Discontinuous Galerkin method for first order hyperbolic problems

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Lecture 3

FVM for nonlinear system

Nonlinear hyperbolic system of 1st order

$$\Omega \subset \mathbb{R}^d, \quad (\boldsymbol{f}_1, \dots, \boldsymbol{f}_d) : \mathbb{R}^n \to \mathbb{R}^n, \quad \text{find } \boldsymbol{u} : \Omega \times (0, T) \to \mathbb{R}^n$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{s=1}^{d} \frac{\partial \boldsymbol{f}_s(\boldsymbol{u})}{\partial x_s} = 0 \quad \text{in } \Omega \times (0, T),$$

- mesh of Ω: $\mathcal{T}_h = \{K_i\}$,
- $\boldsymbol{u}_i(t) := \frac{1}{|K_i|} \int_{K_i} \boldsymbol{u}(x, t) \, \mathrm{d}x, \ t \in [0, T]$
- integration over K_i and the Green theorem

Space semi-discretization

$$\frac{\partial}{\partial t} \boldsymbol{u}_i(t) = \frac{1}{|K_i|} \int_{\partial K_i} \sum_{s=1}^{d} \boldsymbol{f}_s(\boldsymbol{u}(\cdot, t)) n_s \, \mathrm{d}S, \qquad t \in (0, T) \qquad (1)$$

FVM for nonlinear system (2)

Space semi-discretization

$$\frac{\partial}{\partial t}\boldsymbol{u}_i(t) = \frac{1}{|K_i|}\sum_{\Gamma\subset\partial K_i}\int_\Gamma \sum_{s=1}^{d}\boldsymbol{f}_s(\boldsymbol{u}(\cdot,t))n_s\,\mathrm{d}S, \quad t\in(0,T) \qquad (2)$$

Numerical flux

$$\sum_{s=1}^{d}\boldsymbol{f}_i(\boldsymbol{u}(\cdot,t_k))n_s|_\Gamma \approx \mathbf{H}(\boldsymbol{u}_\Gamma^{(L)},\boldsymbol{u}_\Gamma^{(R)},\boldsymbol{n}_\Gamma)$$

$\boldsymbol{u}_\Gamma^{(L)}$ and $\boldsymbol{u}_\Gamma^{(R)}$ are values on $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)}$ sharing Γ, normal $\boldsymbol{n}_\Gamma$

FV scheme

$$\frac{\partial}{\partial t}\boldsymbol{u}_i(t) = \frac{1}{|K_i|}\sum_{\Gamma\subset\partial K_i}\int_\Gamma \sum_{s=1}^{d}\mathbf{H}(\boldsymbol{u}_\Gamma^{(L)},\boldsymbol{u}_\Gamma^{(R)},\boldsymbol{n}_\Gamma)\,\mathrm{d}S, \quad t\in(0,T)$$

Numerical flux $\mathbf{H}(\cdot,\cdot,\cdot)$

Properties of **H**

- consistent: $\mathbf{H}(\boldsymbol{u},\boldsymbol{u},\boldsymbol{n}) = \sum_{s=1}^{d} \boldsymbol{f}_s(\boldsymbol{u})n_s$,
- conservative: $\mathbf{H}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{n}) = -\mathbf{H}(\boldsymbol{u}_2,\boldsymbol{u}_1 - \boldsymbol{n})$.
- (local) Lipschitz continuity, monotonicity, etc.

Examples of **H**

- Lax-Friedrichs

$$\mathbf{H}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{n}) = \frac{1}{2}\sum_{s=1}^{d}\left(\boldsymbol{f}_s(\boldsymbol{u}_1) + \boldsymbol{f}_s(\boldsymbol{u}_2)\right) n_s - \frac{1}{\lambda}(\boldsymbol{u}_1 - \boldsymbol{u}_2),$$

- Vijayasundaram

$$\mathbf{H}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{n}) = \mathbb{P}^{+}(\langle\boldsymbol{u}\rangle,\boldsymbol{n})\boldsymbol{u}_1 + \mathbb{P}^{-}(\langle\boldsymbol{u}\rangle,\boldsymbol{n})\boldsymbol{u}_2,$$

where $\mathbb{P}^{\pm}$ are the pos/neg parts of $\mathbb{P} := \frac{D}{D\boldsymbol{u}}(\sum_{s=1}^{d} \boldsymbol{f}_s(\boldsymbol{u})n_s)$

DGM for nonlinear system

Nonlinear hyperbolic system of 1st order

$$\text{find } \boldsymbol{u}(x,t): \quad \frac{\partial \boldsymbol{u}}{\partial t} + \sum_{s=1}^{d} \frac{\partial \boldsymbol{f}_s(\boldsymbol{u})}{\partial x_s} = 0 \quad \text{in } \Omega \times (0,T), \qquad (3)$$

- mesh of Ω: $\mathcal{T}_h = \{K_i\}$,
- multiply (3) by $\boldsymbol{\varphi} \in \boldsymbol{H}^1(\Omega, \mathcal{T}_h)$, $\int_K$, Greens theorem, $\sum_{K\in\mathcal{T}_h}$

Space semi-discretization

$$\frac{\partial}{\partial t}\int_\Omega \boldsymbol{u}(\cdot,t)\boldsymbol{\varphi}\,\mathrm{d}x - \sum_{K\in\mathcal{T}_h}\int_K \sum_{s=1}^{d} \boldsymbol{f}_s(\boldsymbol{u}(\cdot,t))\frac{\partial\boldsymbol{\varphi}}{\partial x_s}\,\mathrm{d}x \qquad (4)$$

$$+ \sum_{K\in\mathcal{T}_h}\int_{\partial K}\sum_{s=1}^{d}\boldsymbol{f}_s(\boldsymbol{u}(\cdot,t))n_s\boldsymbol{\varphi}\,\mathrm{d}S = 0 \quad \forall t$$

Employing the identity: $\sum_{K\in\mathcal{T}_h}\int_{\partial K} = \sum_{K\in\mathcal{T}_h}\sum_{\Gamma\in\partial K}\int_\Gamma$

DGM for nonlinear system

$$\frac{\partial}{\partial t}\int_{\Omega}\boldsymbol{u}(\cdot,t)\boldsymbol{\varphi}\,\mathrm{d}x - \sum_{K\in\mathcal{T}_h}\int_K \sum_{s=1}^d \boldsymbol{f}_s(\boldsymbol{u}(\cdot,t))\frac{\partial\boldsymbol{\varphi}}{\partial x_s}\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\sum_{\Gamma\in\partial K}\int_{\Gamma}\sum_{s=1}^d \boldsymbol{f}_s(\boldsymbol{u}(\cdot,t))n_s\,\varphi\,\mathrm{d}S = 0 \tag{5}$$

If $\boldsymbol{u}$ is continuous over all Γ then a manipulation gives

$$\frac{\partial}{\partial t}\int_{\Omega}\boldsymbol{u}(\cdot,t)\boldsymbol{\varphi}\,\mathrm{d}x - \sum_{K\in\mathcal{T}_h}\int_K \sum_{s=1}^d \boldsymbol{f}_s(\boldsymbol{u}(\cdot,t))\frac{\partial\boldsymbol{\varphi}}{\partial x_s} + \sum_{\Gamma\in\mathcal{F}_h}\int_{\Gamma}\sum_{s=1}^d \boldsymbol{f}_s(\boldsymbol{u}(\cdot,t))n_s\,[\varphi]\,\mathrm{d}S = 0,\ \forall t, \tag{6}$$

Physical flux ≈ numerical flux

$$\sum_{s=1}^d \boldsymbol{f}_i(\boldsymbol{u}(\cdot,t_k))n_s|_{\Gamma} \approx \mathbf{H}(\boldsymbol{u}_{\Gamma}^{(L)},\boldsymbol{u}_{\Gamma}^{(R)},\boldsymbol{n}_{\Gamma})$$

DGM for nonlinear system – an alternative

back to relation (5)

$$\frac{\partial}{\partial t}\int_\Omega \boldsymbol{u}(\cdot,t)\boldsymbol{\varphi}\,\mathrm{d}x - \sum_{K\in\mathcal{T}_h}\int_K \sum\nolimits_{s=1}^{d} \boldsymbol{f}_s(\boldsymbol{u}(\cdot,t))\frac{\partial\boldsymbol{\varphi}}{\partial x_s}\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\sum_{\Gamma\in\partial K}\int_\Gamma \sum\nolimits_{s=1}^{d} \boldsymbol{f}_s(\boldsymbol{u}(\cdot,t))n_s\,\boldsymbol{\varphi}\,\mathrm{d}S = 0$$

$$\sum\nolimits_{s=1}^{d} \boldsymbol{f}_i(\boldsymbol{u}(\cdot,t_k))n_s|_\Gamma \approx \mathbf{H}(\boldsymbol{u}_\Gamma^{(L)},\boldsymbol{u}_\Gamma^{(R)},\boldsymbol{n}_\Gamma)$$

$$\frac{\partial}{\partial t}\int_\Omega \boldsymbol{u}(\cdot,t)\boldsymbol{\varphi}\,\mathrm{d}x - \sum_{K\in\mathcal{T}_h}\int_K \sum\nolimits_{s=1}^{d} \boldsymbol{f}_s(\boldsymbol{u}(\cdot,t))\frac{\partial\boldsymbol{\varphi}}{\partial x_s}\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K} \sum\nolimits_{s=1}^{d} \mathbf{H}(\boldsymbol{u}_\Gamma^{(L)},\boldsymbol{u}_\Gamma^{(R)},\boldsymbol{n}_\Gamma)\,\boldsymbol{\varphi}\,\mathrm{d}S = 0 \tag{7}$$

relation (7):

$$\frac{\partial}{\partial t}\int_\Omega \boldsymbol{u}(\cdot,t)\varphi\,\mathrm{d}x - \sum_{K\in\mathcal{T}_h}\int_K \sum\nolimits_{s=1}^{d} \boldsymbol{f}_s(\boldsymbol{u}(\cdot,t))\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K} \sum\nolimits_{s=1}^{d} \mathbf{H}(\boldsymbol{u}_\Gamma^{(L)},\boldsymbol{u}_\Gamma^{(R)},\boldsymbol{n}_\Gamma)\,\varphi\,\mathrm{d}S = 0$$

Using the conservativity: $\mathbf{H}(\boldsymbol{u}_\Gamma^{(L)},\boldsymbol{u}_\Gamma^{(R)},\boldsymbol{n}_\Gamma) = -\mathbf{H}(\boldsymbol{u}_\Gamma^{(R)},\boldsymbol{u}_\Gamma^{(L)},-\boldsymbol{n}_\Gamma)$

$$\frac{\partial}{\partial t}\int_\Omega \boldsymbol{u}(\cdot,t)\varphi\,\mathrm{d}x - \sum_{K\in\mathcal{T}_h}\int_K \sum\nolimits_{s=1}^{d} \boldsymbol{f}_s(\boldsymbol{u}(\cdot,t))\frac{\partial\varphi}{\partial x_s} + \sum_{\Gamma\in\mathcal{F}_h}\int_\Gamma \mathbf{H}(\boldsymbol{u}_\Gamma^{(L)},\boldsymbol{u}_\Gamma^{(R)},\boldsymbol{n}_\Gamma)\,[\varphi]\,\mathrm{d}S = 0,\ \ \forall t, \tag{8}$$

Discontinuous Galerkin method for nonlinear system (3)

Approximate solution by DGM

- $\boldsymbol{S}_{hp} \subset \boldsymbol{H}^1(\Omega, \mathcal{T}_h)$ ...discontinuous pw polynomial functions
- $\boldsymbol{u}_h \in C^1([0, T]; \boldsymbol{S}_{hp})$
- such that

$$\frac{\partial}{\partial t}(\boldsymbol{u}_h(t), \boldsymbol{\varphi}_h) + \boldsymbol{b}_h(\boldsymbol{u}_h(t), \boldsymbol{\varphi}_h) = 0 \qquad \forall \boldsymbol{\varphi}_h \in \boldsymbol{S}_{hp}\ \forall t \in (0, T)$$

where

$$\boldsymbol{b}_h(\boldsymbol{u}_h, \boldsymbol{\varphi}_h) = -\sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^{d} \boldsymbol{f}_s(\boldsymbol{u}_h) \frac{\partial \boldsymbol{\varphi}_h}{\partial x_s} \, \mathrm{d}x + \sum_{\Gamma \in \mathcal{F}_h} \int_\Gamma \sum_{s=1}^{d} \mathbf{H}(\boldsymbol{u}_h|_\Gamma^{(L)}, \boldsymbol{u}_h|_\Gamma^{(R)}, \boldsymbol{n}_\Gamma)[\varphi_h] \mathrm{d}S.$$

- $\boldsymbol{u}_h(0)$ is given by initial condition

System of ODEs, has to be solved

DGM – boundary conditions

Numerical flux through $\partial\Omega$

$$\sum_{\Gamma\in\mathcal{F}_h}\int_\Gamma \sum\nolimits_{s=1}^{d} \mathbf{H}(\boldsymbol{u}_h|_\Gamma^{(L)}, \boldsymbol{u}_h|_\Gamma^{(R)}, \boldsymbol{n}_\Gamma)[\varphi_h]\mathrm{d}S.$$

- if $\Gamma \subset \partial\Omega$, how to set $\boldsymbol{u}_h|_\Gamma^{(R)}$?
- number of BCs $=$ the number of incoming characteristics
- matrix

$$\mathbb{P}(\boldsymbol{u}, \boldsymbol{n}) := \sum_{s=1}^{2} \frac{D}{D\boldsymbol{u}} \boldsymbol{f}_s(\boldsymbol{u})\boldsymbol{n}_s$$

- if eigenvalue of $\mathbb{P}$ on $\partial\Omega$ is negative then the corresponding characteristic is incomming

Boundary conditions for DGM – Euler equations

Number of BCs

- 2D inviscid compressible flow, $\boldsymbol{u} = (\rho, v_1, v_2, e)^{\mathrm{T}}$
- eigenvalues: $\boldsymbol{v}\cdot\boldsymbol{n}, \quad \boldsymbol{v}\cdot\boldsymbol{n}, \quad \boldsymbol{v}\cdot\boldsymbol{n} - a, \quad \boldsymbol{v}\cdot\boldsymbol{n} + a,$
  a speed of sound
- we distinguish:

subsonic inlet	$(\boldsymbol{v}\cdot\boldsymbol{n} < 0,\ \lvert\boldsymbol{v}\cdot\boldsymbol{n}\rvert < a)$:	3 BCs
supersonic inlet	$(\boldsymbol{v}\cdot\boldsymbol{n} < 0,\ \lvert\boldsymbol{v}\cdot\boldsymbol{n}\rvert > a)$:	4 BCs
subsonic outlet	$(\boldsymbol{v}\cdot\boldsymbol{n} > 0,\ \lvert\boldsymbol{v}\cdot\boldsymbol{n}\rvert < a)$:	1 BCs
supersonic outlet	$(\boldsymbol{v}\cdot\boldsymbol{n} > 0,\ \lvert\boldsymbol{v}\cdot\boldsymbol{n}\rvert > a)$:	0 BCs

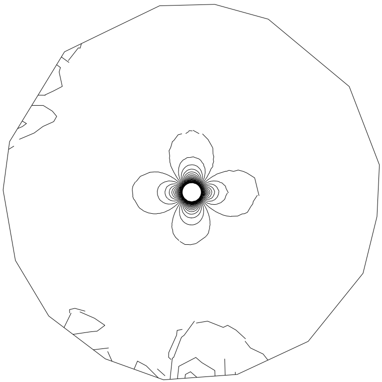
Setting of BC

Problem dependent

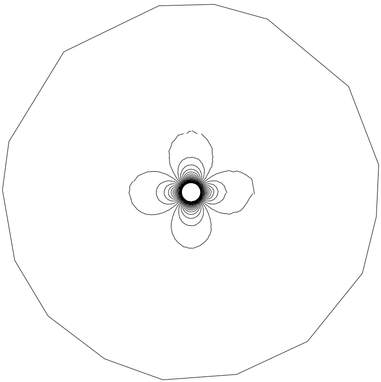
- flow around isolated profile: far-field BC (flow at ∞)
- flow in a channel (tube): ration between inlet/outlet pressure
-

Example of far-filed BCs: flow around isolated circle

artificial boundary 20 times larger then diameter of the circle



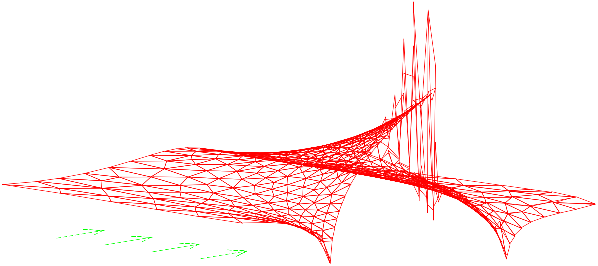
not correct BC



correct BC

Observations

- nonlinear hyperbolic problems: solution can contain discontinuities (shock waves)
- numerical approximation by FVM is fine (but smeared)
- numerical approximation by higher order methods can oscillate



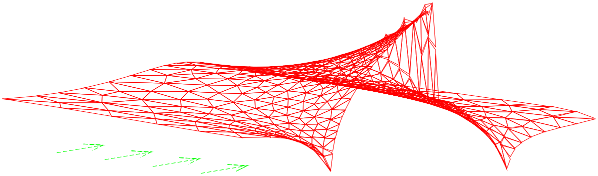
flow through a channel

DGM – stabilization techniques (2)

Artificial diffusion approach

$$\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{s=1}^{d} \frac{\partial \boldsymbol{f}_s(\boldsymbol{u})}{\partial x_s} = G(\boldsymbol{u})\Delta \boldsymbol{u} \qquad (9)$$

$G(\boldsymbol{u})$ vanish for smooth $\boldsymbol{u}$



flow through a channel