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Abstract

This paper is concerned with the analysis of the discontinuous Galerkin finite element method applied to the space semidiscretization of nonlinear nonstationary convection-diffusion problems. The attention is paid to the derivation of the method and the investigation of error estimates for schemes with a symmetric as well as nonsymmetric discretization of diffusion terms and with the interior and boundary penalty. The theoretical results are verified by numerical experiments.

Keywords: nonlinear convection-diffusion equation, discontinuous Galerkin finite element method, symmetric and nonsymmetric formulation of diffusion terms, interior and boundary penalty, method of lines, error estimates, experimental order of convergence
MSC Classification 2000: 65M60, 65M15, 65M12

1 Introduction

In the finite element method (FEM), an important question arises: should we prefer to use *conforming* or *nonconforming* finite elements? Conforming (i.e., continuous) finite element approximations are suitable for problems with sufficiently regular solutions. However, singularly perturbed problems or nonlinear conservation laws have solutions with steep gradients or discontinuities and their approximations by conforming finite elements may suffer from the Gibbs phenomenon manifested by spurious oscillations propagating from boundary or interior layers into the computational domain. One way how to avoid this drawback is to use a suitable stabilization as, e.g. the streamline diffusion method or Galerkin least squares method and shock capturing stabilization. (For a survey, see [24], Chapter 4.) If these techniques are applied to systems of partial differential equations, e.g. of compressible flow, the form of the stabilization terms is rather complex and several parameters have to be tuned.

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In computational fluid dynamics, the finite volume (FV) method is rather popular. (Cf., [15], [20], [21], [22], [24], [29] for references.) It seems that for conservation laws with discontinuous solutions the finite volume method, using piecewise constant approximations, is more suitable, because the FV approximations are discontinuous on interelement interfaces, which allows good resolution of shock waves and contact discontinuities. On the other hand, the increase of accuracy in finite volume schemes applied on unstructured and/or anisotropic meshes is problematic.

A combination of ideas and techniques of the FV and FE methods yields the *discontinuous Galerkin finite element method* (DGFEM) using advantages of both approaches and allowing to obtain schemes with a higher order accuracy in a natural way. The DGFEM is based on the idea to approximate the solution of an initial-boundary value problem by piecewise polynomial functions over a finite element mesh without any requirement on interelement continuity. The DGFEM, first used in 1973 in [33], was applied to nonlinear conservation laws in 1989 by Cockburn and Shu ([10]). Later it was used for the numerical simulation of compressible flow by Bassi and Rebay in [2] and [3]. During several recent years the DGFE schemes have been extensively developed and become more and more popular. Some aspects of the DGFEM are investigated, e.g. in [1], [5], [6], [12], [16], [26], [30], [28], [34]. Applications to gas dynamics are discussed, e.g. in [5], [11], [12], [13], [18], [19], [27], [35], [36]. For a survey of various DGFE techniques, see, e. g. [8] and [9].

In [25], the space DG semidiscretization (called the method of lines) was applied to a nonstationary linear convection-diffusion equation equipped with Dirichlet–Neumann boundary conditions and initial condition and the error estimates analysis was presented there. The present paper is concerned with the discontinuous Galerkin finite element method for the space discretization of nonlinear nonstationary convection-diffusion problems. First, we pay the attention to the derivation of DGFE schemes and discuss symmetric and nonsymmetric variants of the discretization of diffusion terms. Then the attention is paid to the error analysis of the method using nonsymmetric as well as symmetric approximation of diffusion terms and the interior and boundary penalty. (Often we speak about the NIPG or SIPG methods – see [30].) Finally, the theoretical results are verified by numerical tests and experimental orders of convergence are computed for piecewise linear, quadratic and cubic elements.

2 Scalar convection-diffusion problem

2.1 Continuous problem

Let us consider the nonstationary nonlinear convection-diffusion problem to find $u : Q_T = \Omega \times (0,T) \to I\!R$ such that

$$\text{a)} \qquad \frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} = \varepsilon \Delta u + g \quad \text{in } Q_T, \tag{2.1}$$

$$\text{b)} \qquad u|_{\Gamma_D \times (0,T)} = u_D,$$

$$
\begin{aligned}
&\text{c)} \qquad \varepsilon \frac{\partial u}{\partial n}|_{\Gamma_N \times (0,T)} = g_N, \\
&\text{d)} \qquad u(x,0) = u^0(x), \quad x \in \Omega.
\end{aligned}
$$

We assume that $\Omega \subset I\!R^d$, $d = 2, 3$, is a bounded polygonal (if $d = 2$) or polyhedral (if $d = 3$) domain with a Lipschitz-continuous boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$ and $T > 0$. The diffusion coefficient $\varepsilon > 0$ is a given constant, $g : Q_T \to I\!R$, $u_D : \Gamma_D \times (0,T) \to I\!R$, $g_N : \Gamma_N \times (0,T) \to I\!R$ and $u^0 : \Omega \to I\!R$ are given functions, $f_s \in C^1(I\!R)$, $s = 1, \dots, d$, are given inviscid fluxes. Without the loss of generality we assume that $f_s(0) = 0, s = 1, \dots, d$.

In what follows we shall use the standard notation for function spaces: $L^p(\omega)$ – Lebesgue space, $W^{k,p}(\omega)$ – Sobolev space, $L^p(0,T;X)$ – Bochner spaces of functions defined in $(0,T)$ with values in a Banach space X, $C^k([0,T];X)$ – space of k-times continuously differentiable mappings of the interval $[0,T]$ with values in X. (ω is a domain, $k \geq 0$ integer, $p \in [1, \infty]$) – see, e.g. [32]. By $(\cdot,\cdot)$ we define the scalar product in the space $L^2(\Omega)$. The symbols $\|\cdot\|_{L^2(\omega)}$ and $|\cdot|_{H^k(\omega)}$ denote the $L^2(\omega)$-norm and $H^k(\omega)$-seminorm, respectively.

Let us assume the existence of a solution of problem (2.1), a)–d) satisfying the following *regularity conditions*:

$$
u \in L^2(0,T; H^{p+1}(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2(0,T; H^p(\Omega)) \tag{2.2}
$$

for some integer $p \geq 1$. Then $u \in C([0,T]; H^p(\Omega))$ satisfies (2.1), a)–d) pointwise (almost everywhere).

2.2 Discretization

Let $\mathcal{T}_h$ ($h > 0$) denote a partition of the closure $\overline{\Omega}$ of the domain Ω into a finite number of closed triangles (if $d = 2$) or tetrahedra (if $d = 3$) K with mutually disjoint interiors. We call $\mathcal{T}_h$ a triangulation of Ω, but *do not* require the usual conforming properties from the finite element method (cf. [7]), see Fig. 1. [2]

We set $h_K = \mathrm{diam}(K)$, $h = \max_{K \in \mathcal{T}_h} h_K$. By $|K|$ and ρ_K we denote the d-dimensional Lebesgue measure of K and the radius of the largeest ball inscribed in K, respectively. All elements of $\mathcal{T}_h$ will be numbered so that $\mathcal{T}_h = \{K_i\}_{i \in I}$, where $I \subset Z^+ = \{0, 1, 2, \dots\}$ is a suitable index set. If two elements K_i, $K_j \in \mathcal{T}_h$ contain a nonempty open face, we call them *neighbours*. In this case we put $\Gamma_{ij} = \Gamma_{ji} = \partial K_i \cap \partial K_j$. For $i \in I$ we set $s(i) = \{j \in I; K_j \text{ is a neighbour of } K_i\}$. The boundary $\partial\Omega$ is formed by a finite number of faces of elements K_i adjacent to $\partial\Omega$. We denote all these boundary faces by S_j, where $j \in I_b \subset Z^- = \{-1, -2, \dots\}$ and set $\gamma(i) = \{j \in I_b; S_j \text{ is a face of } K_i\}$, $\Gamma_{ij} = S_j$ for $K_i \in \mathcal{T}_h$ such that $S_j \subset \partial K_i$, $j \in I_b$. For K_i not containing any boundary face S_j we put

[2] In [16] we found that in the DGFE discretization we can allow general convex polygonal ($d = 2$) or polyhedral ($d = 3$) elements. However, it is possible to use even nonconvex elements in the form of starshaped polygons ($d = 2$) or polyhedra ($d = 3$). This possibility will be analyzed in [17].

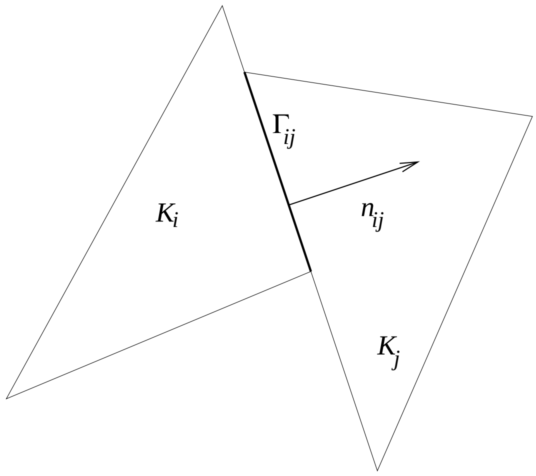

Figure 1: Neighbouring elements $K_i,\ K_j$

$\gamma(i) = \emptyset$. Obviously, $s(i) \cap \gamma(i) = \emptyset$ for all $i \in I$. Now, if we write $S(i) = s(i) \cup \gamma(i)$, we have

$$\partial K_i = \bigcup_{j \in S(i)} \Gamma_{ij}, \qquad \partial K_i \cap \partial\Omega = \bigcup_{j \in \gamma(i)} \Gamma_{ij}. \tag{2.3}$$

Moreover, for $i \in I$, by $\gamma_D(i)$ and $\gamma_N(i)$ we denote the subsets of $\gamma(i)$ formed by such indexes j that the faces Γ_{ij} form the parts Γ_D and Γ_N, respectively, of $\partial\Omega$. Thus, we suppose that

$$\gamma(i) = \gamma_D(i) \cup \gamma_N(i), \quad \gamma_D(i) \cap \gamma_N(i) = \emptyset. \tag{2.4}$$

Furthermore, we use the following notation: $\boldsymbol{n}_{ij} = ((n_{ij})_1, \ldots, (n_{ij})_d)$ = unit outer normal to ∂K_i on the face Γ_{ij} ($\boldsymbol{n}_{ij}$ is a constant vector on Γ_{ij}), $d(\Gamma_{ij}) = \mathrm{diam}(\Gamma_{ij})$, $|\Gamma_{ij}|$ = $(d-1)$-dimensional Lebesgue measure of Γ_{ij}.

Over the triangulation $\mathcal{T}_h$ we define the *broken Sobolev space*

$$H^k(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^k(K)\ \forall\, K \in \mathcal{T}_h\}, \tag{2.5}$$

with seminorm defined by

$$|v|_{H^k(\Omega,\mathcal{T}_h)} = \left(\sum_{K \in \mathcal{T}_h} |v|^2_{H^k(K)} \right)^{1/2}, \quad v \in H^k(\Omega, \mathcal{T}_h). \tag{2.6}$$

For $v \in H^1(\Omega, \mathcal{T}_h)$ we set

$$\begin{gathered} v|_{\Gamma_{ij}} = \text{ trace of } v|_{K_i} \text{ on } \Gamma_{ij}, \quad v|_{\Gamma_{ji}} = \text{ trace of } v|_{K_j} \text{ on } \Gamma_{ji}, \\ \langle v \rangle_{\Gamma_{ij}} = \frac{1}{2}\left(v|_{\Gamma_{ij}} + v|_{\Gamma_{ji}} \right) \quad \text{and} \quad [v]_{\Gamma_{ij}} = v|_{\Gamma_{ij}} - v|_{\Gamma_{ji}}, \end{gathered} \tag{2.7}$$

denoting the *traces, average* and *jump of the traces* of v on $\Gamma_{ij} = \Gamma_{ji}$, respectively. Obviously, $\langle v \rangle_{\Gamma_{ij}} = \langle v \rangle_{\Gamma_{ji}}$, $[v]_{\Gamma_{ij}} = -[v]_{\Gamma_{ji}}$ and $[v]_{\Gamma_{ij}} \boldsymbol{n}_{ij} = [v]_{\Gamma_{ji}} \boldsymbol{n}_{ji}$.

The approximate solution of problem (2.1), a)–d) is sought in the space of discontinuous piecewise polynomial functions S_h defined by

$$S_h = S^{p,-1}(\Omega, \mathcal{T}_h) = \{v; v|_K \in P^p(K)\ \forall\, K \in \mathcal{T}_h\}, \tag{2.8}$$

where p is a positive integer and $P^p(K)$ denotes the space of all polynomials on K of degree $\leq p$.

2.3 DGFE formulation

In order to derive the discrete problem, we assume that u is a solution of problem (2.1), a)–d) satisfying (2.2). Then $u(\cdot,t) \in H^2(\Omega) \subset H^2(\Omega, \mathcal{T}_h)$ for almost all (a.a.) $t \in (0,T)$ and

$$\begin{aligned} &\langle u(\cdot,t)\rangle_{\Gamma_{ij}} = u(\cdot,t)|_{\Gamma_{ij}}, \\ &[u(\cdot,t)]_{\Gamma_{ij}} = 0, \\ &\langle \nabla u(\cdot,t)\rangle_{\Gamma_{ij}} = \nabla u(\cdot,t)|_{\Gamma_{ij}} = \nabla u(\cdot,t)|_{\Gamma_{ji}} \qquad \text{for a.a. } t \in (0,T). \end{aligned} \tag{2.9}$$

We multiply equation (2.1), a) by any $\varphi \in H^2(\Omega, \mathcal{T}_h)$, integrate over $K_i \in \mathcal{T}_h$, apply Green's theorem and sum over all $K_i \in \mathcal{T}_h$. After some manipulation we obtain the identity

$$\begin{aligned} &\int_\Omega \frac{\partial u}{\partial t}\,\varphi\,\mathrm{d}x + \sum_{i\in I}\sum_{j\in S(i)}\int_{\Gamma_{ij}}\sum_{s=1}^{d} f_s(u)\,(n_{ij})_s\,\varphi|_{\Gamma_{ij}}\,\mathrm{d}S \\ &-\sum_{i\in I}\int_{K_i}\sum_{s=1}^{d} f_s(u)\,\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x + \sum_{i\in I}\int_{K_i}\varepsilon\,\nabla u\cdot\nabla\varphi\,\mathrm{d}x \\ &-\sum_{i\in I}\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}}\varepsilon\langle\nabla u\rangle\cdot\boldsymbol{n}_{ij}[\varphi]\,\mathrm{d}S \\ &-\sum_{i\in I}\sum_{j\in\gamma_D(i)}\int_{\Gamma_{ij}}\varepsilon\,\nabla u\cdot\boldsymbol{n}_{ij}\,\varphi\,\mathrm{d}S \\ =\ &\int_\Omega g\,\varphi\,\mathrm{d}x + \sum_{i\in I}\sum_{j\in\gamma_N(i)}\int_{\Gamma_{ij}}\varepsilon\,\nabla\,u\cdot\boldsymbol{n}_{ij}\,\varphi\,\mathrm{d}S. \end{aligned} \tag{2.10}$$

To the left-hand side of (2.10) we add now the terms

$$\pm\sum_{i\in I}\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}}\varepsilon\langle\nabla\varphi\rangle\cdot\boldsymbol{n}_{ij}[u]\,\mathrm{d}S, \tag{2.11}$$

which vanish due to (2.9). Further, to the left-hand side and the right-hand side of (2.10) we add the terms

$$\pm\sum_{i\in I}\sum_{j\in\gamma_D(i)}\int_{\Gamma_{ij}}\varepsilon\,\nabla\varphi\cdot\boldsymbol{n}_{ij}\,u\,\mathrm{d}S \tag{2.12}$$

and

$$\pm \sum_{i\in I} \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \varepsilon\, \nabla\varphi \cdot \boldsymbol{n}_{ij}\, u_D \,\mathrm{d}S,$$

respectively, which are identical due to the Dirichlet condition (2.1), b). We can add these terms equipped with the sign + (the so-called *nonsymmetric DG discretization* of diffusion terms) or with the sign − (*symmetric DG discretization* of diffusion terms).

In view of the Neumann condition (2.1), c), we replace the second term on the right-hand side of (2.10) by

$$\sum_{i\in I} \sum_{j\in\gamma_N(i)} \int_{\Gamma_{ij}} g_N\, \varphi \,\mathrm{d}S. \tag{2.13}$$

Because of the stabilization of the scheme we introduce the *interior penalty*

$$\varepsilon \sum_{i\in I} \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \sigma [u]\, [\varphi] \,\mathrm{d}S, \tag{2.14}$$

vanishing for the exact solution, and the *boundary penalty*

$$\varepsilon \sum_{i\in I} \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \sigma\, u\, \varphi \,\mathrm{d}S = \varepsilon \sum_{i\in I} \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \sigma u_D \varphi \mathrm{d}S, \tag{2.15}$$

where σ is a *weight* defined by

$$\sigma|_{\Gamma_{ij}} = C_W \varepsilon / d(\Gamma_{ij}), \tag{2.16}$$

and $C_W > 0$ is a suitable constant. For the nonsymmetric DG discretization of the diffusion terms we can set $C_W = 1$, but for the symmetric discretization, the choice of C_W is more sophisticated and will be a consequence of the sequel analysis.

On the basis of above considerations we introduce the following forms defined for $u,\, \varphi \in H^2(\Omega, \mathcal{T}_h)$:

$$\begin{aligned} a_h^N(u,\varphi) \;=\;& \sum_{i\in I} \int_{K_i} \varepsilon\, \nabla u \cdot \nabla\varphi \,\mathrm{d}x \\ -\;& \sum_{i\in I} \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \varepsilon \langle\nabla u\rangle \cdot \boldsymbol{n}_{ij} [\varphi] \,\mathrm{d}S + \sum_{i\in I} \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \varepsilon \langle\nabla \varphi\rangle \cdot \boldsymbol{n}_{ij} [u] \,\mathrm{d}S \\ -\;& \sum_{i\in I} \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \varepsilon\, \nabla\, u \cdot \boldsymbol{n}_{ij}\, \varphi \,\mathrm{d}S + \sum_{i\in I} \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \varepsilon\, \nabla\, \varphi \cdot \boldsymbol{n}_{ij}\, u \,\mathrm{d}S \end{aligned} \tag{2.17}$$

(nonsymmetric variant of the diffusion form),

$$\begin{aligned} a_h^S(u,\varphi) \;=\;& \sum_{i\in I} \int_{K_i} \varepsilon\, \nabla u \cdot \nabla\varphi \,\mathrm{d}x \\ -\;& \sum_{i\in I} \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \varepsilon \langle\nabla u\rangle \cdot \boldsymbol{n}_{ij} [\varphi] \,\mathrm{d}S - \sum_{i\in I} \sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \varepsilon \langle\nabla \varphi\rangle \cdot \boldsymbol{n}_{ij} [u] \,\mathrm{d}S \\ -\;& \sum_{i\in I} \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \varepsilon\, \nabla\, u \cdot \boldsymbol{n}_{ij}\, \varphi \,\mathrm{d}S - \sum_{i\in I} \sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \varepsilon\, \nabla\, \varphi \cdot \boldsymbol{n}_{ij}\, u \,\mathrm{d}S \end{aligned} \tag{2.18}$$

(symmetric variant of the diffusion form),

$$J_h^\sigma(u,\varphi) = \sum_{i\in I}\sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \sigma[u]\,[\varphi]\,\mathrm{d}S + \sum_{i\in I}\sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \sigma\, u\, \varphi\,\mathrm{d}S \tag{2.19}$$

(interior and boundary penalty),

$$\begin{aligned} \ell_h^N(\varphi)(t) &= \int_\Omega g(t)\,\varphi\,\mathrm{d}x + \sum_{i\in I}\sum_{j\in\gamma_N(i)} \int_{\Gamma_{ij}} g_N(t)\,\varphi\,\mathrm{d}S \\ &+ \sum_{i\in I}\sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \varepsilon\,\nabla\varphi\cdot\boldsymbol{n}_{ij}\, u_D(t)\,\mathrm{d}S + \varepsilon\sum_{i\in I}\sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \sigma\, u_D(t)\,\varphi\,\mathrm{d}S \end{aligned} \tag{2.20}$$

(nonsymmetric right-hand side form),

$$\begin{aligned} \ell_h^S(\varphi)(t) &= \int_\Omega g(t)\,\varphi\,\mathrm{d}x + \sum_{i\in I}\sum_{j\in\gamma_N(i)} \int_{\Gamma} g_N(t)\,\varphi\,\mathrm{d}S \\ &- \sum_{i\in I}\sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \varepsilon\,\nabla\varphi\cdot\boldsymbol{n}_{ij}\, u_D(t)\,\mathrm{d}S + \varepsilon\sum_{i\in I}\sum_{j\in\gamma_D(i)} \int_{\Gamma_{ij}} \sigma\, u_D(t)\,\varphi\,\mathrm{d}S \end{aligned} \tag{2.21}$$

(symmetric right-hand side form).

Finally, the convective terms are approximated with the aid of a *numerical flux* $H = H(u,v,\boldsymbol{n})$ by the form

$$\begin{aligned} b_h(u,\varphi) &= -\sum_{i\in I}\int_K \sum_{s=1}^d f_s(u)\,\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x \\ &+ \sum_{i\in I}\sum_{j\in S(i)} H\left(u|_{\Gamma_{ij}}, u|_{\Gamma_{ji}}, \boldsymbol{n}_{ij}\right)\varphi|_{\Gamma_{ij}}\,\mathrm{d}S, \quad u,\,\varphi\in H^1(\Omega,\mathcal{T}_h), \quad u\in L^\infty(\Omega). \end{aligned} \tag{2.22}$$

Of course, if $j\in\gamma(i)$ and $\Gamma_{ij}\subset\partial\Omega$, it is necessary to specify the meaning of $u|_{\Gamma_{ji}}$. Here we use the extrapolation, i.e. we set $u|_{\Gamma_{ji}} := u|_{\Gamma_{ij}}$. (In practical computations, usually more sophisticated boundary conditions depending on the behavious of u are used in the definition of the form b_h. Cf., e.g. the case of compressible flow in [11].)

We assume that the numerical flux has the following properties:

Assumptions (H):

1. $H(u,v,\boldsymbol{n})$ is defined in $\mathbb{R}^2\times B_1$, where $B_1 = \{\boldsymbol{n}\in\mathbb{R}^d; |\boldsymbol{n}|=1\}$, and *Lipschitz-continuous* with respect to u, v:

$$\begin{gathered} |H(u,v,\boldsymbol{n}) - H(u^*,v^*,\boldsymbol{n})| \le C_L(|u-u^*| + |v-v^*|), \\ u,\, v,\, u^*,\, v^* \in\mathbb{R},\ \boldsymbol{n}\in B_1. \end{gathered} \tag{2.23}$$

2. $H(u,v,\boldsymbol{n})$ is *consistent*:

$$H(u,u,\boldsymbol{n}) = \sum_{s=1}^d f_s(u)\,n_s, \quad u\in\mathbb{R},\ \boldsymbol{n} = (n_1,\dots,n_d)\in B_1. \tag{2.24}$$

3. $H(u, v, \boldsymbol{n})$ is *conservative*:

$$H(u, v, \boldsymbol{n}) = -H(v, u, -\boldsymbol{n}), \quad u, v \in I\!R, \ \boldsymbol{n} \in B_1. \tag{2.25}$$

In virtue of (2.23) and (2.24), the functions f_s, $s = 1, \dots, d$, are Lipschitz-continuous with constant $L_f = 2C_L$.

Due to the assumption that $f_s(0) = 0$ for $s = 1, \dots, d$, we have

$$H(0, 0, \boldsymbol{n}) = 0 \quad \forall \boldsymbol{n} \in B_1. \tag{2.26}$$

Now we can introduce the *discrete problem*.

Definition 1 *We say that u_h is a DGFE solution of the convection-diffusion problem (2.1), if*

$$\begin{aligned}
&a) \quad u_h \in C^1([0, T]; S_h), && (2.27)\\
&b) \quad (\frac{\partial u_h(t)}{\partial t}, \varphi_h) + b_h(u_h(t), \varphi_h) + a_h(u_h(t), \varphi_h) + \varepsilon J_h^\sigma(u_h(t), \varphi_h)\\
&\qquad = \ell_h(\varphi_h)\,(t) \qquad \forall\, \varphi_h \in S_h, \ \forall\, t \in (0, T),\\
&c) \quad u_h(0) = u_h^0,
\end{aligned}$$

where $a_h = a_h^N, \ell_h = \ell_h^N$ (nonsymmetric variant) or $a_h = a_h^S, \ell_h = \ell_h^S$ (symmetric variant). By u_h^0 we denote an S_h-approximation of the initial condition u^0: $u_h^0 \in S_h$ is the L^2-projection, i.e.

$$(u_h^0 - u^0, \varphi_h) = 0 \quad \forall \varphi_h \in S_h. \tag{2.28}$$

The above discrete problem has been obtained with the aid of the *method of lines*, i. e. the *space semidiscretization*. The full space-time discretization will be the subject of the paper [14].

¿From the above considerations and the consistency of the numerical flux we conclude that the exact solution u satisfies condition (2.27), b):

$$\begin{aligned}
&(\frac{\partial u(t)}{\partial t}, \varphi_h) + b_h(u(t), \varphi_h) + a_h(u(t), \varphi_h) + \varepsilon J_h(u(t), \varphi_h)\\
&\qquad = \ell_h(\varphi_h)\,(t) \qquad \text{for all } \varphi_h \in S_h \text{ and for a.a. } t \in (0, T)
\end{aligned}$$

which implies the *Galerkin orthogonality property* of the error – see Section 3.

2.4 Numerical realization of the discrete problem

Introducing a basis $\{w_\alpha\}_{\alpha=1}^n$ in the space S_h with $n = \dim S_h$, the sought approximate solution can be expressed as

$$u_h(t) = \sum_{\alpha=1}^{n} \xi_\alpha(t)\, w_\alpha, \quad t \in [0, T], \tag{2.29}$$

and the discrete problem (2.27) is equivalent to a system of ordinary differential equations for unknown coefficients $\xi_\alpha(t),\ \alpha = 1,\dots,n$.

In practical computations, suitable time discretization is applied (Euler forward or backward scheme, Runge–Kutta methods or discontinuous Galerkin time discretization) and integrals are evaluated with the aid of numerical integration. The simplest time discretization is the Euler forward scheme. However, it suffers from a rather restrictive time step limitation due to a CFL-stability condition. Also the Runge-Kutta schemes are conditionally stable. Quite popular is therefore a *semi-implicit scheme*, in which the nonlinear convective terms are treated explicitly, whereas the diffusion terms are approximated in an implicit way. (The theoretical analysis will be the subject of the paper [14].) Let $0 = t_0 < t_1 < \dots$ be a partition of the time interval $[0,T]$ and $\tau_k = t_{k+1} - t_k$ denote the time step between t_k and t_{k+1}. Denoting by $\boldsymbol{\xi}^k = (\xi_1^k,\dots,\xi_n^k)^{\mathrm{T}}$ the vector with components ξ_α^k approximating $\xi_\alpha(t_k)$, then the semi-implicit scheme can be written in the form

$$\boldsymbol{M}\boldsymbol{\xi}^{k+1} + \tau_k(\boldsymbol{A} + \boldsymbol{C} \mp \boldsymbol{C}^{\mathrm{T}})\boldsymbol{\xi}^{k+1} = \boldsymbol{M}\boldsymbol{\xi}^k - \tau_k b^k + \tau_k \boldsymbol{L}^{k+1}. \tag{2.30}$$

Here $\boldsymbol{M}$ is the so-called mass matrix with elements (w_α, w_β), $\alpha,\beta = 1,\dots,n$, the matrix $\boldsymbol{A}$ corresponds to the term

$$\sum_{i\in I}\int_{K_i} \varepsilon\nabla u_h\cdot\nabla\varphi_h \mathrm{d}x + \varepsilon J_h(u_h,\varphi_h),$$

the matrix $\boldsymbol{C}$ corresponds to the expression

$$-\sum_{i\in I}\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}} \varepsilon\langle\nabla u_h\rangle\cdot\boldsymbol{n}_{ij}[\varphi_h]\,\mathrm{d}S - \sum_{i\in I}\sum_{j\in\gamma_D(i)}\int_{\Gamma_{ij}} \varepsilon\,\nabla\, u_h\cdot\boldsymbol{n}_{ij}\,\varphi_h\,\mathrm{d}S,$$

$\boldsymbol{b}^k$ is the vector representing the form $b_h(u_h(t_k),\varphi_h)$ and the vector $\boldsymbol{L}^{k+1}$ represents the right-hand side $\ell_h(\varphi_h)(t_{k+1})$ of (2.27), b). The matrices $\boldsymbol{A}$ and $\boldsymbol{M}$ are symmetric. The sum $\boldsymbol{C}+\boldsymbol{C}^{\mathrm{T}}$ corresponds to the symmetric variant of the diffusion form. In this case, on each time level t_{k+1} we solve a linear algebraic system for the unknown vector $\boldsymbol{\xi}^{k+1}$ with the symmetric matrix $\boldsymbol{M}+\tau_k(\boldsymbol{A}+\boldsymbol{C}+\boldsymbol{C}^{\mathrm{T}})$. The nonsymmetric variant of the diffusion form leads to a linear system with the matrix $\boldsymbol{M}+\tau_k(\boldsymbol{A}+\boldsymbol{C}-\boldsymbol{C}^{\mathrm{T}})$. This matrix is nonsymmetric, but positive definite, as can be easily found. It is a challenging task to develop efficient iterative methods for the solution of system (2.30).

In order to increas the order of the time discretization, it is possible to apply multistep methods, as, e.g. the following two or three step schemes (written here with a constant time step τ):

$$\begin{aligned}
&u_h^k \in S_h, \quad k = 0,1,\dots r, \\
&\left(\frac{3u_h^{k+1} - 4u_h^k + u_h^{k-1}}{2\tau}, v_h\right) + a_h(u_h^{k+1},v_h) + b_h(\hat{u}_h^{k+1},v_h) + \varepsilon J_h^\sigma(u_h^{k+1},v_h) \\
&\qquad = \ell_h(v_h)\,(t_{k+1}) \quad \forall\, v_h\in S_h, \qquad k = 1,\dots,r-1,
\end{aligned} \tag{2.31}$$

or

$$u_h^k \in S_h, \quad k = 0, 1, \dots r, \tag{2.32}$$
$$\left(\frac{11u_h^{k+1} - 18u_h^k + 9u_h^{k-1} - 2u_h^{k-2}}{6\tau}, v_h \right) + a_h(u_h^{k+1}, v_h) + b_h(\hat{u}_h^{k+1}, v_h)$$
$$+\varepsilon J_h^\sigma(u_h^{k+1}, v_h) = \ell_h(v_h)\,(t_{k+1}) \qquad \forall\, v_h \in S_h, \quad k = 2, \dots r-1,$$

where we set $\hat{u}_h^{k+1} = 2u_h^k - u_h^{k-1}$ or $\hat{u}_h^{k+1} = 3\hat{u}_h^k - 3\hat{u}_h^{k-1} + \hat{u}_h^{k-2}$ for scheme (2.31) or (2.32), respectively. Initial values u_h^1, u_h^2 can be determined with the aid of Runge-Kutta methods. (Similar approach was used for the discretization of compressible flow, see [13].)

3 Error analysis

Let us consider a system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$, $h_0 > 0$, of triangulations of the domain Ω, i.e. $\mathcal{T}_h = \{K_i\}_{i\in I_h}$, $I_h \subset Z^+$. For the sake of simplicity, we shall write I instead of I_h ($h \in (0, h_0)$) and the dependence of index sets $I, I_b, s(i), \gamma(i), \gamma_D(i), \gamma_N(i)$ and $S(i)$ on h will not be emphasized by the notation.

Let us assume that the system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ has the following properties:

(A1) The system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ is *regular*: there exists a constant $C_R > 0$ such that

$$\frac{h_K}{\rho_K} \le C_R \quad \forall K \in \mathcal{T}_h, \quad \forall h \in (0, h_0). \tag{3.1}$$

(A2) There exists a constant $C_D > 0$ such that

$$h_{K_i} \le C_D\, d(\Gamma_{ij}), \quad i \in I, j \in S(i), h \in (0, h_0). \tag{3.2}$$

Condition (A2) means that the faces Γ_{ij} do not degenerate with respect to h_{K_i} for $h \to 0+$.

3.1 Some auxiliary results

Under the above assumptions we shall summarize some important approximation results and establish properties of the form b_h.

Lemma 1 (Multiplicative trace inequality) *There exists a constant $C_M > 0$ independent of v, h and K such that*

$$\|v\|^2_{L^2(\partial K)} \le C_M \left(\|v\|_{L^2(K)}\, |v|_{H^1(K)} + h_K^{-1} \|v\|^2_{L^2(K)} \right), \tag{3.3}$$
$$K \in \mathcal{T}_h,\ v \in H^1(K),\ h \in (0, h_0).$$

For the proof see [16], Lemma 3.1.

Lemma 2 (Inverse inequality) *There exists a constant $C_I > 0$ independent of v, h and K such that*

$$|v|_{H^1(K)} \leq C_I h_K^{-1} \|v\|_{L^2(K)}, \quad v \in P^p(K),\ K \in \mathcal{T}_h,\ h \in (0, h_0). \tag{3.4}$$

Proof is a consequence of standard scaling arguments – see, e.g. [7], proof of Theorem 3.2.6, from which it follows that the constant C_I depends on the polynomial degree p in such a way that it is an increasing function of p.

For $v \in L^2(\Omega)$ we denote by $\Pi_h v$ the $L^2(\Omega)$-projection of v on S_h:

$$\Pi_h v \in S_h, \quad (\Pi_h v - v, \varphi_h) = 0 \quad \forall v_h \in S_h. \tag{3.5}$$

Obviously, if $K \in \mathcal{T}_h$, then the function $\Pi_h v|_K$ is the $L^2(K)$-projection of v on $P^p(K)$.

The operator Π_h has the following approximation properties:

Lemma 3 *There exist a constant $C_A > 0$ independent of v and h such that*

$$\begin{aligned} \|\Pi_h v - v\|_{L^2(K)} &\leq C_A h_K^{k+1} |v|_{H^{k+1}(K)}, \\ |\Pi_h v - v|_{H^1(K)} &\leq C_A h_K^{k} |v|_{H^{k+1}(K)}, \\ |\Pi_h v - v|_{H^2(K)} &\leq C_A h_K^{k-1} |v|_{H^{k+1}(K)} \end{aligned} \tag{3.6}$$

for all $v \in H^{k+1}(K)$, $K \in \mathcal{T}_h$ and $h \in (0, h_0)$, where $k \in [1, p]$ is an integer.

For proof, see [25], Lemma 4.1.

In what follows, by C we shall denote a generic positive constant independent of h, attaining in general different values in different places.

Let assumptions (H), (A1) and (A2) be satisfied. We shall be concerned with the Lipschitz-continuity of the form b_h.

Lemma 4 *There exists a constant $C > 0$ independent of $u, \bar{u}, v, u_h, \bar{u}_h, v_h$ and h such that*

$$\begin{aligned} &|b_h(u, v) - b_h(\bar{u}, v)| \leq C \left(J_h^\sigma(v, v)^{1/2} + |v|_{H^1(\Omega, \mathcal{T}_h)} \right) \\ &\times \left(\|u - \bar{u}\|_{L^2(\Omega)} + \left(\sum_{i \in I} h_{K_i} \|u - \bar{u}\|^2_{L(\partial K_i)} \right)^{1/2} \right), \\ &\qquad u,\ \bar{u},\ v \in H^1(\Omega, \mathcal{T}_h),\ h \in (0, h_0), \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &|b_h(u_h, v_h) - b_h(\bar{u}_h, v_h)| \\ \leq\ & C \left(J_h^\sigma(v_h, v_h)^{1/2} + |v_h|_{H^1(\Omega, \mathcal{T}_h)} \right) \|u_h - \bar{u}_h\|_{L^2(\Omega)}, \\ &\qquad u_h,\ \bar{u}_h,\ v_h \in S_h,\ h \in (0, h_0). \end{aligned} \tag{3.8}$$

P r o o f. In view of the definition of the form b_h, for $u,\ \bar{u},\ v \in H^1(\Omega, \mathcal{T}_h)$ we have

$$b_h(u,v) - b_h(\bar{u},v) = \underbrace{-\sum_{i\in I}\int_{K_i}\sum_{s=1}^{d}(f_s(u)-f_s(\bar{u}))\frac{\partial v}{\partial x_s}\,\mathrm{d}x}_{:=\vartheta_1} \tag{3.9}$$

$$+ \underbrace{\sum_{i\in I}\sum_{j\in s(i)}\int_{\Gamma_{ij}}\Big(H(u|_{\Gamma_{ij}}, u|_{\Gamma_{ji}}, \boldsymbol{n}_{ij}) - H(\bar{u}|_{\Gamma_{ij}}, \bar{u}|_{\Gamma_{ji}}, \boldsymbol{n}_{ij})\Big)\, v|_{\Gamma_{ij}}\,\mathrm{d}S}_{:=\vartheta_2}$$

$$+ \underbrace{\sum_{i\in I}\sum_{j\in \gamma(i)}\int_{\Gamma_{ij}}\Big(H(u|_{\Gamma_{ij}}, u|_{\Gamma_{ij}}, \boldsymbol{n}_{ij}) - H(\bar{u}|_{\Gamma_{ij}}, \bar{u}|_{\Gamma_{ij}}, \boldsymbol{n}_{ij})\Big)\, v|_{\Gamma_{ij}}\,\mathrm{d}S}_{:=\vartheta_3}.$$

Using the Lipschitz-continuity of the functions $f_s,\ s = 1,\ldots,d$, and the Cauchy inequality, we find that

$$|\vartheta_1| \le C\sum_{i\in I}\int_{K_i}\sum_{s=1}^{d}|u-\bar{u}|\left|\frac{\partial v}{\partial x_s}\right|\mathrm{d}x \le C\|u-\bar{u}\|_{L^2(\Omega)}|v|_{H^1(\Omega,\mathcal{T}_h)}. \tag{3.10}$$

The conservativity of H implies that

$$\begin{aligned}
\vartheta_2+\vartheta_3 \;=\;& \sum_{i\in I}\Bigg\{\sum_{j\in s(i)}\int_{\Gamma_{ij}} v|_{\Gamma_{ij}}\Big(H(u|_{\Gamma_{ij}}, u|_{\Gamma_{ji}}, \boldsymbol{n}_{ij}) - H(\bar{u}|_{\Gamma_{ij}}, \bar{u}|_{\Gamma_{ji}}, \boldsymbol{n}_{ij})\Big)\,\mathrm{d}S\\
&+\sum_{j\in\gamma(i)}\int_{\Gamma_{ij}} v|_{\Gamma_{ij}}\Big(H(u|_{\Gamma_{ij}}, u|_{\Gamma_{ij}}, \boldsymbol{n}_{ij}) - H(\bar{u}|_{\Gamma_{ij}}, \bar{u}|_{\Gamma_{ij}}, \boldsymbol{n}_{ij})\Big)\,\mathrm{d}S\Bigg\}\\
=\;& \sum_{i\in I}\Bigg\{\frac{1}{2}\sum_{j\in s(i)}\int_{\Gamma_{ij}}\Big(v|_{\Gamma_{ij}} - v|_{\Gamma_{ji}}\Big)\\
&\times\Big(H(u|_{\Gamma_{ij}}, u|_{\Gamma_{ji}}, \boldsymbol{n}_{ij}) - H(\bar{u}|_{\Gamma_{ij}}, \bar{u}|_{\Gamma_{ji}}, \boldsymbol{n}_{ij})\Big)\,\mathrm{d}S\\
&+\sum_{j\in\gamma(i)}\int_{\Gamma_{ij}} v|_{\Gamma_{ij}}\Big(H(u|_{\Gamma_{ij}}, u|_{\Gamma_{ij}}, \boldsymbol{n}_{ij}) - H(\bar{u}|_{\Gamma_{ij}}, \bar{u}|_{\Gamma_{ij}}, \boldsymbol{n}_{ij})\Big)\,\mathrm{d}S\Bigg\}.
\end{aligned}$$

It follows from the Lipschitz-continuity of H, the Cauchy inequality and the inequality $d(\Gamma_{ij}) \le h_{K_i}$ that

$$\begin{aligned}
|\vartheta_2+\vartheta_3| \;\le\;& C\sum_{i\in I}\Bigg\{\frac{1}{2}\sum_{j\in s(i)}\int_{\Gamma_{ij}}\Big|[v]_{\Gamma_{ij}}\Big|\Big(|u-\bar{u}|_{\Gamma_{ij}} + |u-\bar{u}|_{\Gamma_{ji}}\Big)\,\mathrm{d}S\\
&+2\sum_{j\in\gamma(i)}\int_{\Gamma_{ij}}|v|_{\Gamma_{ij}}|\,|u-\bar{u}|_{\Gamma_{ij}}\,\mathrm{d}S\Bigg\} \qquad (3.11)\\
\le\;& C\sum_{i\in I}\Bigg\{\sum_{j\in s(i)}\int_{\Gamma_{ij}}\Big|[v]_{\Gamma_{ij}}\Big|\,|u-\bar{u}|_{\Gamma_{ij}}\,\mathrm{d}S
\end{aligned}$$

$$
\begin{aligned}
& \left. + \sum_{j\in\gamma(i)} \int_{\Gamma_{ij}} |v|_{\Gamma_{ij}}|\, |u-\bar{u}|_{\Gamma_{ij}}\, \mathrm{d}S \right\} \\
\leq \;& C \sum_{i\in I} \left\{ \left(\sum_{j\in s(i)} \int_{\Gamma_{ij}} \frac{[v]^2_{\Gamma_{ij}}}{d(\Gamma_{ij})}\, \mathrm{d}S + \sum_{j\in\gamma(i)} \int_{\Gamma_{ij}} \frac{v^2}{d(\Gamma_{ij})}\, \mathrm{d}S \right)^{1/2} \right. \\
& \times \left(\sum_{j\in s(i)} d(\Gamma_{ij}) \int_{\Gamma_{ij}} |u-\bar{u}|^2\, \mathrm{d}S \right. \\
& \left. \left. + \sum_{j\in\gamma(i)} d(\Gamma_{ij}) \int_{\Gamma_{ij}} |u-\bar{u}|^2\, \mathrm{d}S \right)^{1/2} \right\} \\
\leq \;& C\, J_h^\sigma(v,v)^{1/2} \left(\sum_{i\in I} h_{K_i} \|u-\bar{u}\|^2_{L^2(\partial K_i)} \right)^{1/2}.
\end{aligned}
$$

Now from (3.10) and (3.11) we get (3.7).

Further, let $u_h,\ \bar{u}_h,\ v_h \in S_h$. In order to prove estimate (3.8), we set $u := u_h,\ \bar{u} := \bar{u}_h$ and $v := v_h$ in (3.7). Using the multiplicative trace inequality (3.3), the discrete Cauchy inequality and the inverse inequality (3.4), for $\varphi \in S_h$ we obtain

$$
\begin{aligned}
\sum_{i\in I} h_{K_i} \|\varphi\|^2_{L^2(\partial K_i)} \;\leq\;& C \sum_{i\in I} \left(\|\varphi\|^2_{L^2(K_i)} + h_{K_i} \|\varphi\|_{L^2(K_i)}\, |\varphi|_{H^1(K_i)} \right) \qquad (3.12) \\
\leq\;& C \left(\|\varphi\|^2_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)} \left(\sum_{i\in I} h^2_{K_i} |\varphi|^2_{H^1(K_i)} \right)^{1/2} \right) \\
\leq\;& C \left(\|\varphi\|^2_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)} C \left(\sum_{i\in I} \|\varphi\|^2_{L^2(K_i)} \right)^{1/2} \right) \\
\leq\;& C \|\varphi\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Now from (3.7) and (3.12) we get (3.8). □

Lemma 5 *There exists a constant $C > 0$ independent of u, v_h and h such that*

$$
\begin{aligned}
& |b_h(u,v_h) - b_h(\Pi_h u, v_h)| \leq \qquad (3.13) \\
\leq\;& C h^{p+1} \left(J_h^\sigma(v_h,v_h)^{1/2} + |v_h|_{H^1(\Omega,\mathcal{T}_h)} \right) |u|_{H^{p+1}(\Omega)}, \\
& u \in H^{p+1}(\Omega),\ v_h \in S_h,\ h \in (0,h_0),
\end{aligned}
$$

where $\Pi_h u$ is the S_h-interpolant of u from Lemma 3.

Proof. Let us use (3.7), where $u \in H^{p+1}(\Omega)$, $\bar{u} := \Pi_h u$ and $v := v_h \in S_h$. Applying the multiplicative trace inequality (3.3), the discrete Cauchy inequality and estimates (3.6), we obtain

$$
\sum_{i\in I} h_{K_i} \|u - \Pi_h u\|^2_{L^2(\partial K_i)} \;\leq\; \sum_{i\in I} C \Big(\|u - \Pi_h u\|^2_{L^2(K_i)}
$$

$$
\begin{aligned}
&\quad +h_{K_i}\|u-\Pi_h u\|_{L^2(K_i)}|u-\Pi_h u|_{H^1(K_i)}\Big) \qquad (3.14)\\
\leq\ & C\sum_{i\in I}\left(h_{K_i}^{2p+2}+h_{K_i}h_{K_i}^{p+1}h_{K_i}^{p}\right)|u|^2_{H^{p+1}(K_i)}\\
\leq\ & Ch^{2p+2}|u|^2_{H^{p+1}(\Omega)}.
\end{aligned}
$$

¿From this estimate, (3.7) and (3.6) we already get (3.13). □

Now let us consider the solution u of problem (2.1) satisfying the regularity assumption (2.2). Let $\Pi_h u(t)$ be the S_h-interpolation of $u(t)$ ($t\in[0,T]$) from Lemma 3. We define the functions

$$
\begin{aligned}
\xi(t) &= u_h(t)-\Pi_h u(t)\in S_h, \quad \eta(t)=\Pi_h u(t)-u(t)\in H^{p+1}(\Omega,\mathcal{T}_h), \qquad (3.15)\\
&\quad \text{for a.a.}\, t\in(0,t).
\end{aligned}
$$

It follows from (3.5) that

$$
\Pi_h\left(\frac{\partial u}{\partial t}\right)=\frac{\partial}{\partial t}(\Pi_h u) \qquad (3.16)
$$

and, thus,

$$
\frac{\partial \eta}{\partial t}(t)=\frac{\partial u}{\partial t}(t)-\Pi_h\frac{\partial u}{\partial t}(t)\in H^p(\Omega,\mathcal{T}_h)\quad \text{for a.a. } t\in(0,T). \qquad (3.17)
$$

Of course, u, u_h, η and ξ are functions of x and t. In the following considerations we shall not emphasize this dependence by notation, if not necessary.

Lemma 6 *For all* $h\in(0,h_0)$ *we have*

$$
\begin{aligned}
\|\eta\|_{L^2(\Omega)} &\leq Ch^{p+1}|u|_{H^{p+1}(\Omega)}, \qquad (3.18)\\
|\eta|_{H^1(\Omega,\mathcal{T}_h)} &\leq Ch^{p}|u|_{H^{p+1}(\Omega)},\\
|\eta|_{H^2(\Omega,\mathcal{T}_h)} &\leq Ch^{p-1}|u|_{H^{p+1}(\Omega)},\\
\left\|\frac{\partial \eta}{\partial t}\right\|_{L^2(\Omega)} &\leq Ch^{p}\left|\frac{\partial u}{\partial t}\right|_{H^p(\Omega)}.
\end{aligned}
$$

where $C>0$ *is a constant independent of* u *and* h.

The proof is an immediate consequence of Lemma 3.

Lemma 7 *For* $h\in(0,h_0)$ *we have*

$$
\begin{aligned}
&|b_h(u,\xi)-b_h(u_h,\xi)| \qquad (3.19)\\
&\leq C\left(J_h^\sigma(\xi,\xi)^{1/2}+|\xi|_{H^1(\Omega,\mathcal{T}_h)}\right)\left(h^{p+1}|u|_{H^{p+1}(\Omega)}+\|\xi\|_{L^2(\Omega)}\right),
\end{aligned}
$$

where $C>0$ *is a constant independent of* h, u, u_h *and* ξ.

Proof. We can write

$$
\begin{aligned}
b_h(u,\xi) - b_h(u_h,\xi) &= (b_h(u,\xi) - b_h(\Pi_h u,\xi)) \\
&+ (b_h(\Pi_h u,\xi) - b_h(u_h,\xi)).
\end{aligned} \tag{3.20}
$$

¿From Lemmas 4 and 5 we find that

$$
\begin{aligned}
|b_h(\Pi_h u,\xi) - b_h(u_h,\xi)| &\le C\left(J_h^\sigma(\xi,\xi)^{1/2} + |\xi|_{H^1(\Omega,\mathcal{T}_h)}\right)\|\Pi_h u - u_h\|_{L^2(\Omega)}, \\
|b_h(u,\xi) - b_h(\Pi_h u,\xi)| &\le Ch^{p+1}\left(J_h^\sigma(\xi,\xi)^{1/2} + |\xi|_{H^1(\Omega,\mathcal{T}_h)}\right)|u|_{H^{p+1}(\Omega)},
\end{aligned} \tag{3.21}
$$

which together with (3.15) and (3.20) yield (3.19). $\square$

Lemma 8 *There exists a constant $C > 0$ independent of u, h, ξ, ε such that*

$$
|a_h(\eta,\xi)| \le C\varepsilon h^p |u|_{H^{p+1}(\Omega)}\left(J_h^\sigma(\xi,\xi)^{1/2} + |\xi|_{H^1(\Omega,\mathcal{T}_h)}\right), \quad h \in (0,h_0). \tag{3.22}
$$

Proof. In view of the definition of the form a_h, we can write

$$
\begin{aligned}
a_h(\eta,\xi) &= \vartheta_1 + \vartheta_2 + \vartheta_3, \\
\vartheta_1 &= \varepsilon \sum_{i\in I}\int_{K_i} \nabla\eta\cdot\nabla\xi \,\mathrm{d}x, \\
\vartheta_2 &= -\varepsilon \sum_{i\in I}\sum_{\substack{j\in s(i)\\ j<i}} \int_{\Gamma_{ij}} \Big(\langle\nabla\eta\rangle\cdot\boldsymbol{n}_{ij}[\xi] \mp \langle\nabla\xi\rangle\cdot\boldsymbol{n}_{ij}[\eta]\Big)\,\mathrm{d}S, \\
\vartheta_3 &= -\varepsilon \sum_{i\in I}\sum_{j\in\gamma(i)} \int_{\Gamma_{ij}} \Big((\nabla\eta\cdot\boldsymbol{n}_{ij})\,\xi \mp (\nabla\xi\cdot\boldsymbol{n}_{ij})\,\eta\Big)\,\mathrm{d}S.
\end{aligned} \tag{3.23}
$$

(The sign $-$ or $+$ in ϑ_2 and ϑ_3 corresponds to the nonsymmetric or symmetric DG formulation of the diffusion terms, respectively.)

Using the Cauchy inequality and estimates (3.18), we get

$$
|\vartheta_1| \le \varepsilon C h^p |u|_{H^{p+1}(\Omega)}|\xi|_{H^1(\Omega,\mathcal{T}_h)}. \tag{3.24}
$$

In order to estimate ϑ_2, we apply the Cauchy inequality and find that

$$
\begin{aligned}
|\vartheta_2| \le\ & \varepsilon\left(\sum_{i\in I}\sum_{j\in s(i)} d(\Gamma_{ij})\|\nabla\eta\|^2_{L^2(\Gamma_{ij})}\right)^{1/2}\left(\sum_{i\in I}\sum_{j\in s(i)} \frac{1}{d(\Gamma_{ij})}\|[\xi]\|^2_{L^2(\Gamma_{ij})}\right)^{1/2} \\
&+\varepsilon\left(\sum_{i\in I}\sum_{j\in s(i)} d(\Gamma_{ij})\|\nabla\xi\|^2_{L^2(\Gamma_{ij})}\right)^{1/2}\left(\sum_{i\in I}\sum_{j\in s(i)} \frac{1}{d(\Gamma_{ij})}\|[\eta]\|^2_{L^2(\Gamma_{ij})}\right)^{1/2} \\
\le\ & \varepsilon\left(\sum_{i\in I} h_{K_i}\|\nabla\eta\|^2_{L^2(\partial K_i)}\right)^{1/2} J_h^\sigma(\xi,\xi)^{1/2} \\
&+\varepsilon\left(\sum_{i\in I} h_{K_i}\|\nabla\xi\|^2_{L^2(\partial K_i)}\right)^{1/2} J_h^\sigma(\eta,\eta)^{1/2}.
\end{aligned} \tag{3.25}
$$

Using a similar process as in (3.14), where we write $\nabla\eta = \nabla(u - \Pi_h u)$ instead of $u - \Pi_h u$ and p instead of $p+1$, we obtain the estimate

$$\left(\sum_{i\in I} h_{K_i}\|\nabla\eta\|^2_{L^2(\partial K_i)}\right)^{1/2} \le C\, h^p|u|_{H^{p+1}(\Omega)}. \tag{3.26}$$

By (3.3) and (3.4)we get

$$\begin{aligned}\sum_{i\in I} h_{K_i}\|\nabla\xi\|^2_{L^2(\partial K_i)} &\le C\sum_{i\in I}\left(\|\nabla\xi\|^2_{L^2(K_i)} + h_{K_i}\|\nabla\xi\|_{L^2(K_i)}|\nabla\xi|_{H^1(K_i)}\right)\\ &\le C\sum_{i\in I}\left(\|\nabla\xi\|^2_{L^2(K_i)} + C\|\nabla\xi\|^2_{L^2(K_i)}\right)\\ &= C|\xi|^2_{H^1(\Omega,\mathcal{T}_h)}.\end{aligned} \tag{3.27}$$

Finally, we estimate the term $J_h^\sigma(\eta,\eta)$. In virtue of (2.19), (2.16), (3.2), (3.3) and (3.6), we have

$$\begin{aligned}J_h^\sigma(\eta,\eta) &\le 4C_W\sum_{i\in I}\sum_{j\in S(i)}\int_{\Gamma_{ij}}\frac{1}{d(\Gamma_{ij})}|\eta|_{\Gamma_{ij}}|^2\,\mathrm{d}S \qquad (3.28)\\ &\le 4C_W C_D\sum_{i\in I}\frac{1}{h_{K_i}}\int_{\partial K_i}|\eta|^2\,\mathrm{d}S\\ &\le C\sum_{i\in I}\frac{1}{h_{K_i}}\left(\|\eta\|_{L^2(K_i)}|\eta|_{H^1(K_i)} + h^{-1}_{K_i}\|\eta\|^2_{L^2(K_i)}\right)\\ &\le Ch^{2p}|u|^2_{H^{p+1}(\Omega)}.\end{aligned}$$

Inequalities (3.25)$-$(3.28) imply that

$$|\vartheta_2| \le \vartheta(\varepsilon,h,u,\xi) := C\,\varepsilon\, h^p|u|_{H^{p+1}(\Omega)}\left(J_h^\sigma(\xi,\xi)^{1/2} + |\xi|_{H^1(\Omega,\mathcal{T}_h)}\right), \tag{3.29}$$

where $C>0$ is a constant independent of u, ξ, h, ε.

Similarly we obtain the estimate of ϑ_3:

$$|\vartheta_3| \le \vartheta(\varepsilon,h,u,\xi). \tag{3.30}$$

Now from (3.23), (3.24), (3.29) and (3.30) we immediately get (3.22). □

Finally, we shall be concerned with the "coercivity" of the form a_h.

Lemma 9 *a) For the nonsymmetric diffusion form $a_h = a_h^N$ we have*

$$\begin{aligned}a_h^N(\varphi_h,\varphi_h) \ge \varepsilon|\varphi_h|^2_{H^1(\Omega,\mathcal{T}_h)}\\ \forall\varphi_h\in S_h,\quad \forall h\in(0,h_0).\end{aligned} \tag{3.31}$$

b) Let

$$C_W \ge 2C_M(1+C_I), \tag{3.32}$$

where C_M and C_I are constants from (3.3) and (3.4), respectively, and let us consider the symmetric diffusion form $a_h = a_h^S$. Then

$$a_h^S(\varphi_h,\varphi_h) \geq \frac{\varepsilon}{2}\left(|\varphi_h|^2_{H^1(\Omega,\mathcal{T}_h)} - J_h^\sigma(\varphi_h,\varphi_h)\right) \qquad (3.33)$$
$$\forall \varphi_h \in S_h, \quad \forall h \in (0,h_0).$$

Proof. a) Inequality (3.31) immediately follows from the definition (2.17) of a_h^N.

b) By (2.18), for $\varphi_h \in S_h$ we have

$$\begin{aligned} a_h^S(\varphi_h,\varphi_h) &= \varepsilon|\varphi_h|^2_{H^1(\Omega,\mathcal{T}_h)} \\ &-2\varepsilon\sum_{i\in I}\left\{\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}}\langle\nabla\varphi_h\rangle\cdot\boldsymbol{n}_{ij}[\varphi_h]\mathrm{d}S + \sum_{j\in\gamma_D(i)}\int_{\Gamma_{ij}}\nabla\varphi_h\cdot\boldsymbol{n}_{ij}\varphi_h\mathrm{d}S\right\}. \end{aligned} \qquad (3.34)$$

This, the Cauchy inequality and Young's inequality imply that for any $\delta > 0$,

$$\begin{aligned} a_h^S(\varphi_h,\varphi_h) &\geq \varepsilon|\varphi|^2_{H^1(\Omega,\mathcal{T}_h)} \\ &-2\varepsilon\left\{\frac{1}{\delta}\sum_{i\in I}\left(\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}} d(\Gamma_{ij})|\langle\nabla\varphi_h\rangle|^2\mathrm{d}S + \sum_{j\in\gamma_D(i)}\int_{\Gamma_{ij}} d(\Gamma_{ij})|\nabla\varphi_h|^2\mathrm{d}S\right)\right\}^{1/2} \\ &\times\left\{\delta\sum_{i\in I}\left(\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}}\frac{1}{d(\Gamma_{ij})}[\varphi_h]^2\mathrm{d}S + \sum_{j\in\gamma_D(i)}\int_{\Gamma_{ij})}\frac{1}{d(\Gamma_{ij})}|\varphi_h|^2\mathrm{d}S\right)\right\}^{1/2} \\ &\geq \varepsilon|\varphi_h|^2_{H^1(\Omega,\mathcal{T}_h)} - \varepsilon\omega - \varepsilon\frac{\delta}{C_W}J_h^\sigma(\varphi_h,\varphi_h), \end{aligned} \qquad (3.35)$$

where

$$\omega = \frac{1}{\delta}\sum_{i\in I}\left(\sum_{\substack{j\in s(i)\\ j<i}}\int_{\Gamma_{ij}} d(\Gamma_{ij})|\langle\nabla\varphi_h\rangle|^2\mathrm{d}S + \sum_{j\in\gamma_D(i)}\int_{\Gamma_{ij}} d(\Gamma_{ij})|\nabla\varphi_h|^2\mathrm{d}S\right) \qquad (3.36)$$

Further, using Lemmas 1 and 2, we get

$$\begin{aligned} \omega &\leq \frac{1}{\delta}\sum_{i\in I} h_{K_i}\int_{\partial K_i}|\nabla\varphi_h|^2\mathrm{d}S \\ &\leq \frac{C_M}{\delta}\sum_{i\in I} h_{K_i}\left(|\varphi_h|_{H^1(K_i)}|\nabla\varphi_h|_{H^1(K_i)} + h_{K_i}^{-1}|\varphi_h|^2_{H^1(K_i)}\right) \\ &\leq \frac{C_M(1+C_I)}{\delta}|\varphi_h|^2_{H^1(\Omega,\mathcal{T}_h)}. \end{aligned} \qquad (3.37)$$

Now let us choose

$$\delta = 2C_M(1+C_I). \qquad (3.38)$$

Then it follows from (3.35) – (3.38) and (3.32) that

$$a_h^S(\varphi_h, \varphi_h) \geq \frac{1}{2}\varepsilon \left(|\varphi_h|^2_{H^1(\Omega,\mathcal{T}_h)} - \frac{2C_M(1+C_I)}{C_W} J_h^\sigma(\varphi_h, \varphi_h) \right)$$
$$\geq \frac{1}{2}\varepsilon \left(|\varphi_h|^2_{H^1(\Omega,\mathcal{T}_h)} - J_h^\sigma(\varphi_h, \varphi_h) \right),$$

which we wanted to prove.

Corollary 1 *In virtue of the above lemma, we get the coercivity inequality*

$$a_h(\varphi_h, \varphi_h) + \varepsilon J_h^\sigma(\varphi_h, \varphi_h) \;\; \geq \;\; \frac{\varepsilon}{2} \left(|\varphi_h|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h^\sigma(\varphi_h \varphi_h) \right), \qquad \varphi_h \in S_h, \quad h \in (0, h_0), \tag{3.39}$$

both for $a_h = a_h^N$, for which we can set $C_W = 1$, and for $a_h = a_h^S$, provided the constant C_W satisfies condition (3.32).

In the convergence proof, Gronwall's lemma will be applied:

Lemma 10 *Let y, q, z, $r \in C([0,T])$, $y, q, r \geq 0$ and let*

$$y(t) + q(t) \leq z(t) + \int_0^t r(s)\, y(s)\, \mathrm{d}s, \quad t \in [0,T]. \tag{3.40}$$

Then

$$y(t) + q(t) \leq z(t) + \int_0^t r(\vartheta)\, z(\vartheta) \exp\left(\int_\vartheta^t r(s)\, \mathrm{d}s \right) \mathrm{d}\vartheta, \quad t \in [0,T]. \tag{3.41}$$

The proof can be carried out in a similar way as in [21], Par. 8.2.29.

3.2 Error estimates for the method of lines

In this paragraph we shall establish error estimates for the schemes introduced in Section 2.3. Our goal is to prove the following *main result*.

Theorem 1 *Let assumptions* (H), (A1) *and* (A2) *be satisfied. Let u be the exact solution of problem (2.1) satisfying (2.2) and let u_h be the approximate solution defined by (2.27). Then the error $e_h = u_h - u$ satisfies the estimate*

$$\max_{t \in [0,T]} \|e_h(t)\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2} \int_0^T \left(|e_h(\vartheta)|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h^\sigma(e_h(\vartheta), e_h(\vartheta)) \right) \mathrm{d}\vartheta \leq \hat{C} h^{2p}, \tag{3.42}$$

with a constant $\hat{C} > 0$ independent of h.

Proof.

As in (3.15), we set $\xi = u_h - \Pi_h u \in S_h$, $\eta = \Pi_h u - u$. Then $e_h = u_h - u = \xi + \eta$. Let us subtract (2.29) from (2.27), use Corollary 1 and the relation

$$\left(\frac{\partial \xi(t)}{\partial t}, \xi(t)\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\|^2_{L^2(\Omega)}. \tag{3.43}$$

Then we get

$$\begin{aligned} & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}|\xi(t)|^2_{H^1(\Omega,\mathcal{T}_h)} + \frac{\varepsilon}{2}J_h^\sigma(\xi(t),\xi(t)) \\ \le\ & b_h(u(t),\xi(t)) - b_h(u_h(t),\xi(t)) - \left(\frac{\partial \eta(t)}{\partial t}, \xi(t)\right) \\ & -a_h(\eta(t),\xi(t)) - \varepsilon J_h^\sigma(\eta(t),\xi(t)) \quad \text{for a.a. } (0,T). \end{aligned} \tag{3.44}$$

Now from (3.18) – (3.22) and (3.28), using Young's inequality, we derive the estimates (we omit the argument t)

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2_{L^2(\Omega)} + \varepsilon|\xi|^2_{H^1(\Omega,\mathcal{T}_h)} + \varepsilon J_h^\sigma(\xi,\xi) \\ \le\ & C\Big\{\left(J_h^\sigma(\xi,\xi)^{1/2} + |\xi|_{H^1(\Omega,\mathcal{T}_h)}\right)\left(\|\xi\|_{L^2(\Omega)} + h^{p+1}|u|_{H^{p+1}(\Omega)}\right) \\ & +h^p\,|\partial u/\partial t|_{H^p(\Omega)}\,\|\xi\|_{L^2(\Omega)} \\ & +\varepsilon\, h^p|u|_{H^{p+1}(\Omega)}\left(J_h^\sigma(\xi,\xi)^{1/2} + |\xi|_{H^1(\Omega,\mathcal{T}_h)}\right) \\ & +\varepsilon h^p|u|_{H^{p+1}(\Omega)}J_h^\sigma(\xi,\xi)^{1/2}\Big\} \\ \le\ & \frac{\varepsilon}{2}\left(J_h^\sigma(\xi,\xi) + |\xi|^2_{H^1(\Omega,\mathcal{T}_h)}\right) + C\Big\{\left(1+\frac{1}{\varepsilon}\right)\|\xi\|^2_{L^2(\Omega)} \\ & +\frac{1}{\varepsilon}\left(\left(\varepsilon^2 h^{2p} + h^{2p+2}\right)|u|^2_{H^{p+1}(\Omega)}\right) + h^{2p}\,|\partial u/\partial t|^2_{H^p(\Omega)}\Big\} \\ & \qquad \text{almost everywhere in } (0,T). \end{aligned} \tag{3.45}$$

Hence,

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\left(|\xi(t)|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h^\sigma(\xi(t),\xi(t))\right) \\ \le\ & C\Big\{\left(1+\frac{1}{\varepsilon}\right)\|\xi(t)\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\left(\left(\varepsilon^2 h^{2p} + h^{2p+2}\right)|u(t)|^2_{H^{p+1}(\Omega)}\right) \\ & + h^{2p}\,|\partial u(t)/\partial t|^2_{H^p(\Omega)}\Big\}, \quad \text{for a.a. } t \in (0,T). \end{aligned} \tag{3.46}$$

The integration of (3.46) from 0 to $t \in [0,T]$ and the relation $\xi(0) = u_h^0 - \Pi_h u^0 = 0$ yield

$$\begin{aligned} & \|\xi(t)\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\int_0^t \left(|\xi(\vartheta)|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h^\sigma(\xi(\vartheta),\xi(\vartheta))\right)\mathrm{d}\vartheta \\ \le\ & C\Big\{\left(1+\frac{1}{\varepsilon}\right)\int_0^t \|\xi(\vartheta)\|^2_{L^2(\Omega)}\,\mathrm{d}\vartheta + \frac{1}{\varepsilon}h^{2p}\int_0^t \left(\left(\varepsilon^2 + h^2\right)|u(\vartheta)|^2_{H^{p+1}(\Omega)}\right)\mathrm{d}\vartheta \\ & + h^{2p}\int_0^t |\partial u(\vartheta)/\partial t|^2_{H^p(\Omega)}\,\mathrm{d}\vartheta\Big\}, \quad t \in [0,T]. \end{aligned} \tag{3.47}$$

Using Gronwall's Lemma 10, we get

$$\begin{aligned} & \|\xi(t)\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\int_0^t \left(|\xi(\vartheta)|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h^\sigma(\xi(\vartheta),\xi(\vartheta)) \right) \mathrm{d}\vartheta \\ \le\ & C\left((\varepsilon + h^2/\varepsilon)\|u\|^2_{L^2(0,T;H^{p+1}(\Omega))} + \|\partial u/\partial t\|^2_{L^2(0,T;H^p(\Omega))} \right) \\ & \times h^{2p} \exp\left(\tilde{C}\frac{1+\varepsilon}{\varepsilon} t \right), \quad t \in [0,T]. \end{aligned} \tag{3.48}$$

(C and $\tilde{C}$ are constants independent of t, h, ε, u.)

Now, since $e_h = \xi + \eta$ and thus,

$$\begin{aligned} \|e_h\|^2_{L^2(\Omega)} &\le 2\left(\|\xi\|^2_{L^2(\Omega)} + \|\eta\|^2_{L^2} \right), \\ |e_h|^2_{H^1(\Omega,\mathcal{T}_h)} &\le 2\left(|\xi|^2_{H^1(\Omega,\mathcal{T}_h)} + |\eta|^2_{H^1(\Omega,\mathcal{T}_h)} \right), \\ J_h^\sigma(e_h,e_h) &\le 2\left(J_h^\sigma(\xi,\xi) + J_h^\sigma(\eta,\eta) \right), \end{aligned} \tag{3.49}$$

it follows from (3.48), (3.28) and (3.18) that

$$\begin{aligned} & \|e_h(t)\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\int_0^t \left(|e_h(\vartheta)|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h^\sigma(e_h(\vartheta),e_h(\vartheta)) \right) \mathrm{d}\vartheta \\ \le\ & Ch^{2p}\Big\{ \left((\varepsilon + h^2/\varepsilon)\|u\|^2_{L^2(0,T;H^{p+1}(\Omega))} + \|\partial u/\partial t\|^2_{L^2(0,T;H^p(\Omega))} \right) \\ & \times \exp\left(\tilde{C}\frac{1+\varepsilon}{\varepsilon} T \right) + h^2\|u\|^2_{L^\infty(0,T;H^{p+1}(\Omega))} \Big\}, \ t \in [0,T]. \end{aligned} \tag{3.50}$$

Since $h \in (0, h_0)$, this already implies (3.42). □

Remark 1 *If we apply the above considerations under the assumption that the exact solution has a higher regularity than (2.2), namely*

$$\frac{\partial u}{\partial t} \in L^2(0,T;H^{p+1}(\Omega)), \tag{3.51}$$

we find that the error estimate becomes

$$\begin{aligned} & \max_{t\in[0,T]} \|e_h(t)\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\int_0^T \left(|e_h(\vartheta)|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h^\sigma(e_h(\vartheta),e_h(\vartheta)) \right) \mathrm{d}\vartheta \\ \le\ & \bar{C}(\varepsilon + h^2/\varepsilon + h^2)h^{2p}, \end{aligned} \tag{3.52}$$

where $\bar{C} > 0$ is a constant independent of h.

However, the constants $\hat{C}$ and $\bar{C}$ from (3.42) and (3.52), respectively, are of the order $O(\exp(\tilde{C}\frac{1+\varepsilon}{\varepsilon}T)$, which blows up for $\varepsilon \to 0+$. This is unfortunately a consequence of the application of necessary tools for overcoming the nonlinear convective terms, namely Young's inequality and Gronwall's lemma. In [25], error estimates not suffering from this drawback were derived for the method of lines applied to a linear nonstationary convection-diffusion problem. In the nonlinear case, this problem remains open.

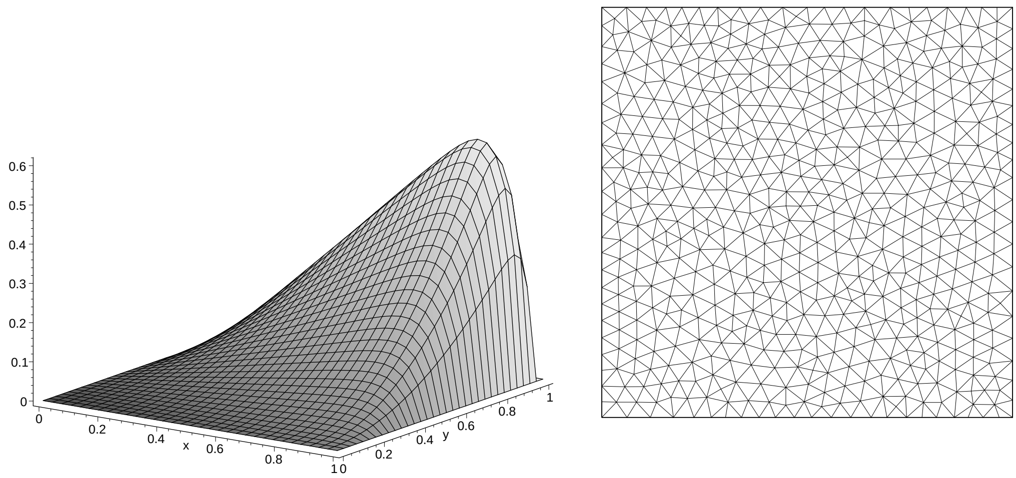

Figure 2: The exact solution (4.2) at $t = 1$ for $\varepsilon = 0.1$ (left) and an example of the mesh $\mathcal{T}_{h_4}$

4 Numerical experiments

In this section we verify theoretical results presented in the previous section. We consider the 2D viscous Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x_1} + u\frac{\partial u}{\partial x_2} = \varepsilon \Delta u + g \quad \text{in} \;\; \Omega \times (0, T), \tag{4.1}$$

with the initial and boundary Dirichlet conditions. Obviously, $f_s(u) = u^2/2$ for $s = 1, 2$. We set $\Omega = (0, 1)^2$, $T = 1$ and define the function g and the initial and boundary conditions in such a way that the exact solution has the form

$$u(x_1, x_2, t) = \left(1 - e^{-10t}\right) \hat{u}(x_1, x_2), \tag{4.2}$$

where

$$\begin{aligned} \hat{u}(x_1, x_2) \;\; &= \;\; x_1 x_2^2 - x_2^2 \exp\left(\frac{2(x_1 - 1)}{\varepsilon}\right) - x_1 \exp\left(\frac{3(x_2 - 1)}{\varepsilon}\right) \\ &+ \;\; \exp\left(\frac{2x_1 + 3x_2 - 5}{\varepsilon}\right) \end{aligned} \tag{4.3}$$

and $\varepsilon = 0.1$. For $t = T = 1$ the solution u differs very little from the steady state solution $\hat{u}$. The function $\hat{u}$ is equal to zero on the top and right parts of $\partial\Omega$ and $|\hat{u}| \le \exp(-20)$ on the rest of $\partial\Omega$. It contains two boundary layers along the right and top sides of the computational domain, see Figure 2.

The above problem is discretized by method (2.27). For the solution of the resulting system of ordinary differential equations we use the semi-implicit scheme (2.32), which is formally of the third order, with a rather small constant time step $\tau = T/r$, time levels $t_k = k\tau$ and initial values $u_h^k = \Pi_h u(t_k), k = 0, 1, 2$. The volume and line integrals appearing in the definition of the discrete problem are evaluated by quadrature formulae exact for polynomials of degree 6 and 7, respectively. The resulting systems of linear algebraic equations were solved by the biconjugate gradient method with SSOR preconditioning with the aid of LASPACK package [31]. (The iterative process was stopped when the residual was less than 10^{-10}.)

In the definition (2.22) of the form b_h the following numerical flux is used:

$$H(u_1, u_2, \boldsymbol{n}) = \begin{cases} \sum_{s=1}^{2} f_s(u_1) n_s, & \text{if } A > 0 \\ \sum_{s=1}^{2} f_s(u_2) n_s, & \text{if } A \le 0 \end{cases}, \tag{4.4}$$

where

$$A = \sum_{s=1}^{2} f_s'(\bar{u}) n_s, \quad \bar{u} = \frac{1}{2}(u_1 + u_2) \text{ and } \boldsymbol{n} = (n_1, n_2). \tag{4.5}$$

Numerical experiments are carried out with the use of piecewise linear (P^1), quadratic (P^2) and cubic (P^3) elements on 7 unstructured triangular meshes ($\mathcal{T}_{h_l},\ l = 1, \dots, 7$). (Figure 2 shows the mesh $\mathcal{T}_{h_4}$.) Since we use unstructured meshes, it appears to be suitable to investigate the dependence of computational errors not on $h = \max_{K \in \mathcal{T}_h} h_K$ but on the number $\#\mathcal{T}_{h_l}$ of elements of the triangulations $\mathcal{T}_{h_l}$. Therefore, we define the parameters

$$\bar{h}_l = \frac{2}{\sqrt{\sqrt{3} \# \mathcal{T}_{h_l}}}, \quad l = 1, \dots, 7, \tag{4.6}$$

where $\#\mathcal{T}_{h_l}$ is the number of elements of $\mathcal{T}_h$. Relation (4.6) follows from the fact that the area of an equilateral triangle with the length of a side equal to h is $\sqrt{3}h^2/4$. The values of h and $\bar{h}_l$ are shown in Table 1.

The time step τ was chosen according to the formula

$$\tau \le \frac{1}{75} h, \tag{4.7}$$

where h is the corresponding mesh size. Numerical experiments indicate that this choice is sufficient. Smaller values of τ do not cause any further decrease of computational errors.

The error of the solution is evaluated at time $T = 1$ over the domain Ω in the $L^2(\Omega)$–norm and in the $H^1(\Omega, \mathcal{T}_h)$–seminorm:

$$\begin{aligned} e_h^0 &\equiv \|u_h(\cdot, T) - u(\cdot, T)\|_{L^2(\Omega)}, \\ e_h^1 &\equiv |u_h(\cdot, T) - u(\cdot, T)|_{H^1(\Omega, \mathcal{T}_h)}, \end{aligned} \tag{4.8}$$

where $u(\cdot, T)$ is the exact solution (4.2) – (4.3) of equation (4.1) at time T and $u_h(\cdot, T)$ is the numerical solution at time T obtained by scheme (2.32). We suppose that the errors behave according to the formula

$$e_h^n \approx D_n h^{\alpha^n}, \quad n = 0, 1, \tag{4.9}$$

l	$\#\mathcal{T}_{\bar{h}_l}$	h	$\bar{h}_l$
1	148	1.733E-01	1.898E-01
2	289	1.277E-01	1.358E-01
3	591	9.008E-02	9.500E-02
4	1056	6.371E-02	7.110E-02
5	2360	4.510E-02	4.754E-02
6	4219	3.194E-02	3.555E-02
7	9372	2.262E-02	2.386E-02

Table 1: Number of elements $\#\mathcal{T}_{h_l}$, maximal diameter h and the parameter $\bar{h}_l$ for the triangulations $\mathcal{T}_{h_l},\ l = 1, \ldots, 7$

where $D_n > 0,\ n = 0, 1$ are constants independent of h and $\alpha^n,\ n = 0, 1$, are the *orders of accuracy* of the method.We define the *local experimental order of convergence* by

$$\alpha_l^n = \frac{\log\left(e^n_{h_l}/e^n_{h_{l-1}}\right)}{\log\left(\bar{h}_l/\bar{h}_{l-1}\right)}, \quad l = 2, \ldots, 7,\ n = 0, 1. \tag{4.10}$$

The *global experimental order of convergence* $\bar{\alpha}^n,\ n = 0, 1$, is obtained by the least squares method.

Tables 2 – 7 show the errors in L^2-norm and H^1-seminorm, the values of α_l^n, for $l = 2, \ldots, 7$ and $\bar{\alpha}^n$ for P^1, P^2 and P^3 approximations. The convergence in L^2-norm and H^1-seminorm is shown in Figures 3 and 4, respectively, where expected optimal error slopes are also plotted. Figures 5 and 6 show the graphs of the exact solution and piecewise linear, quadratic and cubic approximate solutions along the line $x_2 = 0.5$ obtained on the mesh $\mathcal{T}_{h_4}$ using nonsymmetric and symmetric discretization of diffusion terms. These results indicate the improvement of the quality of the approximate solution in dependence on the increasing polynomial degree p. These results demonstrate the convergence of the *p-version* of the DGFEM. Of course, our experimental results may be influenced by some effects as numerical integration errors, round-off errors, time discretization errors of scheme (2.32) and errors caused by the iterative solution of the linear discrete problem on each time level.

¿From the presented computational results we see that the quality of approximate solutions obtained with the aid of symmetric and nonsymmetric formulation is comparable. Nevertheless, for p even ($p = 2$ in our experiments), the order of accuracy in L^2-norm is higher in the case of the symmetric formulation. Comparing Figure 5 with Figure 6 and Figure 7 with Figure 8, we can see that the approximate solution obtained with the aid of the nonsymmetric P^1 scheme suffers from spurious overshoots in boundary layers in contrast to the symmetric P^1 scheme. (For $p > 1$, this effect does not appear.) ¿From this point of view, symmetric formulation of diffusion terms could seem to be more suitable. On the other hand, in the symmetric formulation we meet the problem to determine the weight parameter C_W in the interior and boundary penalty terms. We

l	$\bar{h}_l$	nonsymmetric e_h^0	α_l^0	symmetric e_h^0	α_l^0
1	1.898E-01	3.466 E-01	–	1.485 E-01	–
2	1.358E-01	1.858 E-01	1.864	8.238 E-02	1.761
3	9.500E-02	8.277 E-02	2.260	3.445 E-02	2.438
4	7.110E-02	5.232 E-02	1.580	2.238 E-02	1.485
5	4.754E-02	2.334 E-02	2.008	9.854 E-03	2.040
6	3.555E-02	1.580 E-02	1.342	6.575 E-03	1.393
7	2.386E-02	6.264 E-03	2.319	2.704 E-03	2.226
global order $\bar{\alpha}^0$			1.902		1.912

Table 2: Computational errors in L^2–norm, values of α_l^0, $l = 2, \ldots, 7$, and $\bar{\alpha}^0$ for nonsymmetric and symmetric formulation and P^1 elements

l	$\bar{h}_l$	nonsymmetric e_h^0	α_l^0	symmetric e_h^0	α_l^0
1	1.898E-01	4.012 E-02	–	7.688 E-03	–
2	1.358E-01	1.975 E-02	2.119	3.401 E-03	2.437
3	9.500E-02	9.573 E-03	2.024	1.050 E-03	3.286
4	7.110E-02	5.694 E-03	1.791	6.037 E-04	1.907
5	4.754E-02	2.571 E-03	1.977	2.353 E-04	2.343
6	3.555E-02	1.622 E-03	1.585	1.428 E-04	1.720
7	2.386E-02	6.355 E-04	2.348	4.117 E-05	3.117
global order $\bar{\alpha}^0$			1.955		2.454

Table 3: Computational errors in L^2–norm, values of α_l^0, $l = 2, \ldots, 7$, and $\bar{\alpha}^0$ for nonsymmetric and symmetric formulation and P^2 elements

derived a theoretical condition (3.32) for this parameter, guaranteeing the coercivity of the problem. This and the fact that the constant C_I is an increasing function of p (see comment behind Lemma 2) imply that also C_W increases in dependence on the increasing polynomial degree p. This is in agreement with our numerical experiments. In the case of the symmetric formulation we choose $C_W = 10$ for $p = 1$, $C_W = 50$ for $p = 2$ and $C_W = 200$ for $p = 3$. (Our tests show that, e.g. the choice $C_W = 100$ in the case $p = 3$ leads to a collaps of the computational process.)

5 Conclusion

We have presented an analysis of the discontinuous Galerkin finite element method of lines for the numerical solution of a nonstationary nonlinear convection-diffusion problem. We consider the schemes using symmetric and nonsymmetric formulation of dif-

l	$\bar{h}_l$	nonsymmetric e_h^0	α_l^0	symmetric e_h^0	α_l^0
1	1.898E-01	8.290 E-03	–	4.230 E-03	–
2	1.358E-01	2.929 E-03	3.110	1.700 E-03	2.725
3	9.500E-02	5.899 E-04	4.480	2.935 E-04	4.910
4	7.110E-02	2.621 E-04	2.795	1.420 E-04	2.503
5	4.754E-02	5.463 E-05	3.900	2.789 E-05	4.047
6	3.555E-02	2.596 E-05	2.561	1.175 E-05	2.977
7	2.386E-02	4.191 E-06	4.570	2.512 E-06	3.865
global order $\bar{\alpha}^0$			3.611		3.608

Table 4: Computational errors in L^2–norm, values of $\alpha_l^0,\ l = 2,\dots,7$, and $\bar{\alpha}^0$ for nonsymmetric and symmetric formulation and P^3 elements

l	$\bar{h}_l$	nonsymmetric e_h^1	α_l^1	symmetric e_h^1	α_l^1
1	1.898E-01	3.139 E-00	–	1.674 E-00	–
2	1.358E-01	2.208 E-00	1.051	1.288 E-00	0.784
3	9.500E-02	1.231 E-00	1.633	8.321 E-01	1.222
4	7.110E-02	9.605 E-01	0.855	7.049 E-01	0.571
5	4.754E-02	5.398 E-01	1.433	4.573 E-01	1.076
6	3.555E-02	4.436 E-01	0.675	4.001 E-01	0.461
7	2.386E-02	2.524 E-01	1.413	2.494 E-01	1.184
global order $\bar{\alpha}^1$			1.210		0.903

Table 5: Computational errors in H^1–seminorm, values of $\alpha_l^1,\ l = 2,\dots,7$, and $\bar{\alpha}^1$ for nonsymmetric and symmetric formulation and P^1 elements

l	$\bar{h}_l$	nonsymmetric e_h^1	α_l^1	symmetric e_h^1	α_l^1
1	1.898E-01	7.348 E-01	–	5.938 E-01	–
2	1.358E-01	4.261 E-01	1.628	3.503 E-01	1.577
3	9.500E-02	1.942 E-01	2.197	1.663 E-01	2.082
4	7.110E-02	1.280 E-01	1.437	1.155 E-01	1.257
5	4.754E-02	5.618 E-02	2.048	4.904 E-02	2.130
6	3.555E-02	3.727 E-02	1.413	3.493 E-02	1.168
7	2.386E-02	1.493 E-02	2.292	1.368 E-02	2.350
global order $\bar{\alpha}^1$			1.858		1.790

Table 6: Computational errors in H^1–seminorm, values of $\alpha_l^1,\ l = 2,\dots,7$, and $\bar{\alpha}^1$ for nonsymmetric and symmetric formulation and P^2 elements

l	$\bar{h}_l$	nonsymmetric e_h^1	nonsymmetric α_l^1	symmetric e_h^1	symmetric α_l^1
1	1.898E-01	1.835 E-01	–	1.305 E-01	–
2	1.358E-01	9.704 E-02	1.905	7.584 E-02	1.623
3	9.500E-02	3.250 E-02	3.058	2.455 E-02	3.153
4	7.110E-02	1.494 E-02	2.679	1.275 E-02	2.259
5	4.754E-02	4.783 E-03	2.832	3.846 E-03	2.980
6	3.555E-02	2.126 E-03	2.792	2.031 E-03	2.198
7	2.386E-02	6.040 E-04	3.153	5.367 E-04	3.335
global order $\bar{\alpha}^1$			2.785		2.670

Table 7: Computational errors in H^1–seminorm, values of α_l^1, $l = 2, \ldots, 7$, and $\bar{\alpha}^1$ for nonsymmetric and symmetric formulation and P^3 elements

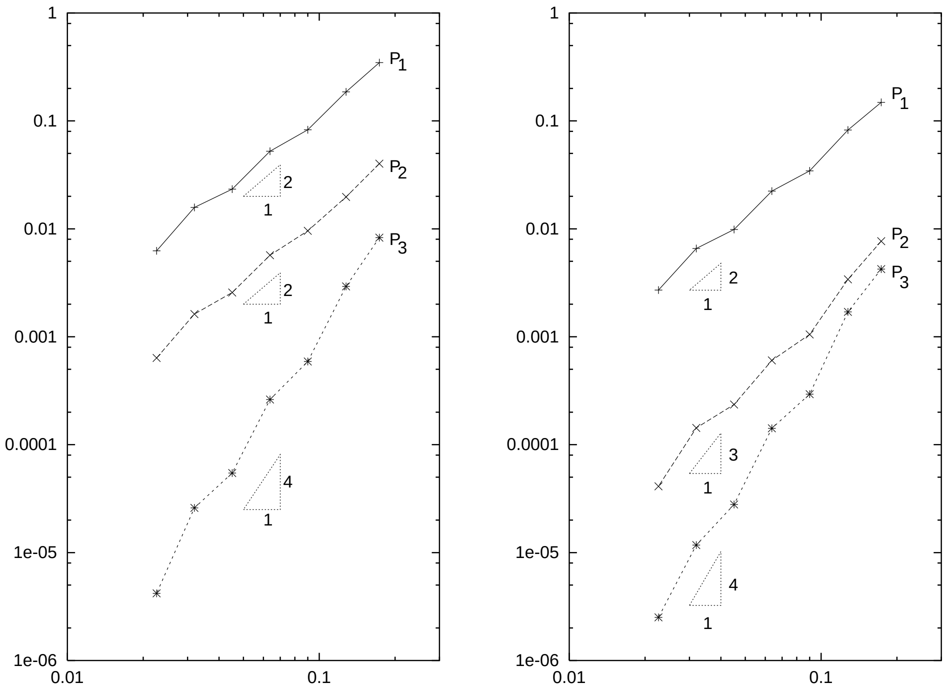

Figure 3: The convergence in L^2–norm with respect to h in logarithmic scale: nonsymmetric formulation (left) and symmetric formulation (right)

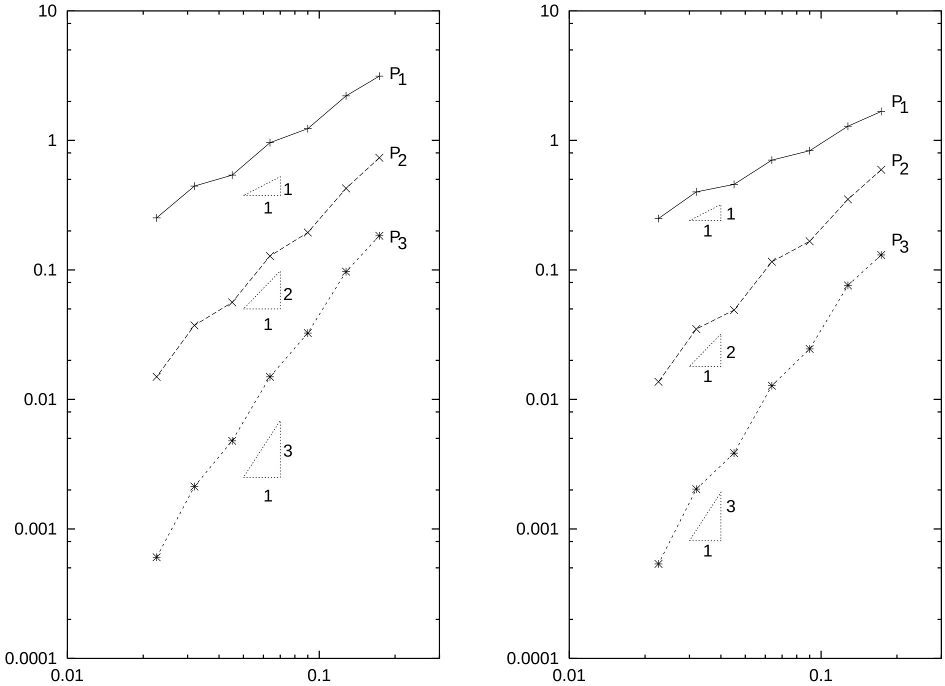

Figure 4: The convergence in H^1–seminorm with respect to h in logarithmic scale: nonsymmetric formulation (left) and symmetric formulation (right)

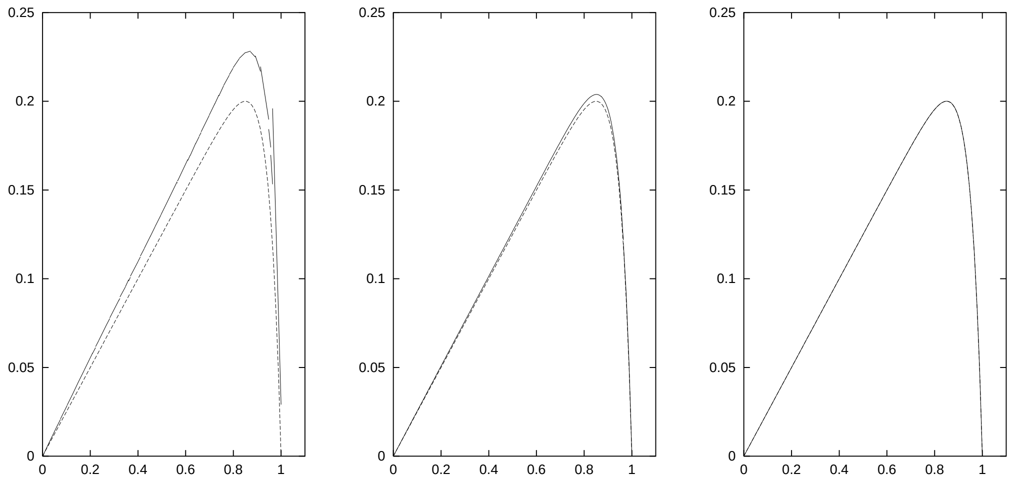

Figure 5: Graphs of the numerical solutions (full lines) along $x_2 = 0.5$ for P^1 (let), P^2 (center) and P^3 (right) elements on the mesh $\mathcal{T}_{h_4}$ in comparison with the exact solution (dashed lines), nonsymmetric formulation

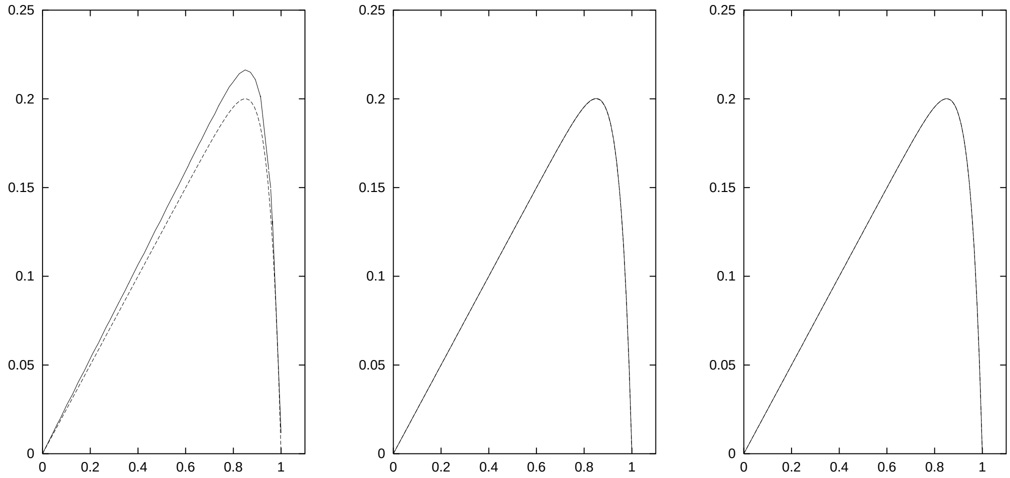

Figure 6: Graphs of the numerical solutions (full lines) along $x_2 = 0.5$ for P^1 (let), P^2 (center) and P^3 (right) elements on the mesh $\mathcal{T}_{h_4}$ in comparison with the exact solution (dashed lines), symmetric formulation

fusion terms, combined with interior and boundary penalties (usually denoted by SIPG and NIPG, respectively). We have derived a priori error estimates, namely $O(h^p)$ in $L^\infty(0,T;L^2(\Omega))$–norm and $L^2(0,T;H^1(\Omega,\mathcal{T}_h))$–seminorm. The presented numerical experiments indicate a better behaviour of the experimental $L^\infty(L^2)$–order of convergence. We can conclude that the SIPG variant yields numerical solutions with a better quality than NIPG. On the other hand, for SIPG it is necessary to tune the penalty parameter C_W in dependence on the degree p of polynomial approximations. In the NIPG case we simply put $C_W = 1$ for arbitrary p.

¿From our results some suggestions for the future work follow:

– derivation of optimal error estimates,

– analysis of the optimal choice of the penalty parameter for SIPG method,

– avoiding the blow up behaviour of estimates with respect to $\varepsilon \to 0+$,

– development of efficient a posteriori error estimates,

– analysis of the effect of numerical integration,

– analysis of the DGFEM for problems with nonlinear convection as well as diffusion,

– development of fast and robust methods for the solution of the discrete problem.

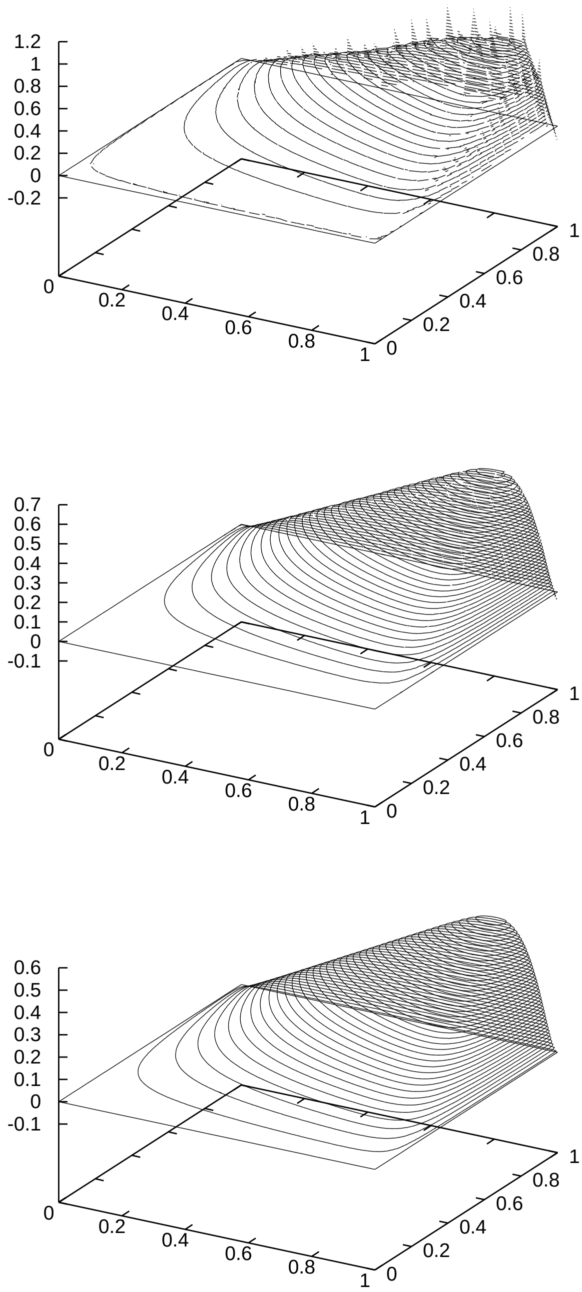

Figure 7: Isolines of the numerical solutions for P^1 (top), P^2 (center) and P^3 (bottom) elements on the mesh $\mathcal{T}_{h_4}$, nonsymmetric formulations

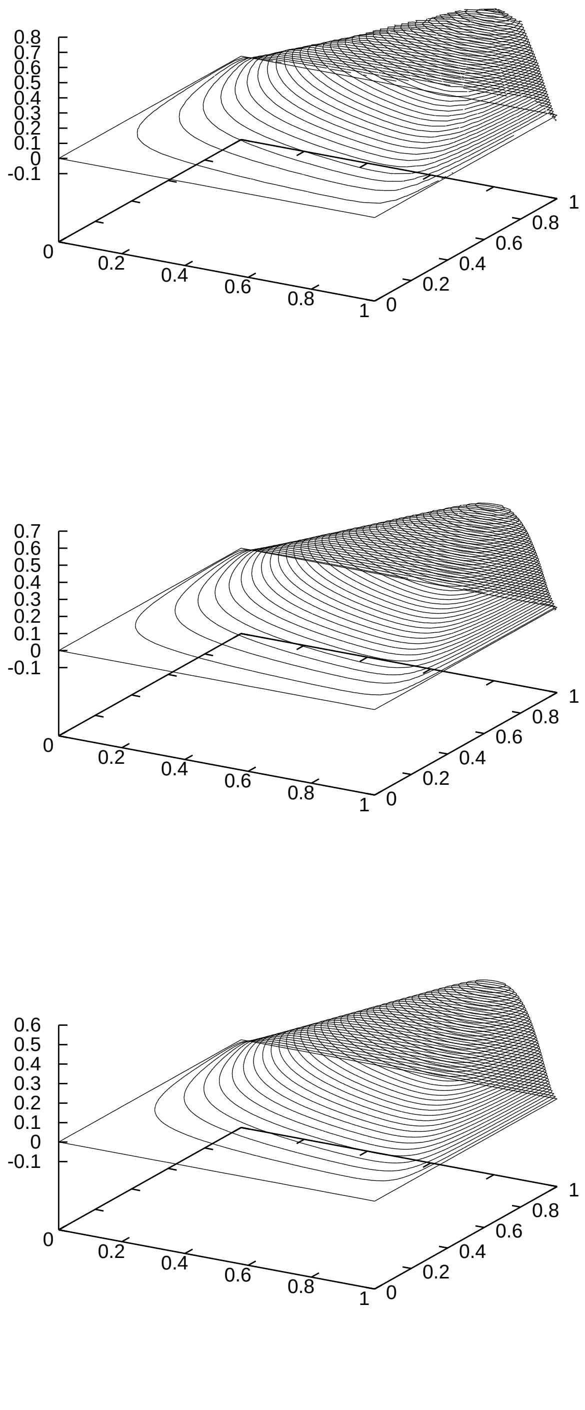

Figure 8: Isolines of the numerical solutions for P^1 (top), P^2 (center) and P^3 (bottom) elements on the mesh $\mathcal{T}_{h_4}$, symmetric formulations

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