rightarrow$) if $A$ is principal $\overset{6.5}{\Longrightarrow}$ $\deg A = 0$

($\Leftarrow$) if $\deg A = 0 \overset{7.6(2)}{\Longrightarrow} \ell(A) = 1 \overset{6.9}{\Longrightarrow}$ $A$ is principal.

(2) $A \sim B \Leftrightarrow A - B \in \mathrm{Princ}(L|k) \overset{(1)}{\Longleftrightarrow} \deg(A-B) = 0 \Longleftrightarrow \deg A = \deg B$

(3) follows from ~~(1)~~ 7.6(1) and 7.7(1)

$\boxed{\text{T\&N}}$ $\mathbb{P}^{(1)}_{L|k} := \{P \in \mathbb{P}_{L|k} \mid \deg P = 1\}$

Lemma 7.9 Let $P \in \mathbb{P}^{(1)}_{L|k} \neq \emptyset$, $h \in \mathbb{Z} : h \geq 0$, $s \in L$.

(1) $k = \tilde{k}$,

(2) $s \in \mathcal{L}(iP) - \mathcal{L}((i-1)P) \Longleftrightarrow (s)_- = iP \quad \forall i \geq 1$,

(3) if $\exists h \geq 0 : \ell(iP) \geq i - h + 1 \ \forall i \geq h \Longrightarrow g \leq h$,

(4) if $\forall i \geq h+1 \ \exists s_i \in L : (s_i)_- = iP \Longrightarrow g \leq h$.

proof: (1) $1 = \deg_{L|k}(P) = [\tilde{k}:k] \deg_{L|\tilde{k}}(P) \Longrightarrow [\tilde{k}:k] = 1$

(2) $\Delta \in \mathcal{L}(iP) \overset{\text{by definition}}{\iff} (\Delta) + iP \geq 0 \iff \begin{cases} V_Q(\Delta) \geq 0 \ \forall Q \neq P, \\ \& \\ V_P(\Delta) \geq -i \end{cases}$

Thus:

$\Delta \in \mathcal{L}(iP) \setminus \mathcal{L}((i-1)P) \iff \left\{ \begin{array}{l} V_Q(\Delta) \geq 0 \ \forall Q \neq P \\ \& \\ V_P(\Delta) = -i \end{array} \right\} \iff (\Delta)_- = iP$

(3) Note that $\deg P = 1 \Rightarrow \deg(iP) = i$; let $\boxed{i \geq \max(h, 2g-1)}$

$\Rightarrow \underbrace{i - g + 1}_{} = \deg(iP) - g + 1 \overset{2.6(2)}{=} \ell(iP) \underbrace{\overset{\text{hypothesis}}{\geq} i - h + 1}_{}, \Rightarrow g \leq h$

(4) Let $i \geq h+1$ : $(\Delta_i)_- = iP \overset{(2)}{\Longrightarrow} \Delta_i \in \mathcal{L}(iP) \setminus \mathcal{L}((i-1)P)$

We have: $\qquad \Rightarrow \mathcal{L}((i-1)P) \subsetneq \mathcal{L}(iP)$

$\widetilde{K} \overset{(1)}{=} K = \mathcal{L}(0) \subseteq \mathcal{L}(hP) \subsetneq \mathcal{L}((h+1)P) \subsetneq \cdots \subsetneq \mathcal{L}((h+i-1)P) \subsetneq \mathcal{L}((h+i)P) \subsetneq$

dimensions $\underbrace{1 = \ell(0) \leq \ell(hP) < \ell((h+1)P) < \cdots < \ell((h+i-1)P) < \ell((h+i)P) < \cdots}_{}$

$\Longrightarrow \ell(iP) \geq i - h + 1 \ \forall i \geq h+1 \overset{(3)}{\Longrightarrow} g \leq h \qquad \square$

Recall that $g \geq 0$.

Example 7.10 Let $x$ be a variable $\Rightarrow$

$K(x)$ is an AFF over $K$

By 2.14 $\mathbb{P}_{K(x)|K} = \{P_p \mid p \in K[x] \text{ irreducible}\} \cup \{P_\infty\}$

where $P_p = (p) = \{q \in K(x) \mid v_p(q) > 0\}$ is the maximal ideal of VR $K[x]_{(p)}$ (the localization in $(p)$)

$P_\infty = \{q \in K(x) \mid v_\infty(q) > 0\}$ for $v_\infty(\frac{a}{b}) = \deg_x b - \deg_x a$ $(a, b \in K[x] \setminus \{0\})$

$\Rightarrow$ $v_{P_p} = v_p$ & $v_p(x^i) \geq 0$ $\forall i \geq 0$ & $p \in K[x]$ irreducible

$v_\infty(x^i) = -\deg_x x^i = -i$ -u- $\Rightarrow$ $\boxed{(x^i)_- = i P_\infty}$ $\forall i \geq 1$

$\Rightarrow \deg P_\infty = \deg (x)_- \overset{6.5}{=} [K(x) : K(x)] = \underline{1}$

$\overset{7.9(4)}{\Longrightarrow}$ $0 \leq g \leq 0$ $\Rightarrow$ genus of $K(x)$ is $\underline{0}$.