ON RINGS DETERMINED BY THEIR IDEMPOTENTS AND UNITS

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Abstract. This paper describes properties of three particular classes of rings determined by their idempotents and units. It is shown that right UG rings, i.e. rings in which any two generators of each principal right ideal are associated, contains local rings and regular rings of stable range 1. Semiperfect and von Neumann regular rings satisfy necessarily the condition P_r, which says that every principal right ideal is generated by a sum of a unit and an idempotent. Finally, idun-semicommutative rings, generalizing semicommutative condition by restriction on sums of units and of idempotents, contains all local and abelian regular rings.

1. Introduction

Units and idempotents present key tools for description and understanding structure of important classes of rings, such as clean, von Neumann regular or local rings. The main goal of this paper is to describe three particular classes of rings determined by properties of their idempotents and units on background of von Neumann regular and local rings. All these classes of rings include classical and widely studied ones, namely, right UG rings, defined by the condition $aR = bR$ implies $a = bu$ for some $u \in U(R)$, generalizes domains and von Neumann regular rings, rings satisfying the condition P_r, which says that every right ideal is generated by a sum of an idempotent and an invertible element, generalizes the notion of clean rings and, finally, idun-semicommutative rings satisfying the condition $xy = 0$ whenever $xvy = 0$ for every idempotent and unit v, generalizes semicommutative rings.

If we consider the class of all von Neumann regular rings, note that while P_r-rings generalize von Neumann regular ones (Example 3.3(2)), regular UG rings are precisely unit regular ones (Theorem 2.1) and regular idun-semicommutative rings are characterized

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as abelian regular (Theorem 4.7). Local rings forms one extreme class from the point of view of idempotents and units, since all their elements outside from the Jacobson radical are units and they contain trivial idempotents only. Note that local rings are UG rings and they satisfies the condition P_r (Example 3.3(1), Proposition 2.4(2)) and idun-semicommutative local rings are precisely semicommutative local rings.

Throughout this paper, R will be an associative ring with identity, $U(R)$ its group of units, $J(R)$ its Jacobson radical and $Id(R)$ its set of idempotents. The left and right annihilators of a subset X of a ring are denoted by $r_R(x)$ and $l_R(x)$, respectively. Recall that an element $a \in R$ is *(unit-)regular* if $a = aba$ ($a = aua$) for some $b \in R$ ($u \in U(R)$) and R is called a *(unit-)regular ring* if every element is (unit-)regular.

If a, b are elements of a ring R and $u \in U(R)$ such that $a = bu$, then a and b are called *right associated*. Clearly, right associated elements a and b are right multiples of each other, or they generate the same principal right ideals aR and bR. Note that R is a *UG ring* provided its every principal right ideal is uniquely generated up to associativity, i.e., $a, b \in R$ are right associated whenever $aR = bR$. The research of UG rings was started by Kaplansky ([9]) (see also [10]). By Marks [11], a von Neumann regular ring is unit-regular if and only if it is a left (right) UG ring (see also [10, Corollary 2.10]) and Theorem 2.1 extends this observation for any von Neumann regular ring R, namely R is unit-regular if and only if R has stable range 1 if and only if R is left(right) UG. Although the ring $R = End(V_D)$ is not left UG, where V_D is a vector space of countably infinite dimension over a division ring D, it holds true for each $a, b \in R$ that the condition $Ra = Rb$ implies $a = (e + u)b$ for a unit u and an idempotent e (cf. Example 2.2).

This motivates the definition of the conditions P_r and P_l. Recall that R satisfies P_r (or P_l) if for every $r \in R$ there exists $u \in U(R)$ and $e \in Id(R)$ such that $rR = (e + u)R$ (or $Rr = R(e + u)$). It is easy to see that local and regular rings satisfy the properties P_r and P_l (cf. Example 3.3). Recall that a ring R is *clean* if every element r of R is clean, i.e. there exist an idempotent $e \in R$ and an element $t \in U(R)$ such that $r = e + t$ [13]. Note that every clean ring satisfies the properties P_r and P_l.

A ring R is semiperfect if $R/J(R)$ is semisimple and all idempotent of $R/J(R)$ lifts modulo $J(R)$. We prove in Theorem 3.8 that every semiperfect ring satisfies the properties

P_r and P_l. We also give description of the properties P_r and P_l in particular classes of rings. Namely, a domain R satisfies the property P_r if and only if $R = U(R) \cup (U(R) + U(R))$ (Proposition 3.4). Furthermore, if $R/J(R)$ is unit regular and every idempotent of $R/J(R)$ lifts modulo $J(R)$, then R satisfies the properties P_r and P_l Proposition 3.7.

Recall that a ring R is said to be *semicommutative* if $xy = 0$ implies $xRy = 0$ for each $x, y \in R$ or, equivalently, if the right (left) annihilator of each element of R is an ideal. We introduce the notions of idun-semicommutative rings which are obtained by formally replacing the whole ring in the above definitions, by $Id(R) + U(R)$; a ring R is called *idun-semicommutative* if $xy = 0$ implies $x(Id(R) + U(R))y = 0$ for all $x, y \in R$. As it is remarked we show that every local ring is idun-semicommutative (Corollary 4.12) and regular idun-semicommutative rings are precisely abelian regular ones (Theorem 4.7)

Finally, let A and B be two rings with identity, K an ideal of B and $f : A \to B$ a ring homomorphism. We consider the subring of $A \times B$, defined by $A \bowtie^f K := \{(a, f(a)+k) | a \in A, k \in K\}$ which is called *amalgamated* construction of with A with B along K with respect to f. In [7], clean-like properties of the amalgamation ring $A \bowtie^f K$ of A with B along K with respect to f. We examine closure properties of the amalgamation construction $A \bowtie^f K$ and decomposition for the classes of UG, P_r and semicommutative rings.

2. UG rings

Recall that a ring R every principal right ideal is uniquely generated up to associativity We should point out that it is apparently unknown whether right UG is equivalent to left UG. For instance, all domains have (left and right) UG.

Let us recall that a ring R has *stable range 1* if for any $a, b \in R$ with $aR + bR = R$, there exists $x \in R$ such that $a + bx \in U(R)$. We show that every ring with stable range 1 is left (and right) UG, while the converse fails as $\mathbb{Z}$ is UG:

Theorem 2.1. *The following are equivalent for any von Neumann regular ring R:*

(1) *R is unit-regular.*
(2) *R has stable range 1.*
(3) *R is left(right) UG.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from [6, Proposition 4.12] and the equivalence (1) $\Leftrightarrow$ (3) is proved in [11, Theorem]. $\square$

Example 2.2. We have shown that von Neumann regular rings need not be UG. Suppose V_D is a vector space of countably infinite dimension over a division ring D. Hence $End(V_D)$ is a von Neumann regular ring which is not unit-regular, which implies that $End(V_D)$ is not a left UG ring by Theorem 2.1.

A topological space X is called *strongly zero-dimensional* if X is a non-empty completely regular Hausdorff space and every finite functionally open cover $\{U_i\}_{i=1}^k$ of the space X has a finite open refinement $\{V_i\}_{i=1}^m$ such that $V_i \bigcap V_j = \emptyset$, whenever $i \neq j$. A T_1-space X which has a base consisting of closed sets is called *zero-dimensional*.

Let us construct an example of a non-regular UG ring.

Example 2.3. Every unit-regular ring is left (right) UG by Theorem 2.1. For the converse, let X be a zero-dimensional space which is not a strongly zero-dimensional (e.g., Dowker's example, see [5, 6.2.20]). In [4], Canfell showed that if X is a zero-dimensional space then $C(X)$ is a UG ring. On the other hand, $C(X)$ is not clean (see, [2, Theorem 2.5] or [12, Theorem 13]). Hence $C(X)$ is a UG ring which is not unit-regular.

Now, we formulate several closure properties of the class of all UG rings.

Proposition 2.4. *Let R_i, $i \in \Lambda$, and R be rings.*

1. *If R is local, then it is left and right UG.*
2. *$\prod_{i\in\Lambda} R_i$ is a UG ring if and only if R_i is UG for each $i \in \Lambda$.*
3. *Every commutative perfect left and right UG.*
4. *Every domain is left and right UG.*

Proof. (1) Let R be a local ring and $a, b \in R$ such that $aR = bR \neq 0$. Denote by $\pi : R \to aR$ the projection $\pi(r) = ar$ for each $r \in R$ and put $I = \operatorname{Ker}\pi$. If $b = au$ and $a = bv$, then $\pi(uv) = a$. Since $bR = aR$, we have $\pi(1) = \pi(uv)$, hence $uv + I = 1 + I \in R/I$ and so $1 - uv \in I \subseteq J(R)$. Then $uv \notin J(R)$ which means that uv is a unit in R. The symmetric argument says that vu is a unit, hence $u, v \in U(R)$.

(2) This follows from the fact that $a\prod_{i\in\Lambda} R_i = \prod_{i\in\Lambda} \pi_i(a)R_i$ for natural projections π_i.

(3) Since commutative perfect rings are isomorphic to finite products of local rings, the assertion follows from (1) and (2).

(4) If $aR = bR$ for nonzero $a, b \in R$, then there are $r, s \in R$ for which $ar = b$ and $a = bs$. Hence $a(rs - 1) = 0$ and $b(rs - 1) = 0$, which implies $r, s \in U(R)$. □

Let us make one easy observation on the amalgamation construction and then a description of closure properties of UG rings in case of the amalgamated rings.

Lemma 2.5. *If A and B be a pair of rings, $f : A \to B$ be an injective ring homomorphism and K be a proper ideal of B with $f(A) \cap K = 0$, then the rings $A \bowtie^f K$ and $f(A) + K$ are isomorphic.*

Proof. Desired isomorphism is induced by the canonical projection $\pi_B : A \times B \to B$. □

Theorem 2.6. *Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and K be a proper ideal of B. Then the followings hold.*

1. *Let A be a UG ring. If, for every $a \in A$ and $i, j \in K$ such that $(f(a)+i)(f(A)+K) = (f(a) + j)(f(A) + K)$, there exists $k \in K$ satisfying $1 + k \in U(f(A) + K)$ and $j - i = (f(a) + i)k$, then $A \bowtie^f K$ is a UG ring.*
2. *If $A \bowtie^f K$ is a UG ring, then so is A.*
3. *If f is injective, A reduced and K is a nil ideal, then $A \bowtie^f K$ is a UG ring if and only if $f(A) + K$ is a UG ring.*

Proof. Let $R := A \bowtie^f K$.

(1) Suppose that $(a, f(a) + i)R = (b, f(b) + j)R$ for $a, b \in A$ and $i, j \in K$. Then there exists $u \in U(A)$ such that $au - b$ by the hypothesis, hence $(u, f(u)) \in U(R)$ and

$$(a, f(a) + i)(u, f(u)) = (au, f(a)f(u) + if(u)) = (b, f(b) + if(u)).$$

Since $(b, f(b) + if(u))R = (b, f(b) + j)R$, we get

$$(f(b) + j)(f(A) + K) = (f(b) + if(u))(f(A) + K).$$

Then, by the hypothesis, there there exists $k \in K$ satisfying $1 + k \in U(f(A) + K)$ and $j - if(u) = (f(b) + if(u))k$, hence $f(b) + j = (f(b) + if(u))(1 + k)$. Now, the element

$v = (u, f(u)) \cdot (1, 1 + k) = (u, f(u) + f(u)k) \in U(R)$ satisfies $(a, f(a) + i)v = (b, f(b) + j)$ as desired.

(2) Suppose that R is a UG ring and let $xA = yA$ for $x, y \in A$. Then it is easy to compute that $(x, f(x))R = (y, f(y))R$, hence there $(u_1, u_2) \in U(R)$ such that $(x, f(x)) = (y, f(y))(u_1, u_2)$ by the hypothesis. Since $u_1 \in U(A)$ and $xu_1 = y$, the ring A is UG.

(3) Since $f(A) \cap K = 0$, Lemma 2.5 implies that $A \bowtie^f K \cong f(A) + K$. $\square$

Finally, the following example illustrates possible relations between rings $A \bowtie^f K$ and $f(A) + K$.

Example 2.7. Let E be any ring which is not UG, e.g. the endomorphism ring from Example 2.2 and $A = \mathbb{Z}\langle x_r, r \in E\rangle$ the free polynomial ring in noncommuting variables $\{v_r \mid r \in E\}$. Then a map $x_r \to r$ induces a surjective ring homomorphism $f : A \to E$, where A is a UG ring by Proposition 2.4(4). It shows that a factor of a UG ring need not be UG and, moreover, the example of a UG ring $A \bowtie^f 0 \cong A$ such that $f(A) + 0 \cong E$ is not UG.

3. Rings satisfying the condition P_r

Recall the definition of the properties P_r and P_l:

P_r: For every $r \in R$ there exists $u \in U(R)$ and $e \in Id(R)$ such that $rR = (e + u)R$

P_l: For every $r \in R$ there exists $u \in U(R)$ and $e \in Id(R)$ such that $Rr = R(e + u)$

We start the section with an easy reformulation of the definition.

Lemma 3.1. *A ring R satisfies the (right) property P_r if and only if for every $r \in R$ there exist $u \in U(R)$ and $e \in Id(R)$ such that $rR = (1 + eu^{-1})R$.*

Proof. Clearly, there exists $u \in U(R)$ and $e \in Id(R)$ such that $rR = (e + u)R$ if and only if $rR = (1 + eu^{-1})R$. $\square$

Let us formulate several elementary observation about sets $Id(R)$, $U(R)$ and $J(R)$.

Lemma 3.2. *Let R be a ring.*

(1) *If $j \in J(R)$, then $jR = (1 + (j - 1))R$ where $1 \in Id(R)$ and $j - 1 \in U(R)$.*

(2) *If* $u \in U(R)$*, then* $uR = (0 + u)R$ *where* $0 \in Id(R)$ *and* $u \in U(R)$*.*

(3) *If* $e \in Id(R)$*, then* $eR = (-1 + (1 - e))R$ *where* $1 - e \in Id(R)$ *and* $-1 \in U(R)$*.*

(4) *If* $r = u + v \in U(R) + U(R)$*, then* $rR = (1 + vu^{-1})R$ *where* $1 \in Id(R)$ *and* $vu^{-1} \in U(R)$*.*

Example 3.3. (1) Every clean ring satisfies the properties P_r and P_l.

(2) Every local ring satisfies the properties P_r and P_l. Indeed, since $R = U(R) \cup J(R)$ for a local ring R, the assertion follows from Lemma 3.2(1) and (2).

(3) Every von Neumann regular ring satisfies the properties P_r and P_l by Lemma 3.2(3).

(4) According to Zhang and Tong [15] an element of ring is *G-clean* if it is the sum of a unit regular element and a unit. Equivalently, a ring is G-clean if every element is a unit multiple of a clean element. Remark that if a ring has the property that every element has a right unit multiple that is clean, then this also satisfying P_r. Hence due to the facts about the right UG property, there are rings satisfying P_r which are not G-clean.

Now we can describe domains satisfying P_r:

Proposition 3.4. *Let* R *be a ring.*

(1) *If* $Id(R) = \{0, 1\}$*, then* R *satisfies the property* P_r *if and only if for each* $r \in R \setminus U(R)$ *there exists* $u \in U(R)$ *such that* $rR = (1 + u)R$*.*

(2) *If* R *is a domain, then* R *satisfies* P_r *if and only if* $R = U(R) \cup (U(R) + U(R))$*.*

Proof. (1) It follows from the definition and Lemma 3.2(2).

(2) Note that $Id(R) = \{0, 1\}$ since R is a domain and that $0 = 1 + (-1) \in U(R) + U(R)$. If $r \in R\setminus(U(R) \cup \{0\})$ and R satisfies the property P_r, then there exists $u \in U(R)$ such that $rR = (1 + u)R$ by (1). Hence there exist $s, t \in R$ such that $rs = (1 + u)$ and $(1 + u)t = r$ which implies $rst = r$ and $(1 + u)ts = (1 + u)$. As R is a domain we get that $st = 1$ and $ts = 1$, hence $t \in U(R)$. Now $r = t + ut \in U(R) + U(R)$.

If $R = U(R) \cup (U(R) + U(R))$, then R satisfies the property P_r by Lemma 3.2(2) and (4). □

Example 3.5. Let X be a connected space. $C(X)$ satisfies the property P_r if and only if is UG. Indeed, it is well-known that X is connected if and only if 0 and 1 are the only idempotents in $C(X)$, so $C(X)$ satisfies the property P_r if and only if is UG by Proposition 3.4.

Example 3.6. (1) The ring of integers $\mathbb{Z}$ is a UG ring which is not P_r by Proposition 3.4(2).

(2) The von Neumann regular ring $End(V_D)$ in Example 2.2 satisfies P_r which is not UG .

For a ring R, we recall that $u \in U(R)$ if and only if $u + J(R) \in U(R/J(R))$.

The following closure properties of the class of rings satisfying P_r will then be used in the description of semiperfect and amalgamated ring.

Proposition 3.7. *Let R, R_i $(i \in \Lambda)$ be rings and I be an ideal of R.*

1. *If R satisfies P_r, then R/I satisfies the property P_r .*
2. *$\prod_{i\in\Lambda} R_i$ satisfies the property P_r if and only if each R_i satisfies P_r for each $i \in \Lambda$.*
3. *If $R/J(R)$ is unit regular and every idempotent of $R/J(R)$ lifts modulo $J(R)$, then R satisfies both the properties P_r and P_l .*
4. *If R is a commutative ring, then the polynomial ring $R[x]$ does not satisfy P_r.*

Proof. (1) The assertion follows from the fact that the homomorphic images of idempotents are idempotents and the homomorphic images of units are units.

(2) Let $R = \prod_{i\in\Lambda} R_i$. By (1), it suffices to prove the reverse implication. If $r = (r_i)_{i\in\Lambda} \in R$, then there are $e_i \in Id(R_i)$ and $u_i \in U(R_i)$ such that $r_iR = (e_i+u_i)R$. Now, it is obvious that $rR = ((e_i)_{i\in\Lambda} + (u_i)_{i\in\Lambda})R = (e_i + u_i)_{i\in\Lambda}R$ where $(e_i)_{i\in\Lambda} \in Id(R)$ and $(u_i)_{i\in\Lambda} \in U(R)$.

(3) Let $a \in R$. Then there exists $u \in U(R)$, $e \in Id(R)$, and $j \in J(R)$ such that $au = e + j$, Thus $a(-u) = (1 - e) - (1 + j)$, and so $aR = ((1 - e) - (1 + j))R$ with $(1 - e) \in Id(R)$ and $-(1 + j) \in U(R)$ which proves that R satisfies the property P_r.

(4) Assume that $R[x]$ satisfies the property P_r. Then there exists a maximal ideal I of R such that $R/I[x]$ is a domain which satisfies the property P_r by (1). Clearly $x \notin U(R/I[x]) \cup \{0\}$. If $x = u + v \in U(R/I[x]) + U(R/I[x])$, then $xv^{-1} - uv^{-1} = 1$, a contradiction. □

By Example 3.3(1), every von Neumann regular ring satisfies the properties P_r and P_l. Recall that a ring R is *semiperfect* if $R/J(R)$ is semisimple and all idempotent of $R/J(R)$ lifts modulo $J(R)$ and R is right *max* if every nonzero right module contains a maximal submodule. An ideal $I \subset R$ is *right T-nilpotent*, provided for every sequence $a_1, a_2, \cdots, \in I$ there exists n such that $a_n a_{n-1} \cdots a_1 = 0$.

Theorem 3.8. *Every semiperfect ring and every commutative max ring satisfy both the properties P_r and P_l.*

Proof. Since any semisimple ring is unit regular, we can easily say that semiperfect rings satisfy the properties P_r and P_l by Proposition 3.7(3).

If R is a commutative max ring, then $J(R)$ is T-nilpotent by [1, Remark 28.5] and $R/J(R)$ is commutative regular by [14], hence R satisfies the properties P_r and P_l by Proposition 3.7(3) again. □

Corollary 3.9. *Every right perfect ring satisfies P_r and P_l.*

Theorem 3.10. *Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and K be a proper ideal of B. Then the followings hold.*

(1) *If A is a clean ring and K is nil, then $A \bowtie^f K$ is clean and so it satisfies P_r.*
(2) *If $A \bowtie^f K$ satisfies the property P_r, then both A and $f(A) + K$ satisfy P_r.*

Proof. Put $R := A \bowtie^f K$.

(1) Suppose that A is a clean ring and let $(x, f(x)+t) \in R$. Then $x = e+u$ for $e \in Id(A)$, $u \in U(A)$, hence $f(x) = f(e+u) = f(e) + f(u)$ satisfying

$$f(e) \in Id(f(A)) \subseteq Id(f(A) + K) \text{ and } f(u) \in U(f(A)) \subseteq U(f(A) + K).$$

Note that K is a nil ideal of the ring $f(A)+K$, which implies $f(u)+t \in\subseteq U(f(A)+K)$, and so $(u, f(u)+t) \in U(R)$ by [7, Lemma 2.5(2)]. Thus $(x, f(x)+t) = (e, f(e))+(u, f(u)+t) \in Id(R) + U(R)$.

(2) Consider the canonical projections $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$. Clearly, $\pi_A(R) = A$ and $\pi_B(R) = f(A) + K$. Thus A and $f(A) + K$ satisfy the property P_r by Proposition 3.7(1). □

4. Idun-semicommutative Rings

Recall that a ring R is called *idun-semicommutative* if $xy = 0$ implies $x(Id(R)+U(R))y = 0$ for all $x, y \in R$.

Proposition 4.1. *Semicommutative rings are idun-semicommutative. The converse is true for clean rings.*

Proof. Let R be a semicommutative ring and let $xy = 0$ for any $x, y \in R$. Semicommutativity of R gives us $x(Id(R) + U(R))y \subseteq xRy = 0$, as desired.

Since $R = Id(R) + U(R)$ for any clean ring R, the converse is clear. □

A ring R is said to be *reduced* if it contains no non-zero nilpotent elements. Equivalently, a ring is reduced if it has no non-zero elements with square zero, that is, $x^2 = 0$ implies $x = 0$.

Example 4.2. Every reduced ring is semi-commutative, hence idun-semicommutative. In, particular, every product of domains is idun-semicommutative.

Now, we formulate equivalent conditions to the idun-semicommutativity.

Proposition 4.3. *Let R be a ring. The following statements are equivalent:*

(1) *R is an idun-semicommutative ring,*
(2) *for any $x \in R$, $(Id(R) + U(R))r_R(x) \subseteq r_R(x)$,*
(3) *for any $x \in R$, $l_R(x)(Id(R) + U(R)) \subseteq l_R(x)$,*
(4) *for any $x \in R$, $(Id(R) + U(R))r_R(x) = r_R(x)$,*
(5) *for any $x \in R$, $l_R(x)(Id(R) + U(R)) = l_R(x)$,*
(6) *for any $x \in R$, $(Id(R) + U(R))r_R(x) = r_R(x)(Id(R) + U(R))$,*
(7) *for any $x \in R$, $l_R(x)(Id(R) + U(R)) = (Id(R) + U(R))l_R(x)$.*

Proof. It is enough to prove the equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (6), as the UG condition is left-right symmetric, hence the equivalence (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) $\Leftrightarrow$ (7) follows from the symmetric argument.

(1) $\Rightarrow$ (2) Let $x \in R$, $e \in Id(R)$ and $u \in U(R)$ with $r \in r_R(x)$. Since $xr = 0$ and R is idun-semicommutative, we get $x(e + u)r = 0$. Thus $(e + u)r \in r_R(x)$, as desired.

(2) $\Rightarrow$ (4) As $1 = 0+1 \in Id(R)+U(R)$, we obtain $r_R(x) \subseteq (Id(R)+U(R))r_R(x) \subseteq r_R(x)$ which implies the equality.

(4) $\Rightarrow$ (6) Since $r_R(x)$ is a right ideal, we obtain the inclusion $r_R(x)(Id(R) + U(R)) \subseteq r_R(x)$. As $1 = 0 + 1 \in Id(R) + U(R)$ we have $r_R(x)(Id(R) + U(R)) = r_R(x) = (Id(R) + U(R))r_R(x)$.

(6) $\Rightarrow$ (1) Let $xy = 0$ for $x, y \in R$ and $e \in Id(R)$, $u \in U(R)$. As $y \in r_R(x)$, we get $(e + u)y \in r_R(x)(Id(R) + U(R))$, hence $x(e + u)y = 0$. □

Proposition 4.4. *Idun-semicommutative rings are abelian.*

Proof. Let R be an idun-semicommutative ring and $e \in Id(R)$. We have $r_R(1 - e) = eR$. By Lemma 4.3(4), we obtain $(Id(R) + U(R))eR = eR$. Clearly, $1 - (1 - e)re$ is a unit for every $r \in R$. Then $(e + (1 - (1 - e)re) \in (Id(R) + U(R))$ and $(e + (1 - (1 - e)re)e = (e+(e-(1-e)re) \in eR = r_R(1-e)$. Thus we obtain $(1-e)(e+(e-(1-e)re) = 0$ which implies $-(1 - e)re = 0$ and $re = ere$. Similarly, we get $er = ere$ by $r_R(e) = (1 - e)R$. Hence $er = re$, i.e., the idempotent e is central. □

We can formulate an easy consequence of the last assertion:

Corollary 4.5. *Let R be a ring and n be a natural number. Then the matrix ring $M_n(R)$ is idun-semicommutative if and only if R is idun-semicommutative and $n = 1$.*

In the following two observations, we give some conclusion for the converse of 4.4.

Proposition 4.6. *The following conditions are equivalent for a ring R:*

1. *R is idun-semicommutative,*
2. *R is abelian and $ur_R(x) = r_R(x)$ for each $u \in U(R)$ and $x \in R$,*
3. *R is abelian and $l_R(x)u = l_R(x)$ for each $u \in U(R)$ and $x \in R$.*

Proof. (1) $\Rightarrow$ (2) R is abelian by Proposition 4.4. If $u \in U(R)$ and $x \in R$, then $ur_R(x) \subseteq r_R(x)$ and $u^{-1}r_R(x) \subseteq r_R(x)$ by Proposition 4.3(2). Hence $r_R(x) \subseteq ur_R(x)$ and so $ur_R(x) = r_R(x)$.

(2) $\Rightarrow$ (1) Since $Id(R)r_R(x) = Id(R)r_R(x) \subseteq r_R(x)$ and $U(R)r_R(x) \subseteq r_R(x)$, then $(Id(R) + U(R))r_R(x) \subseteq r_R(x)$ for each $x \in R$ as $r_R(x)$ is a right ideal, R is idun-semicommutative by Proposition 4.3(2).

(1) $\Leftrightarrow$ (3) The argument is symmetric using Proposition 4.3(3). $\square$

Theorem 4.7. *The following conditions are equivalent for a Von Neumann regular ring R:*

(1) *R is idun-semicommutative,*
(2) *R is abelian,*
(3) *R is semicommutative.*

Proof. (1) $\Rightarrow$ (2) The implication follows from Proposition 4.4.

(2) $\Rightarrow$ (3) If $xy = 0$, then there exist a central idempotent e such that $yR = eR = Re$, hence $0 = xyR = xReR = xRy$.

(3) $\Rightarrow$ (1) The implication is clear. $\square$

We recall two natural generalizations of commutative rings.

A ring R is called *symmetric* if $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$, and R is called is *reversible* if, for any $a, b \in R$, $ab = 0$ if and only if $ba = 0$.

Proposition 4.8. *Every reversible rings is idun-semicommutative.*

Proof. Let be R reversible ring. Assume that $ab = 0$ for any $a, b \in R$. Since R is reversible, we get $ba = 0$. Hence $ba(e + u) = 0$ for $e \in Id(R)$ and $u \in U(R)$. Clearly, $a(e + u)b = 0$, which implies that R is an idun-semicommutative ring. $\square$

As each symmetric ring is reversible we obtain the following consequence.

Corollary 4.9. *Every symmetric ring is idun-semicommutative.*

In the following assertion, we collect three several algebraic properties of idun-semicommutative rings.

Proposition 4.10. *Let R, R_i $(i \in \Lambda)$ be rings and I be an ideal of R.*

(1) *Any subring of an idun-semicommutative ring is idun-semicommutative as well.*
(2) *$\prod_{i\in\Lambda} R_i$ is idun-semicommutative if and only if R_i is idun-semicommutative for each $i \in \Lambda$.*

(3) *If R is an idun-semicommutative ring, then $J(R)r_R(x) \subseteq r_R(x)$ for each $x \in R$.*

(4) *If R is an idun-semicommutative ring and $e \in Id(R)$, then the corner ring eRe is idun-semicommutative.*

Proof. (1) If S is a subring of an idun-semicommutative ring R and $x, y \in S \subseteq R$ such that $xy = 0$, then $x(Id(S) + U(S))y \subseteq x(Id(R) + U(R))y = 0$, since $Id(S) \subseteq Id(R)$ and $U(S) \subseteq U(R)$.

(2) Let us denote by $\pi_i$the canonical projection for each $i \in I$.

The necessity: Let $\alpha_i, \beta_i \in R_i$ and $\alpha_i\beta_i = 0$ and let $\alpha, \beta \in \prod_{n\in\Lambda} R_n$ for which $\pi_i(\alpha) = \alpha_i$, $\pi_i(\beta) = \beta_i$ and $\pi_j(\alpha) = \pi_j(\beta) = 0$ for all $j \neq i$. Then $\alpha\beta = 0$, hence $\alpha(Id(R)+U(R))\beta = 0$ by the hypothesis. Since $\pi_i(Id(R)) = R_i$ and $\pi_i(U(R)) = U(R_i)$, we have $\alpha_i\pi_i(Id(R) + U(R))\beta_i = 0$ as desired.

The sufficiency: Suppose that $\alpha = (\alpha_i)_{i\in\Lambda}, \beta = (\beta_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} R_i$ such that $\alpha\beta = 0$. Then $\alpha_i\beta_i = 0$ for each $i \in \Lambda$. Since R_i is idun-semicommutative, we get $\alpha_i(Id(R_i) + U(R_i))\beta_i = 0$. Hence $\alpha(Id(R)+U(R))\beta = 0$, it shows that $\prod_{i\in\Lambda} R_i$ is idun-semicommutative.

(3) Let R be an idun-semicommutative ring. Assume that $j \in J(R)$ and $y \in r_R(x)$. Then $1 - j = 0 + (1 - j) \in Id(R) + U(R)$. By the hypothesis , we get $x(1 - j)y = 0$. Therefore $xjy = 0$, so $jy \in r_R(x)$. It shows that $J(R)r_R(x) \subseteq r_R(x)$.

(4) Let R be an idun-semicommutative ring. Suppose $exe, eye \in eRe$ and $(exe)(eye) = 0$. Since R is an idun-semicommutative ring, $(exe)(Id(R) + U(R))(eye) = 0$. Then

$$(exe)(e(Id(R) + U(R))e)(eye) = 0.$$

Thus

$$(exe)(Id(eRe) + U(eRe))(eye) \subseteq (exe)(eId(R)e + eU(R)e)(eye) = 0,$$

so eRe is an idun-semicommutative ring. □

If X is a subset of R we denote by $\langle X\rangle$ the subgroup of the abelian group $(R, +, -, 0)$ generated by the set X.

Theorem 4.11. *Let $R = \langle U(R) \cup Id(R) \cup J(R)\rangle$. Then $R = \langle U(R) \cup Id(R)\rangle$ and R is idun-semicommutative if and only if R is semicommutative.*

Proof. If $j \in J(R)$, then $j - 1 \in U(R)$, hence $j \in \langle U(R) \cup Id(R)\rangle$. It proves that $J(R) \subseteq \langle U(R) \cup Id(R)\rangle$.

Let R be idun-semicommutative and $x \in R$. It is enough to prove that $r_R(x)$ is two sided ideal. Clearly it is left ideal, hence $\langle r_R(x)\rangle = r_R(x)$. Then by Proposition 4.10(3) we have $(Id(R) + U(R))r_R(x) = r_R(x)$, hence

$$Rr_R(x) = \langle U(R) \cup Id(R)\rangle r_R(x) = \langle (Id(R) + U(R))r_R(x)\rangle = \langle r_R(x)\rangle = r_R(x).$$

We have proved that $r_R(x)$ is two-sided ideal for each $x \in R$, thus R is semicommutative.

The reverse implication is obvious. □

Corollary 4.12. *A local ring is idun-semicommutative if and only if it is semicommutative.*

Example 4.13. By Gerasimov and Sakhaev's Example ([8]) there exist some semilocal rings with no non-trivial idempotents satisfying $U(R)r_R(x) \subseteq r_R(x)$ for each $x \in R$ which is not local containing x, y such that $xy = 0$ and $xRy \neq 0$ is idun-semicommutative but non-semicommutative.

Proposition 4.14. *A semiperfect ring is idun-semicommutative if and only if it is semicommutative.*

Proof. By Proposition 4.1 it is enough to prove the direct implication. Since R semiperfect and idun-semicommutative, it is semiperfect and abelian by Proposition 4.4, hence there exists a sequence of central orthogonal idempotents $e_1, \dots e_n$ such that $R \cong \prod_i e_i R$ where $e_i R$ is a local idun-semicommutative ring by Proposition 4.10(4). Since $e_i R$ is semicommutative by Corollary 4.12, then it is easy to see that $R \cong \prod_i e_i R$ is semicommutative as well. □

To conclude the paper we formulate an observation and an example on closure properties of amalgamation constructions and decompositions of idun-semicommutative rings.

Theorem 4.15. *Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and K be a proper ideal of B. Then the followings hold.*

(1) *If A and $f(A) + K$ are idun-semicommutative, then so is $A \bowtie^f K$.*
(2) *If $A \bowtie^f K$ is idun-semicommutative, then so is A.*

(3) *If f is injective, A reduced and K is a nil ideal, then $A \bowtie^f K$ is idun-semicommutative if and only if $f(A) + K$ is so.*

Proof. (1) If A and $f(A)+K$ are idun-semicommutative, then the product $A \times (f(A)+K)$ is idun-semicommutative by Proposition 4.10(2). Since $A \bowtie^f K$ is a subring of the ring $A \times (f(A) + K)$, it is idun-semicommutative by Proposition 4.10(1).

(2) As A is isomorphic to a subring of the idun-semicommutative ring $A \bowtie^f K$, it is is idun-semicommutative by Proposition 4.10(1).

(3) $A \bowtie^f K \cong f(A) + K$ by Lemma 2.5 since $f(A) \cap K = 0$. □

Example 4.16. Similarly as in Example 2.7 we denote by E a ring which is not idun-semicommutative, for example a matrix ring over a field F consisting 2×2 where $xy = 0$ and $xuy \neq 0$ for

$$x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in Id(E),\ u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in U(E).$$

If $A = F\langle x_{11}, x_{12}, x_{21}, x_{22}\rangle$ denotes the free polynomial ring in noncommuting variables over F, then any mapping of variables onto a F-basis of E can be extended to a surjective ring homomorphism $f : A \to E$, where A is a domain, hence an idun-semicommutative ring. Repeating the argument of Example 2.7 $A \bowtie^f 0 \cong A$ is an idun-semicommutative ring with a non-idun-semicommutative factor $f(A) + 0 \cong E$.

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