Proposition 9.2 Let $H, F \in K[X_0, X_1, X_2]$, F irreducible.

(1) Then either - $H \in (F)$ and $H(a) = 0$ $\forall a \in V_F$ or
- $H \notin (F)$ and $V_F \cap V_H$ is finite.

(2) If $X_j \notin (F)$ $\left(\Leftrightarrow F \notin (X_j)\right)$ for $j = 0, 1, 2$ $\Rightarrow |\{(a_0 : a_1 : a_2) \in V_F \mid a_j = 0\}| < \infty$

Proof: Put $d = \deg F$ and $\hat{V}_f := \{\hat{b} \in \mathbb{P}^2 \mid b \in V_f\}$ for $f \in K[x_1, x_2]$

(1) Suppose that $H \notin (F) \Rightarrow d \geq 1$

(a) if $F \notin (X_0) \Rightarrow \deg(\Pi_0(F)) = d \Rightarrow \widehat{\Pi_0(F)} = F$

if $F \in (X_0) \overset{F\text{-irreducible}}{\Longrightarrow} \exists \lambda \in K^* : F = \lambda X_0 \Rightarrow F \notin (X_1) \cup (X_2)$

we may switch X_i's w.l.o.g. so let's suppose $\boxed{F \notin (X_0) \ (\Rightarrow) \ \& \ f \in K[x_1, x_2] : F = \hat{f}}$

(b) Put $G := F(0, X_1, X_2) \in K[X_1, X_2]$, $\deg G = d$ $(\neq 0)$

$\Rightarrow$ either $\deg_{x_1} G > 0$ or $\deg_{x_2} G > 0$: w.l.o.g. $\boxed{\deg_{x_1} G > 0}$

Then $|\{\lambda \in \bar{K} \mid G((\lambda:1))=0\}| < \infty$
$(0:a_1:1) \in V_F \Leftrightarrow G((a_1:1))=0$
$|\{0:a_1:0 \mid a_1 \in k\}| = 1$
$\Rightarrow V_F \setminus \hat{V}_f$ is finite $\Rightarrow$? it remains to prove $\boxed{|\hat{V}_f \cap V_H| < \infty}$

(c) by (a) $\exists h \in K[x_1,x_2], \exists i \geq 0 : H = X_0^i \hat{h} \Rightarrow p_0(V_H) \underset{\text{Ob. B(5)}}{=} V_h$

$H \notin (F) \overset{F=\hat{f}}{\Rightarrow} h \notin (f) \Rightarrow |\hat{V}_f \cap V_H| = |p_0(\hat{V}_f) \cap p_0(V_H)| = |V_f \cap V_h| < \infty$ by 4.4(3) as $GCD(h,f) = 1$

(2) follows from (1) putting $H := X_j$.

<u>Corollary 9.3</u> Let $F, G \in K[X_0,X_1,X_2]$ be irreducible, $V_F = V_G$, $a \in V_F$.
Then (1) $\exists \lambda \in K^*: F = \lambda G$, (2) F is smooth at $a \Leftrightarrow G$ is smooth at a.

<u>Proposition 9.4</u>: Let $f \in K[x_1,x_2]$ be irreducible and $F = \hat{f}$. Define $\varepsilon_f : K(V_f) \to K(V_F)$ by $\varepsilon_f\left(\frac{g+(f)}{h+(f)}\right) := \frac{\hat{g} X_0^{\deg h} + (F)}{\hat{h} X_0^{\deg g} + (F)}$, $\varepsilon_f(0) = 0$

$\varepsilon : K(x_1) \to K(\mathbb{P}^1)$ by $\varepsilon\left(\frac{g}{h}\right) := \frac{\hat{g} X_0^{\deg h}}{\hat{h} X_0^{\deg g}}$, $\varepsilon(0) = 0$

Then ε_f & ε are K-isomorphisms.

Proof of 9.4: By Observation B (1), (2) $\mathcal{E}_f$ & $\mathcal{E}$ are K-homomorphisms

Let $R \in K(V_F) \Rightarrow \exists\, g, h \in K[x_1, x_2]\ \exists\, r, s \in \mathbb{N}: r + \deg h = s + \deg g$

such that $R = \dfrac{\hat{g} X_0^s + (F)}{\hat{h} X_0^r + (F)} \Rightarrow R = \mathcal{E}_f\left(\dfrac{g + (f)}{h + (f)}\right) \Rightarrow \mathcal{E}_f$ is onto

surjectivity of $\mathcal{E}$ can be proved the same way on $(F)/(f)$.

$\boxed{\text{T\&N}}$ Let $G \in K[X_0, X_1]$. Then $\forall\, A, B \in K[X_0, X_1] \setminus \{0\}$

define $v_G(A) := \max\{ l \geq 0 \mid G^l | A \}$, $v_G\left(\frac{A}{B}\right) = v_G(A) - v_G(B)$

$v_G(0) = \infty$

Lemma 9.5 Let v be a normalised discrete valuation (NDV) of the AFF (by 9.4) $K(\mathbb{P}^1)$ over K. Then

(0) v_F is a NDV $\forall$ irreducible F

(1) $\exists$ irreducible $G \in K[X_0, X_1]$ such that $v = v_G$,

(2) degree of the place $\{ u \in K(\mathbb{P}^1) \mid v_G(u) > 0 \}$ is $\deg G$,

(3) the mapping $(a_0 : a_1) \to \{u \in K(\mathbb{P}^1) \mid v_{a_1 x_0 - a_0 x_1}(u) > 0\}$
is a bijection $\mathbb{P}^1 \to \mathbb{P}^{(1)}_{K(\mathbb{P}^1)/K}$.

proof: (1) $9.4 \Rightarrow v\varepsilon$ is a NDV upon $K(x_1) \overset{3.14.}{\Longrightarrow}$
$\Rightarrow$ either (a) $v\varepsilon = v_\infty$ or (b) $v\varepsilon = v_g$ for irreducible $g \in K[x_1]$

(a) $v(\varepsilon(\frac{a}{b})) = v_\infty(\frac{a}{b}) = \deg b - \deg a = v_{x_0}\left(\frac{\hat{a}\, x_0^{\deg b}}{\hat{b}\, x_0^{\deg a}}\right)$

(b) $v(\varepsilon(\frac{a}{b})) = v_g(\frac{a}{b}) = v_g(a) - v_g(b) = v_{\hat{g}}(\hat{a}) - v_{\hat{g}}(\hat{b})$
$= v_{\hat{g}}\left(\frac{\hat{a}\, \cancel{x_0^{\deg b}}}{\hat{b}\, \cancel{x_0^{\deg a}}}\right)$

$\boxed{\text{(a) \& (b)} \Rightarrow v_{\hat{f}} \text{ \& } v_{x_0} \text{ are NDV} \Rightarrow (0)}$

(2) Using 9.4 & the proof of (1):

$$\frac{\deg\{u \in K(\mathbb{P}^1) \mid v_{\hat{g}}(u) > 0\}}{\deg\{u \in K(\mathbb{P}^1) \mid v_{x_0}(u) > 0\}} \quad \begin{matrix} = \deg\{u \in K(x_1) \mid v_g(u) > 0\} = \deg g = \deg \hat{g} \\ = \deg\{u \in K(x_1) \mid v_\infty(u) > 0\} = 1 = \deg x_0 \end{matrix}$$

(3) follows from (1), (2) & 9.3(1)

$\boxed{\text{In the rest of the lecture } F \in K[X_0, X_1, X_2] \text{ is irreducible}}$

$\boxed{\text{T\&N}}$ Let $a \in V_F \subseteq \mathbb{P}^2$

$$\mathcal{O}_a := \left\{ \frac{G + (F)}{H + (F)} \in K(V_F) \mid H(a) \neq 0 \right\}$$

$$P_a := \left\{ \frac{G + (F)}{H + (F)} \in \mathcal{O}_a \mid G(a) = 0 \right\}$$

<u>Observation C</u> Let $a \in V_F$

(1) if $f \in K[x_1, x_2]$: $F = \hat{f}$ and $\exists \gamma \in V_f$ $\boxed{\hat{\gamma} = a}$ $\Rightarrow$

for $g + (f)/h + (f) \in K(V_f)$: $\left.\begin{array}{l} h(\gamma) \neq 0 \iff \hat{h} X_0^{\deg g}(a) \neq 0 \\ g(\gamma) = 0 \iff \hat{g} X_0^{\deg h}(a) = 0 \end{array}\right\} \Rightarrow$

$\Rightarrow \boxed{\mathcal{E}_f(P_\gamma) = P_a}$

(2) if $F \neq \hat{f}$ $\forall f \in K[x_1, x_2] \Rightarrow \exists \lambda \in K^*: F = \lambda X_0$

$$\Rightarrow K(V_F) = K(V_{\lambda X_0}) \cong K(V_{X_2}) \cong K(\mathbb{P}^1)$$

<u>Theorem 9.6</u> Let $P \in \mathbb{P}_{K(V_F)|K}$, $a \in V_F$ (where F irreducible)

(1) $\exists\, b \in V_F$ such that $P_b \subseteq P$,

(2) if $\deg P = 1$ & $P_a \subseteq P \Rightarrow a \in V_F(K)$,

(3) if F is smooth at $a \in V_F(K) \Rightarrow P_a = P$ & $\deg P_a = 1$.

<u>Proof</u>: If $F = \lambda X_j$ for $j \in \{0,1,2\}$ & $\lambda \in K^*$ $\overset{\text{Obs } C(2)}{\Longrightarrow} K(V_F) \cong K(\mathbb{P}_1) \Rightarrow$

$\Rightarrow$ the assertion follows from 9.5 $\Rightarrow$ $\boxed{\text{Let } F \notin (X_j)\ \forall j = 0,1,2}$

$\Rightarrow \exists f \in K[x_1, x_2]$ such that $F = \hat{f}$ and f is irreducible by Obsen. B(3)

(1) Put $\xi_j := X_j + (F)$ and $m := \max\{v_P(\xi_i/\xi_j) \mid i \neq j\}$

Note $v_P(\xi_i/\xi_j) = -v_P(\xi_j/\xi_i)$, $v_P(\xi_1/\xi_0) + v_P(\xi_0/\xi_2) + v_P(\xi_2/\xi_1) = v_P(1) = 0$

w.l.o.g. $m = v_P(\xi_1/\xi_0) \geq 0$?? $v_P(\xi_0/\xi_2) > 0 \Rightarrow \underbrace{v_P(\xi_1/\xi_2)}_{\boxed{>m}} = \underbrace{v_P(\xi_1/\xi_0)}_{m} + \underbrace{v_P(\xi_0/\xi_2)}_{>0}$ $\Leftarrow$

a contradiction

$\Rightarrow v_P(\xi_2/\xi_0) \geq 0$

Applying K-isomorphism ε_f from 9.4 we get:

$Q := \varepsilon_f^{-1}(P) \in \mathbb{P}_{K(V_f)|K}$ $\quad x_1 + (f) = \varepsilon_f^{-1}(\xi_1/\xi_0)$, $x_2 + (f) = \varepsilon_f^{-1}(\xi_2/\xi_0) \in \mathcal{O}_Q$

$\Rightarrow K[V_f] \subseteq \mathcal{O}_Q \overset{5.15}{\Longrightarrow} \tilde{Q} := K[V_f] \cap Q$ is maximal in $K[V_f]$

cf. $\boxed{T\&N}$ between 5.10 – 5.11

$\overset{\text{Hilbert's}}{\underset{\text{Nullstellensatz}}{\Longrightarrow}} \exists\, \gamma \in \mathbb{A}^2 : \omega(I_\gamma) = \tilde{Q} \subseteq K[V_f]$

Since $(f) \subseteq I_\gamma = \{h \in K[x_1, x_2] \mid h(\gamma) = 0\} \Rightarrow \gamma \in V_f$

Since $\gamma \in V_f \overset{\text{by the definition}}{\Longrightarrow} P_\gamma \subseteq Q\,,\ \mathcal{O}_\gamma \subseteq \mathcal{O}_Q \Rightarrow \underbrace{P_{\hat{\gamma}}}_{= P_a \subseteq \varepsilon_f(P_\gamma)} \subseteq \underset{\varepsilon_f(Q)}{\overset{\shortparallel}{P}}$

(2) if $\deg P = 1 \Rightarrow$

$$\boxed{\dim_k k[V_f]/\tilde{Q} \leq \dim_k \mathcal{O}_Q/Q \overset{9.4}{=} \dim_k \varepsilon_f(\mathcal{O}_Q)/\varepsilon_f(Q) = \dim_k \mathcal{O}_P/P = 1}$$

Then $\gamma \in V_f(k)$ & Obser. (4) before 5.15 $\Rightarrow a = \hat{\gamma} \in V_f(k)$

(3) follows from 9.1 & 5.13 repeating the argument of the proof of 8.3(4).