ON THE STRUCTURE OF HYPERSIMPLE RINGS

M. TAMER KOŞAN, ÖZGÜR TAŞDEMIR, AND JAN ŽEMLIČKA

Abstract. The aim of the present paper is to describe the structure of hypersimple rings, i.e. rings over which injective hulls of simple modules are cyclic. In particular, hypersimple duo rings are characterized in terms of local pseudo-Frobenius rings, and commutative Noetherian hypersimple rings are proven to be exactly quasi-Frobenius rings. Various examples and further divisions of the class of all hypersimple rings are also presented in the paper.

1. Introduction

While Barbara Osofsky in the paper [17] investigated the class of rings over which cyclic modules are injective and showed that such rings are precisely semisimple Artinian ones, the work [3] generalized this problem and introduced the notion of a right *hypercyclic* ring, as a ring over which cyclic right modules have cyclic injective hulls. In particular, it is proved there that left perfect, right hypercyclic rings are Artinian and uniserial. Recently, Lomp et.al [16] obtained various interesting results about hypersimple rings, which generalize the notion of a hypercyclic ring; a module is said to be *hypersimple* if it is simple and its injective hull is cyclic, and a ring R is left (resp. right) *hypersimple*, provided all left (right) simple modules are hypersimple. Clearly, any hypercyclic ring is hypersimple.

In the first half of this note, we continue to describe the classes of hypersimple rings and modules. In particular, we formulate several conditions under which a simple module is injective that presents a borderline example of a hypersimple module (Proposition 2.2 and 2.3). Next, we prove a criterion for duo hypersimple rings in terms of local pseudo-Frobenius rings (Theorem 2.7), which are rings such that the injective hull of a simple module is projective [16, Theorem 2.6]. As a consequence we describe semilocal hypersimple commutative rings as precisely pseudo-Frobenius rings (Corollary 2.8), and Noetherian hypersimple commutative rings, which are quasi-Frobenius (Theorem 2.11). Recall that a ring is said to be right *max* if every non-zero right module over the ring contains a maximal submodule. Although it is much less known about the structure of non-commutative hypersimple rings, we prove that right Noetherian right hypersimple rings are right max by Proposition 2.10.

In [12], Köthe considered those rings over which each right module is a direct sum of cyclic modules, and hence such rings are called right Köthe rings in the literature. In that paper, Köthe showed that right modules over Artinian principal ideal rings are direct sums of cyclic modules. In the commutative case the class of Köthe rings was determined by Cohen and Kaplansky [4]: commutative Köthe rings are Artinian principal ideal rings. In the series of papers [9, 10, 11], Kawada concerned algebras for which every indecomposable module is cyclic. Recall that a ring R is of finite representation type if R is a right Artinian ring which has, up to isomorphism, only finite number of finitely generated indecomposable (left) right

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modules. Note that Köethe rings present examples of rings of finite representation type.

Inspired by these results we focus on two subclasses of the class of all hypersimple rings; rings whose injective indecomposable right R-modules are cyclic, such rings are said to be right *IIMC*, and rings whose all indecomposable right R-modules are cyclic called *IMC* rings. Clearly, any right IMC ring is a right IIMC ring and it is right hypersimple. However Examples 3.4 and 3.20 witness that both implication cannot be reversed in general, Proposition 3.5 shows that a right semiartinian (and so left perfect) ring is hypersimple if and only if it is IIMC. Finally, we characterize the structure of duo IIMC and commutative IMC rings; right duo IIMC rings are described in terms of injectivity of right uniform factor rings (Theorem 3.10), while commutative IMC rings are proven to be exactly max rings whose maximal local factors are chain Frobenius (Theorem 3.18). Finally note that all three studied classes of rings are closed under taking finite products and direct summands (Proposition 3.15)

Throughout the paper, R is a ring with identity and all modules are unital right (left) R-modules. For a module M, we write M_R to indicate that M is a right R-module, and we write $N \leq M$ if N is a submodule of M. We also denote by $E(M) = E_R(M)$, $\mathrm{End}_R(M)$ and $\mathrm{Soc}(M)$ the injective hull, the endomorphism ring and the socle of M, respectively. For two modules X, Y over a ring R, the set of all R-homomorphisms from X to Y is denoted by $\mathrm{Hom}_R(X, Y)$ or $\mathrm{Hom}(X, Y)$. If N is a non-empty subset of a right R-module M, we denote by $\mathrm{Ann}_R(N) =: \{r \in R \mid nr = 0 \text{ for all } n \in N\}$ the annihilator ideal of N in R. Finally, $J(M)$ denotes the Jacobson radical of a module M and $N(R)$ means the nil radical of commutative ring R.

2. Hypersimple modules and rings

We start the section by recalling a few notions which will allow us to find examples of hypersimple modules.

A module M is called *dual-Rickart* [15] if, for any $\rho \in End(M)$, $Im\rho \leq^{\oplus} M$, and M is called an *(n-)epi-retractable module* if every (n-generated) submodule of M is a homomorphic image of M [6]. Azumaya [2], a module M is said to be *regular* if every submodule of M is locally split in M (equivalently, every cyclic submodule of M is a direct summand of M [2, Proposition 5]).

Lemma 2.1. *Let R be a ring.*

(1) Any dual-Rickart epi-retractable R-module is semisimple.
(2) Any dual-Rickart 1-epi-retractable R-module is regular.

Proof. (1) Let N be a submodule of a dual-Rickart epi-retractable module M. By the epi-retractability of M, there exists an epimorphism $f : M \to N$. The dual-Rickartness of M implies that $f(M) = N$ is a direct summand of M. Hence M is semisimple.
(2) Let N be a cyclic submodule of a dual-Rickart 1-epi-retractable R-module M and let $f : M \to N$ be an epimorphism which exists by the 1-epi-retractability of M. Then the dual-Rickartness of M implies $f(M) = N$ is a direct summand of M, and so M is regular. □

Proposition 2.2. *Consider that for a simple module S over a ring R holds one from the following conditions:*

(1) $E(S)$ is 1-epi-retractable and R is right hereditary;
(2) $E(S)$ is a regular;
(3) $E(S)$ is dual-Rickart and 1-epi-retractable.

Then S is injective, hence it is a hypersimple module.

Proof. First, note that $E(S)$ is indecomposable by [18, Corollary 2] and S is a cyclic module.
(1) Since $E(S)$ is 1-epi-retractable, there exists an epimorphism $f : E(S) \to S$. As R is a right hereditary ring, we obtain $f(E(S)) = S$ is injective, and so S is a direct summand of $E(S)$. But $E(S)$ is indecomposable, which implies that $E(S) = S$.
(2) Since $E(S)$ is regular, we have that S is a a direct summand of $E(S)$, i.e $E(S) = S$.
(3) The claim follows from (2) and Lemma 2.1(2). □

Let M be a module over a ring R. Then by a *subfactor of* M, we mean a submodule of a factor module of M, and any module, which is isomorphic to some subfactor of a direct sum of copies of M, is called an M*-subgenerated* module. The full subcategory of the category of all right R-modules whose objects are all M-subgenerated modules is denoted by $\sigma[M_R]$ and called the Wisbauer category of the module M (for details see [20]).

Proposition 2.3. *Let M be a module over a hereditary ring R and let R_1 be the endomorphism ring of a progenerator in $\sigma[M_R]$. Assume that every non-zero factor ring of R_1 is a quasi Frobenius ring and $N \in \sigma[M_R]$. If S is simple such that $E(S) \in \sigma[N_R]$, then S is injective, and so hypersimple.*

Proof. Let S be a simple right R-module. Then $E(S)$ is epi-retractable by [6, Theorem 3.5], hence there exists an epimorphism $f : E(S) \to S$. Since R is right hereditary, any homomorphic image of an injective module is injective, in particular, $f(E(S)) = S$ is injective, and so $E(S) = S$. □

Recall from [8, Example 2.1] that an Artinian ring is an *FGC* ring if and only if every indecomposable module R is cyclic.

Example 2.4. *Any Artinian FGC rings are right hypersimple, since $E(S)$ is indecomposable module for each simple right module S.*

Let us formulate several elementary observations about (indecomposable) injective modules, which we will use later.

Lemma 2.5. *Let R be a ring, E an injective R-module, $A := Ann_R(E)$ and S be a simple R-module. Then*

(1) $Hom(B, E) = 0$ for each right ideal $B \leq A$,
(2) if $S \leq E$ then A contains no subfactor isomorphic to S,
(3) if E is indecomposable then $End(E)$ is a local ring,
(4) if $E(S) \cong R/A$, then R/A is a local right self-injective ring with an essential socle and $Soc(R/A) \cong S \cong (R/A)/J(R/A)$.

Proof. (1) Let $B \leq A$ be a right ideal and $f \in \mathrm{Hom}(B, E)$. Then f can be extended to $\tilde{f} \in \mathrm{Hom}(R, E)$, and hence

$$f(B) = \tilde{f}(B) = \tilde{f}(1)(B) \subseteq EB \subseteq EA = 0,$$

because A annihilates E.
(2) If A contains a subfactor isomorphic to S, there exist nonzero $x \in A$ and an epimorphism $xR \to S$, which contradicts to (1).
(3) It is well-known that $\mathrm{End}(E)$ is a local ring with the unique maximal ideal formed by all non-injective endomorphisms (see e.g. in [14, Theorem 3.52]).
(4) Since $E(S) \cong R/A$ is injective and R/A has a structure of an R-module, we obtain that R/A is right self-injective and $\mathrm{End}(E(S)) \cong \mathrm{End}(R/A) \cong R/A$. As

$E(S)$ is injective, we get R/A is a local by (3), and it is the unique simple module up to an isomorphism over the local ring R/A as S forms a simple socle of R/A. □

Before we formulate a characterization of duo hypersimple ring we prove an easy description of local duo hypersimple rings. First, let us recall that a ring R is called right *pseudo-Frobenius* if R is an injective cogenerator in Mod-R; equivalently if R is a semiperfect right self injective ring with essential right socle (cf. [16, Theorem 2.6]). Note that any right pseudo-Frobenius ring presents an example of a right hypersimple ring by [16, Examples 2.4 and 2.5].

Proposition 2.6. *Let R be a local right duo ring. Then R is is right hypersimple if and only if it is right pseudo-Frobenius.*

Proof. Let R be a hypersimple local duo ring with the maximal ideal I and $A = \mathrm{Ann}(E(R/I))$. Since $E(R/I)$ is an injective cogenerator of the category of all modules over the local ring R, it is enough to prove that $R \cong E(R/I)$. The hypothesis that R is right hypersimple implies that there exists a right ideal A for which $E(R/I) \cong R/A$. As R is right duo, A is two-sided, which means that $A = \mathrm{Ann}_R(E(S))$. Since every non-zero module of a local ring contains a subfactor isomorphic a to the simple module, Lemma 2.5(2) implies that $A = 0$, hence $E(R/I) \cong R/A \cong R$.

The converse follows from [16, Examples 2.4 and 2.5]. □

Theorem 2.7. *The following conditions are equivalent for a right duo ring R:*

(1) R is right hypersimple;
(2) for each maximal right ideal I, there exists an ideal A_I such that R/A_I is local right hypersimple and A_I contains no subfactor isomorphic to R/I;
(3) for each maximal right ideal I, there exists an ideal A_I such that R/A_I is local right pseudo-Frobenius and A_I contains no subfactor isomorphic to R/I.

Proof. (1) ⇒ (2) Let I be a maximal right ideal and put $A = \mathrm{Ann}_R(E(R/I))$. Since R is right duo hypersimple, the same argument as in the proof of Proposition 2.6 shows that $E(R/I) \cong R/A$. Then R/A is a local ring by Lemma 2.5(4) and A contains no subfactor isomorphic to R/I by Lemma 2.5(2). Now, it is easy to see that $E_{R/A}(R/I) = E_R(R/I) = R/A$, which proves that R/A R is right hypersimple.
(2) ⇒ (3) It follows immediately from Proposition 2.6.
(3) ⇒ (1) Since R/I embeds into the cyclic module R/A_I, it is enough to prove that R/A_I is an injective R-module for arbitrary maximal right ideal I by applying the Bear criterion.

Let I be a maximal right ideal, U be an arbitrary right ideal of R and $f \in \mathrm{Hom}(U, R/A_I)$. Note that $f(U \cap A_I) \cap \mathrm{Soc}(R/A_I) = 0$ by the hypothesis, which implies that $f(U \cap A_I) = 0$. Thus f can be lifted modulo the natural projection

$$\pi_U : U \to (U + A_I)/A_I \cong U/(A_I \cap U),$$

i.e. there exists $\tilde{f} \in \mathrm{Hom}((U + A_I)/A_I, R/A_I)$ satisfying $f = \tilde{f}\pi_U$. Furthermore, $\tilde{f}$ can be extended to an endomorphism $\hat{f} \in \mathrm{End}(R/A_I)$. Now, $\hat{f}\pi_R$ is the desired extension of the monomorphism f, where $\pi_R : R \to R/A_I$ denotes the natural projection again. Hence $E(R/I)$ is a direct summand of the cyclic injective R-module R/A_I and we are done. □

For semilocal right duo rings the notions of hypersimple and pseudo-Frobenius rings coincide:

Corollary 2.8. *Let R be a semilocal right duo ring. Then the following conditions are equivalent:*

(1) R is right hypersimple;
(2) R is isomorphic to a finite product of local right pseudo-Frobenius rings;
(3) R is right pseudo-Frobenius.

Proof. (1) ⇒ (2) Let R be semilocal right hypersimple right duo ring. Then, by Theorem 2.7, there exist finitely many maximal right ideals I_i ($i \le n$), corresponding ideals A_i for which $\bigcap_{i\le n} I_i = J(R)$ and it holds for every $i \neq j$ that simple modules R/I_i and R/I_j are non-isomorphic. Furthermore, the factor-rings R/A_i are local right pseudo-Frobenius, and $\bigcap_{i\le n} A_i = 0$, since it contains no simple subfactor. Note that if $i \neq j$, then R/A_i and R/A_j are local with different maximal ideals, which implies that $R/(A_i + A_j)$ is zero, and so A_i and A_j are coprime ideals. Now, the general Chinese reminder theorem implies that $R \cong \prod_{i\le n} R/A_i$.
(2) ⇒ (3) It is clear immediately from the definition of a right pseudo-Frobenius ring.
(3) ⇒ (1) It follows from [16, Examples 2.4 and 2.5]. □

Recall that a ring is said to be right *max*, provided every non-zero right module over the ring contains a maximal submodule.

Lemma 2.9. *Let R be a right hypersimple ring. Then R is right max if and only if each non-zero submodule of each indecomposable injective cyclic module contains a maximal submodule.*

Proof. It is enough to prove the reverse implication. Suppose that R is not right max. Hence there exists a non-zero module M such that $M = J(M)$. It is easy to see that there exists a simple module S and a non-zero homomorphism $\varphi \in \mathrm{Hom}(M, E(S))$. Since $E(S)$ is cyclic by the hypothesis, we obtain that $\varphi(M)$ is a non-zero submodule of a cyclic module without maximal submodules, a contradiction. □

The previous observation allows us to prove useful property of Noetherian hypersimple rings.

Proposition 2.10. *Right Noetherian right hypersimple rings are right max.*

Proof. Let R be a right Noetherian right hypersimple ring. Since each non-zero submodule of any cyclic module over R is finitely generated, it contains a maximal submodule. Hence R is right max by Lemma 2.9. □

In the commutative case we obtain a criterion.

Theorem 2.11. *The following conditions are equivalent for a commutative Noetherian ring R:*

(1) R is hypersimple;
(2) R is Artinian and $E(R/J(R))$ is cyclic;
(3) R is quasi-Frobenius;
(4) $R \cong \prod_{i=1}^{n} R_i$, where each R_i, $i = 1, \dots, n$, is a local Frobenius ring.

Proof. (1)⇒(4) By Proposition 2.10, R is a max ring. Hence $R/J(R)$ is abelian regular and $J(R)$ is T-nilpotent by [7]. Since R is Noetherian, we get that $R/J(R)$ is semisimple and $J(R)$ is a finitely generated nil ideal, hence it is nilpotent. Thus R is an Artinian ring. As R is commutative, $R \cong \prod_{i=1}^{n} R_i$, where each R_i, $i = 1, \dots, n$, is a local Artinian hypersimple ring by Theorem 2.7. Finally R_i is Frobenius for each $i = 1, \dots, n$ by Lemma 2.5(4) and [14, Theorem 16.14].
(3)⇔(4) This follows from [14, Theorems 15.27 and 16.14].
(3)⇒(2) By [14, Theorem 15.1], R is Artinian and $E(R/J(R)) \cong R$ is cyclic.
(2)⇒(1) Since R is Artinian, arbitrary simple R-module S is embeddable into

$R/J(R)$ and so $E(S)$ is a direct summand of $E(R/J(R))$. As $E(R/J(R))$ is cyclic, $E(S)$ is cyclic as well. □

3. Stronger conditions

In this section we will consider two stronger ring conditions than this one defining hypersimple rings. First, define the notion of a *right IIMC* ring by the condition that all injective indecomposable right modules over it are cyclic (the abbreviation means injective indecomposable modules are cyclic).

Before we formulate the following elementary observation, recall that a module is said to be *uniform* if all its non-zero submodules are essential.

Lemma 3.1. *Let R be a ring. Then R is right IIMC if and only if $E(C)$ is cyclic for each uniform (cyclic) right R-module C.*

Proof. The direct implication is obvious. Suppose $E(C)$ is cyclic for each indecomposable cyclic right R-module C. Let E be an injective indecomposable right R-module and C an arbitrary non-zero cyclic submodule of E. Clearly, $E \cong E(C)$ and C is indecomposable, hence $E(C)$ is cyclic, as desired. □

Proposition 3.2. *Let R be a ring.*

(1) If R is quasi-Frobenius, then is right IIMC,
(2) If R is right IIMC, then R is right hypersimple.

Proof. (1) Let M be an arbitrary injective indecomposable R-module. Since R is quasi-Frobenius and $\mathrm{Soc}(M)$ is essential in M, $\mathrm{Soc}(M)$ is simple and the both modules $\mathrm{Soc}(M)$ and M are embeddable into R. As M is injective, it is isomorphic to a direct summand of R. Hence it is cyclic, as desired.
(2) Let S be a simple right R-module. Since $E(S)$ is indecomposable, $E(S)$ is cyclic by Lemma 3.1. □

The next examples show that no one implication from Proposition 3.2 can be reversed. First, we need to recall that R is said to be a (right) *V-ring*, if each simple (right) R-module is injective and note that right V-rings are right hypersimple by [16, Example 2.8].

Example 3.3. *Let R be an abelian regular ring and C be a cyclic right R-module. Note that C has a structure of an abelian regular ring, which implies that C is indecomposable if and only if it is simple. Since R is a (right) V-ring, any simple (right) R-module is injective, hence R is a right IIMC ring by Lemma 3.1, which is not quasi-Frobenius whenever R is not semisimple.*

A ring R is said to be right *semiartinian* if every non-zero right R-module has a non-zero socle ([5]).

Example 3.4. [14, Example 19.24] *If R is a complete discrete valuation ring with the unique simple module S, then the trivial extension $T = R \oplus E(S)$ of the ring R by the bimodule $E(S)$ is a local pseudo-Frobenius by* [14, Example 19.24 and Theorem 19.25]*, hence it is hypersimple by Proposition 2.6. Note that $R = T/E(S)$ is a domain and $E(R)$ is indecomposable injective, which is annihilated by the ideal $E(S)$ of T since $Soc(E(R)) = 0$ and $E(S)$ is semiartinian. If $E(R)$ was cyclic, it would be a homomorphic image of R, but no factor of R is injective. This is why T is not IIMC.*

Although the classes of all IIMC and all hypersimple differ, in the case of semiartinian rings they coincide:

Proposition 3.5. *Let R be a right semiartinian ring. Then R is right IIMC if and only if R is right hypersimple.*

Proof. By Proposition 3.2 it suffices to prove the reverse implication. Let E be an injective indecomposable right R-module and let R be right hypersimple. Since $\mathrm{Soc}(E)$ is essential in E, it is simple and $E \cong E(\mathrm{Soc}(E))$ is cyclic. □

Since any left perfect ring is right semiartinian (see, [1, Theorem 28.4]), we get the following consequence.

Corollary 3.6. *A left perfect ring is right IIMC if and only if it is right hypersimple.*

As every commutative noetherian hypersimple ring is Artinian and so perfect by Theorem 2.11, we can easily see that hypersimple and IIMC rings coincides in this case:

Corollary 3.7. *Let R be a commutative noetherian ring. Then R is IIMC if and only if R is hypersimple.*

Now, we focus on the case of duo IIMC rings.

We say that a two-sided ideal P of a ring R is a *right weak prime*, if R/P is right uniform, equivalently, if $P \neq R$ and for each $a, b \notin P$ we get $(aR+P)\cap(bR+P) \nsubseteq P$. Denote by $\mathcal{W}(R)$ the set of all right weak prime ideals of R. If $P \in \mathcal{W}(R)$, we say that $A_P \in \mathcal{W}(R)$ is *minimal with respect to* P, if R/P is embeddable as a right module into R/A_P and for each $Q \in \mathcal{W}(R)$, it holds that $A_P \subseteq Q$ whenever R/P is embeddable into R/Q.

Example 3.8. *Let R be a right duo ring. Then every right maximal ideal of R belongs to $\mathcal{W}(R)$. Furthermore, if I is a maximal ideal, then the ideal A_I from Theorem 2.7 is minimal with respect to I.*

Lemma 3.9. *Let R be a right duo ring, M an injective indecomposable module, $a \in M$ a non-zero element, $P = Ann(aR)$, and let $A_P \in mathcalW(R)$ be minimal with respect to P.*

1. *(1) $P \in \mathcal{W}(R)$ and R/A_P is embeddable into M.*
2. *(2) If $b \in R$ is non-zero, $Q = Ann(bR)$ and A_Q is minimal with respect to Q such that R/A_Q is right self-injective, then $A_P = A_Q$.*
3. *(3) If M is cyclic, then R/P is a local right self-injective ring.*

Proof. (1) Since M is uniform $R/P \cong aR$ is uniform as well, hence $P \in \mathcal{W}(R)$. As M is injective and we have monomorphisms $f \in \mathrm{Hom}(R/P, R/A_P)$ and $g \in \mathrm{Hom}(R/P, M)$, there exists a homomorphism $h \in \mathrm{Hom}(R/A_P, M)$ such that $hf = g$, which is an embedding since R/A_P is uniform and so $f(R/P)$ is essential in R/A_P.

(2) Obviously, $bR \cong R/Q$ embeds into $aR \cong R/P$, which implies that $A_Q \subseteq A_P$ by the definition. Note that $A_Q \subseteq P, Q$, which implies R/P and R/Q can be viewed as modules over the right self-injective ring R/A_Q. Hence an embedding $R/Q \to R/A_Q$ factorizes through the monomorphism $R/Q \to R/P$, and since R/P is uniform, there exists an embedding of R/P into R/A_Q. Now $A_P \subseteq A_Q$, by minimality of A_P.

(3) Put $N_P = \{m \in M \mid mP = 0\}$. Clearly, N_P has a structure of a module over the ring $\tilde{R} = R/P$ and using the Baer criterion it is easy to see that N_P is injective over $\tilde{R}$. To show that N_P is cyclic, fix an arbitrary $n \in N_P$. Since $\mathrm{Ann}(aR) \subseteq \mathrm{Ann}(nR)$, there exists $\varphi \in \mathrm{Hom}(aR, nR)$ for which $\varphi(a) = n$. As M is injective, φ can be extended to $\tilde{\varphi} \in \mathrm{End}(M)$. Moreover, M is a cyclic module over a right duo ring, hence $\mathrm{End}(M) \cong R/\mathrm{Ann}(M)$ and there exists $r \in \tilde{R}$ such that $\tilde{\varphi}(m) = ar = n$. Thus $N_P \cong \tilde{R}$ is a cyclic injective uniform module over the ring $\tilde{R}$, which proves that $\tilde{R}$ i a right self-injective ring. Finally, $\tilde{R} \cong \mathrm{End}(\tilde{R})$ is local by Lemma 2.5. □

Now we are ready to formulate characterization of right duo IIMC rings:

Theorem 3.10. *The following conditions are equivalent for a right duo ring R:*

1. *R is IIMC;*
2. *R/P is local right self-injective for each $P \in \mathcal{W}(R)$, and for each maximal right ideal I there exists an ideal $A_I \in \mathcal{W}(R)$ such that R/A_I is local right IIMC and A_I contains no subfactor isomorphic to R/I;*
3. *for each $P \in \mathcal{W}(R)$ there exists $A_P \in \mathcal{W}(R)$ such that A_P is minimal with respect to P and R/A_P is local right self-injective.*

Proof. First, recall that every right maximal ideal of a right duo ring is weak prime (cf. Example 3.8). We will use freely in the proof that if I is a maximal right ideal and $A, B \subseteq I$ are ideals such that R/B is local and A contains no subfactor isomorphic to R/I, then $A \subseteq B$.

(1) ⇒ (2) Let $P \in \mathcal{W}(R)$ and $A = \mathrm{Ann}(E(R/P))$. Since $E(R/P)$ is indecomposable injective, it is cyclic by the hypothesis and so $R/A \cong \mathrm{End}(E(R/P))$ is local by Lemma 2.5(3). Moreover, $R/A \cong E(R/P)$ as right modules, thus R/P is right self-injective by Lemma 3.9(3).

Suppose that I is a maximal right ideal of R. As R is a right hypersimple by Proposition 3.2, we may apply Theorem 2.7, which implies that there exists an ideal A_I such that R/A_I is local, A_I contains no subfactor isomorphic to R/I, and $R/A_I \cong \mathrm{End}(R/I)$. It remains to prove that R/A_I is local right IIMC. Let $\tilde{R} = R/A_I$ and fix an indecomposable injective module $\tilde{M}$ over the ring $\tilde{R}$. Then by Lemma 3.1 there exists $\tilde{P} \in \mathcal{W}(\tilde{R})$, and so $P \in \mathcal{W}(R)$, such that $\tilde{M} = E_{\tilde{R}}(\tilde{R}/\tilde{P}) = E_{\tilde{R}}(R/P)$. Put $M = E_R(R/P)$ and $A = \mathrm{Ann}_R(M)$, i.e. M is an injective hull of R/P over the ring R. Then M is cyclic by the hypothesis, hence $R/A \cong End_R(M)$ is local by Lemma 2.5(3) and so $A_I \subseteq A$. It follows that A_I annihilates M, hence $M \cong \tilde{M}$ is cyclic.

(2) ⇒ (3) Let $P \in \mathcal{W}(R)$. Then R/P is local by the hypothesis, hence there exists a maximal right ideal $I \in \mathcal{W}(R)$ and an ideal $A_I \in \mathcal{W}(R)$ such that $A_I \subseteq P \subseteq I$ and A_I contains no subfactor isomorphic to R/I. Since R/A_I is right IIMC , we get that $E_{R/A_I}(R/P)$ is cyclic. Define $A_P := \mathrm{Ann}_R(E_{R/A_I}(R/P))$. Clearly, R/P is embeddable into R/A_P and to prove the minimality of A_P let us suppose that for $Q \in \mathcal{W}(R)$ there exists an embedding of R/P into R/Q. As R/Q is local by the hypothesis and R/P is a factor ring of R/Q we get that $A_I \subseteq Q \subseteq I$. Now, the injectivity of $R/A_P \cong E_{R/A_I}(R/P)$ implies that R/Q embeds into R/A_I, from which it follows that $A_P \subseteq Q$. Thus A_P is minimal with respect to P.

(3) ⇒ (1) Suppose that M is an injective indecomposable R-module, let $a_1, a_2 \in M$ be non-zero elements, and put $P_i := \mathrm{Ann}(a_iR)$, $i = 1, 2$. Then there exists a non-zero element $b \in a_1 \cap a_2R$. Put $Q = \mathrm{Ann}(bR)$ and denote by A_{P_1}, A_{P_1}, and A_Q, respectively, ideals minimal with respect to P_1, P_2, and Q, respectively, existing by the hypothesis. Thus by Lemma 3.9(2) $A_{P_1} = A_Q = A_{P_1}$ because R/A_Q is right self-injective. It means that there exists a unique ideal $A \in \mathcal{W}(R)$ which is minimal with respect to $\mathrm{Ann}(m)$ for each non-zero $m \in M$. Note that R/A is a local right self-injective ring and M has a structure of a module over R/A since A annihilates M. Now, Lemma 3.9(1) implies that R/A is embeddable into M. Since R/A is injective module isomorphic to a submodule of indecomposable R/A-module M, which forms a direct summand of M. hence $R/A \cong M$. □

Since a semilocal right duo ring is isomorphic to a finite product of local right self-injective rings by Corollary 2.8 we obtain the following consequence of Theorem 3.10:

Corollary 3.11. *Let R be a semilocal right duo ring. Then it is right IIMC ring if and only if R/P is local right self-injective for each weak prime ideal P.*

We finish the description of IIMC ring by the following observation on commutative ones.

Lemma 3.12. *Let R be a commutative ring.*

(1) If P is a prime ideal of R and $E(R/P)$ is cyclic, then P is maximal.
(2) If R is IIMC, then the Krull dimension of R is zero (i.e. all prime ideals are maximal) and $J(R) = N(R)$, which is a nil ideal.

Proof. (1) Let $a, b \in R/P$ be non-zero elements. As P is prime, $\mathrm{Ann}(aR) = P = \mathrm{Ann}(bR)$, which implies that $aR \cong R/P \cong bR$ and exists $f \in \mathrm{End}(E(R/P))$ such that $f(a) = b$. Since $\mathrm{End}(E(R/P)) \cong R/\mathrm{Ann}(E(R/P))$ is a commutative ring, there exists $r \in R$ for which $ar = b$. Thus R/P is a simple module, and so P is maximal.

(2) Since $E(R/P)$ is cyclic for every prime ideal P of a IIMC ring R, all prime ideals of R are maximal by (1). Thus the Krull dimension of R is zero and so the Jacobson radical and the prime radical coincide. Finally, note that the prime radical of any commutative ring is nil. □

Proposition 3.13. *A Jacobson radical of each factor-ring of a commutative IIMC ring is nil.*

Proof. Let R be a commutative IIMC ring and U be an ideal. Since every prime ideal of R/U can be lifted to a prime ideal of R, all prime ideals of R/U are maximal by Lemma 3.12(2), which implies that $J(R/U) = N(R/U)$, which is nil. □

As the second subclass of the class of all hypersimple rings we will consider rings R whose all indecomposable right R-modules are cyclic. Such rings will be called *right IMC*. Clearly, any right IMC ring is a right IIMC ring and it follows immediately from Proposition 3.2 that right IMC rings are hypersimple. Furthermore, any right Köethe ring is right IMC, nevertheless, the class of all right IMC rings is larger:

Example 3.14. *Every abelian regular ring is IMC. Indeed, R is an abelian regular ring, M is an indecomposable module and $A = Ann(M)$, then R is simple, since otherwise there exists a non-trivial idempotent of R/A. Thus M is simple and so cyclic.*

We next give the following observation related to how the defining properties hypersimple, IIMC and IMC rings behave with respect to products.

Proposition 3.15. *Let R_i, $i \leq n$ and $R = \prod_{i\leq n}$ be rings. Then R is hypersimple (respectively, IIMC and IMC) if and only of each R_i, $i \leq n$, is hypersimple (respectively, IIMC and IMC) for each $i \leq n$.*

Proof. Let as denote by e_i central idempotents for which $R_i \cong e_i R$ for each $i \leq n$.
($:\Rightarrow$) Let i be arbitrary and let M_i be an R_i-module. Define $M_j = 0$ for all remaining $j \neq i$ and put $M = \prod_{j\leq n} M_j \cong \prod_{j\leq n} Me_j$. Clearly, M is indecomposable over R if and only if M_i is indecomposable over R_i. Furthermore, M is injective (or has a simple essential submodule) over R if and only if each M_i is injective (or has simple essential submodule) over R_i, which finishes the proof.
($\Leftarrow:$) Let M be an R-module and put $M_i = Me_i$, where $i \leq n$. Clearly, $M \cong \prod_{i\leq n} M_i$ is indecomposable if and only if there exists i such that M_i is indecomposable and $M_j = 0$ for each $j \neq i$. Now, by a similar argument as in the proof of direct implication, it is easy to see that M is indecomposable or injective or a module with a simple essential socle over R if and only if M_i is so over R_i. □

Recall that a module is *chain* if the lattice of its submodules is linearly ordered. It can be easily verified that a module turns out to be a chain module if its cyclic submodules form a chain. A ring is right (respectively, left) *chain* if it is right (left) chain module over itself. Note that commutative chain rings are usually called *uniserial* rings.

Example 3.16. *Let R be a right chain right Artinian ring wit the maximal ideal I. It is easy to see that R is right self-injective. Let M be an indecomposable module and denote by s the socle length of M, i.e. the minimal $s \geq 0$ for which $MI^s = 0$. If n is the socle length of R, then M has a structure of a module over the right self-injective ring R/I^{n-s}. Since M contains o cyclic submodule of the socle length s, which is injective, M is isomorphic to R/I^{n-s}. Thus any right chain right Artinian ring, for instance the uniserial ring $\mathbb{Z}_{2^n}$, is an example of an IMC ring.*

Lemma 3.17. *If R is a right IMC ring, then it is right max such that every submodule of any uniform module, in particular, indecomposable injective one, is cyclic.*

Proof. Since every non-zero submodule of a uniform module is indecomposable, it is cyclic by the hypothesis, hence it contains a maximal submodule. As any right IMC ring is hypersimple and indecomposable injective modules are uniform, it remains to apply Lemma 2.9. □

We are ready to describe the structure of commutative IMC rings.

Theorem 3.18. *The following conditions are equivalent for a commutative ring R:*

(1) R is IMC;
(2) R is a max ring and $E(R/I)$ is a chain Artinian module for each maximal ideal I;
(3) $R/J(R)$ is abelian regular, $J(R)$ is T-nilpotent and, for every maximal ideal I there exists $n < \omega$ such that $I^n = I^{n+1}$, the ring R/I^n is uniserial Frobenius and I^n contains no subfactor isomorphic to R/I.

Proof. (1)⇒(2) Let I be a maximal ideal. By Lemma 3.17, the ring R is max and all submodules of $E(R/I)$ are cyclic because $E(R/I)$ is an indecomposable injective module. Put $A := \mathrm{Ann}(E(R/I))$ and $\tilde{R} := R/A$. Then $\tilde{R} \cong E(R/I)$ has a structure of a self-injective ring, which is local by Lemma 2.5(3). As $\tilde{R}$ is a factor of a max ring, it is max as well, and then, by Lemma 3.17, $\tilde{R}$ is a local Noetherian perfect ring with the T-nilpotent cyclic maximal ideal, say $\tilde{I} := I/A$. This means that $\tilde{I}$ is a nilpotent cyclic ideal, $\tilde{R}$ is a uniserial Artinian ring, and so $E(R/I)$ is a chain Artinian module.
(2)⇒(3) By [7], $R/J(R)$ is abelian regular and $J(R)$ is T-nilpotent because R is max. Let I be a maximal ideal and put again $A := \mathrm{Ann}(E(R/I))$. Then A contains no subfactor isomorphic to R/I by Lemma 2.5(2). Since $R/A \cong E(R/I)$ is a self-injective local Artinian ring by Example 3.16, it is Frobenius by [14, Theorem 16.14]. Furthermore, there exists $n < \omega$ such that $I^n \subseteq A$. Note that R/I^n is a local perfect ring as R is max and I is a maximal ideal. Furthermore, if $s \in R \setminus I^n$, then $sI^n \neq sR$, and so $sI \neq sR$, which means that $s \in R \setminus A$ since A contains no subfactor isomorphic to $R/I \cong sR/sI$. This proves that $I^n = A$. Now, if we assume that $I^{n+1} = AI \neq A = I^n$, then A/AI and so A contains a subfactor isomorphic to $R/I \cong sR/sI$, a contradiction. We have proved that $A = I^n = I^{n+1}$ and it contains no subfactor isomorphic to R/I.
(3)⇒(1) Let M be a non-zero indecomposable R-module and put $A = \mathrm{Ann}(M)$. Note that $\overline{R} = R/(A + J(R)) \cong \frac{R/A}{(J(R)+A)/A}$ is a factor of the abelian regular ring

$R/J(R)$, hence it is abelian regular. Let $\overline{e}$ be an idempotent of $\overline{R}$. Obviously, it can be lifted to an idempotent e of the ring R/A modulo the nil ideal $(J(R)+A)/A$ by [13, Theorem 21.28]. Since $M = Me \oplus M(1-e)$ and M is indecomposable, $\overline{e}$ is equal either to 1 or 0, hence $\overline{R}$ a field. This implies that R/A is local with the maximal ideal which is isomorphic to I/A for a suitable maximal ideal I of the ring R. Thus M has a structure of an indecomposable module over R/A. Since R is max, R/A is local max, hence perfect with the unique maximal ideal I/A. Note that by the hypothesis, there exists n for which $I^n = I^{n+1}$ and R/I^n is a uniserial Frobenius ring with $sI = sR$ for each $s \in I^n$. If we assume that there exists $s \in I^n \setminus K$, then $sI \neq sR$ as R/K is semiartinian, which contradicts to the hypothesis. Thus $I^n \subseteq K$ and R/K is a factor of the uniserial Artinian ring R/I^n, which implies that M can be seen as a module over the uniserial Artinian ring R/K. It is well known that any module over a commutative Artinian principal ideal ring decomposes into a direct sum of cyclic modules, and hence the irreducible module M is cyclic, which proves that R is IMC. □

Corollary 3.19. *Let R be a commutative ring which is either semilocal or noetherian. Then R is IMC if and only if it is serial artinian.*

Proof. If R is IMC, then it is a finite product of pseudo-Frobenius rings by Corollary 2.8 and Proposition 2.11, which are chain artinian by Theorem 3.18. The converse follows from Example 3.16 and Proposition 3.15. □

Let us present two examples, the first one shows that not every IIMC ring is IMC and the second one presents non-trivial example of an IMC ring.

Example 3.20. *Any local commutative Frobenius ring which is not uniserial, e.g. $\mathbb{F}_2[x]/(x^2, y^2)$, is an example of an IIMC ring by Proposition 3.2 which is not IMC by Theorem 3.18.*

Example 3.21. *Let F be a field and consider an F-subalgebra R of the F-algebra $\prod_{n<\omega} F[x]/(x^n)$ generated by the ideal $I_\infty = \bigoplus_n F[x]/(x^n)$. If we denote by e_n idempotents satisfying $Re_n \cong F[x]/(x^n)$, then R is an IIMC ring by Theorem 3.18 with maximal ideals $I_n = R(1-e_n)$, $n < \omega$ and $I_\infty = \bigoplus_n Re_n$ for which $I_n^n = I_n^{n+1}$ and $I_\infty = I_\infty^2$.*

We finish the paper by formulating several open question:

(1) Is any semilocal (semiperfect) right hypersimple ring necessarily right pseudo-Frobenius? It is true in duo case by Corollary 2.8.
(2) Is any right Noetherian right hypersimple ring necessarily right quasi-Frobenius? It holds for commutative rings by Theorem 2.11.
(3) Is any semilocal or right Noetherian IMC ring necessarily right serial Artinian? It holds in commutative case by Corollary 3.19.
(4) More generally, over which ring-theoretical conditions do criteria of Corollary 2.8, Theorem 2.11, Corollary 3.11, and Corollary 3.19 remain to hold?

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Faculty Sciences, Department of Mathematics, Gazi University, Ankara, Turkey
E-mail address: mtamerkosan@gazi.edu.tr tkosan@gmail.com

Department of Business Administration, Trakya University, Faculty of Economics and Administrative Sciences, Balkan Campus, Edirne, Turkey
E-mail address: ozgurtasdemir@trakya.edu.tr

Department of Algebra, Charles University, Faculty of Mathematics and Physics Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail address: zemlicka@karlin.mff.cuni.cz