

Numerical quadratures

Vít Dolejší

Charles University Prague
Faculty of Mathematics and Physics

Quiz # 1

Question #1

Let us consider numerical quadrature

- $I(f) := \int_a^b f(x) dx \approx Q(f) := \sum_{i=1}^n w_i f(x_i)$,
- x_i , $i = 1, \dots, n$ are the **nodes**, $w_i \in \mathbb{R}$, $i = 1, \dots, n$ are the **weights**.
- it is **exact** for polynomial functions of **degree $\leq p$** , i.e.
 $I(x^q) = Q(x^q)$ for $q = 0, 1, \dots, p$.

What is the order of error of the corresponding **composite formula** Q_h with the step h ?

- (A) $|I(f) - Q_h(f)| = O(h^{p-1})$
- (B) $|I(f) - Q_h(f)| = O(h^p)$
- (C) $|I(f) - Q_h(f)| = O(h^{p+1})$

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Idea of proof: Taylor expansion on each sub-interval

$$f(x) = \phi_p(x) + O(h^{p+1})$$

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- x_i , $i = 1, \dots, n$ are the nodes, $w_i \in \mathbb{R}$, $i = 1, \dots, n$ are the weights.

This numerical quadrature has order ≥ 1 . Which of the following conditions are necessary? (Multiple answers are possible)

- (A) $w_1 + w_2 + \dots + w_n = 1$
- (B) $w_i \geq 0$ for $i = 1, \dots, n$
- (C) $a \leq x_i \leq b$ for $i = 1, \dots, n$
- (D) $w_1 x_1 + w_2 x_2 + \dots + w_n x_n = \frac{1}{2}$

$Q(f)$ is exact for $f = 1$ (A) and $f = x$ (D),
weights can be negative, nodes can be outside of interval.

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How are defined the **Newton-Cotes formulas** for the given n ?

- (A) The nodes and weights are chosen in such a way that the **order** of $Q(f)$ is the maximal possible.
- (B) The nodes are chosen equidistantly and the weights are chosen in such a way that the **order** of $Q(f)$ is the maximal possible.
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Question #4

We integrate $\int_0^1 \exp(2\sqrt{x}) dx$ numerically by the composite **midpoint formula** and the composite **trapezoid formula**. We obtain the results

- $M_h(f) = 4.21$
- $T_h(f) = 4.24$

- What is the estimate of the error (EST) of these results?
- What is the results obtained by the Simpson rule ($S_h(f)$)?

Outputs are two numbers.

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Answer

- $EST = \frac{1}{3}(M_h(f) - T_h(f)) = 0.01$ (estimate of the error of $M_h(f)$)
- $S_h(f) = \frac{1}{3}(2M_h(f) + T_h(f)) = 4.22$

Question #5

We integrate $\int_0^1 f(x) dx$ numerically by the composite **Simpson formula**. We obtain the following results:

- for $h = 0.2$, $S_h(f) = 2.220$
- for $h = 0.1$, $S_h(f) = 2.234$

- What is the estimate of the error of the result with $h = 0.1$?

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Answer

- Simpson formula has order = 3
- estimate of the error by the half-step size method is

$$EST = \frac{Q_h - Q_{h/2}}{2^{p+1} - 1} = \frac{2.234 - 2.220}{2^4 - 1} = \frac{0.014}{15} \approx 10^{-3}$$

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Fill the following table which contains the **order** of the **Newton-Cotes** and **Gauss** formulas for n integration nodes.

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$n = 2$		
$n = 3$		
$n = 4$		
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$n = 6$		
$n = 7$		

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$n = 3$	3	5
$n = 4$	3	7
$n = 5$	5	9
$n = 6$	5	11
$n = 7$	7	13

Question #8

Why the **half-step size** method **can not be used** for the estimation of the error of the Gauss quadrature?

- (A) this method is unstable since the nodes of a Gauss quadrature are not distributed equidistantly,
- (B) this method significantly over-estimates the error (it is not sufficiently accurate) since the error of the Gauss quadrature is too small due to its high order,
- (C) it can be used but it is less efficient (it requires too many evaluation in integration nodes).

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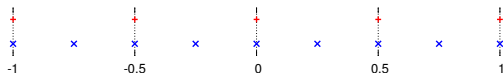
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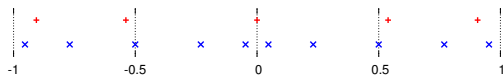
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Newton-Cotes



Gauss



Question #9

- Which assertion about Gauss-Kronrod quadrature formulae is true?

(Multiple answers are possible)

- (A) The pair of quadrature formulas where the Gauss quadrature G_n has order $2n - 1$ and the Kronrod quadrature K_{2n+1} has order $3n + 1$.
- (B) The pair of quadrature formulas $G_n K_{2n+1}$ is suitable for the estimation of the error of the Gauss quadrature.
- (C) The quadrature formulas where the Gauss quadrature G_n is enhanced by additional $n + 1$ nodes in such a way that the resulting formula has the maximal order of accuracy.
- (D) The pair of quadrature formulas which are open (i.e., $a \neq x_i \neq b$, $i = 1, 2, 3 \dots$) and the weight are irrational numbers in general.

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