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On some aspects of the Discontinuous Galerkin finite element method for conservation laws ⋆

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Abstract

The paper is concerned with the numerical solution of nonlinear conservation laws and nonlinear convection-diffusion problems. We discuss two versions of this method: a) Finite volume discontinuous Galerkin method, which is a generalization of the combined finite volume – finite element method. Its advantage is the use of only one mesh (in contrast to the combined finite volume – finite element schemes). However, it is of the first order only. b) Further, the pure discontinuous Galerkin finite element method of higher order is considered. In this case a new limiting of order of accuracy is developed in order to avoid spurious oscillations in the vicinity of discontinuities of the solution.

Key words: discontinuous Galerkin finite element method, numerical flux, conservation laws, convection-diffusion problems, limiting of order of accuracy
PACS: 65M10, 76M10, 76M12

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1 Introduction

During the last decade a number of various numerical techniques for the solution of nonlinear conservation laws, nonlinear convection-diffusion problems and compressible flow were developed. In principle, all numerical methods for the solution of partial differential equations can be applied to the mentioned problems. The most popular are now the finite element (FE) and finite volume (FV) methods. The finite volume schemes are suitable for the discretization of conservation law equations, whereas the FE methods are mainly used for diffusion problems. In order to employ advantages of both these techniques, combined FV - FE methods for the solution convection-diffusion problems and compressible viscous flows were developed. For analysis and applications, see, e.g., [1], [5], [6], [7], [8], [9]. These methods give good results in many cases of complicated technically relevant problems. However, their drawback is the necessity to construct two mutually associated meshes, which is rather complicated particularly in 3D ([10]). Moreover, these methods have the first order of accuracy only.

Another approach representing a simultaneous generalization of the FV as well as FE methods is the Discontinuous Galerkin finite element (DG FE) method. It is based on the idea to approximate the solution by piecewise polynomial functions over a FE mesh without any requirement on interelement continuity. The DG FE methods become more and more popular. For a survey, see, [2] or [3]. The DG FE method in connection with an approximate Riemann solver in convective terms leads to higher order numerical schemes, applied on one mesh only, but for problems with solutions having steep gradients or discontinuous solutions (containing shock waves of contact discontinuities) the well-known disease of higher order schemes, called the Gibbs effect, appears: spurious oscillations in the vicinity of discontinuities. The main goal of this paper is to describe two methods how to cure it. The first possibility uses a FV approximation of the convective terms applied in the framework of the DG FE method. This method requires only one mesh, but its order of accuracy is one. The second method is based on a new type of the *limiting* of the order of accuracy in the vicinity of discontinuities. In contrast to [2], where the author introduces a slope limiter quite in analogy with the FV MUSCL type schemes, we propose and analyze a different new method based on a suitable identification of a discontinuity and the decrease of the order of the method to one in a narrow neighbourhood of the discontinuity. The numerical experiments show that this method yields very accurate solutions to nonlinear conservation laws and convection-diffusion problems.

2 DG FE method for a nonlinear convection-diffusion nonstationary problem

2.1 Continuous problem

Let $\Omega \subset I\!R^d$ ($d = 2$ or 3) be a bounded polyhedral domain and $T > 0$. We set $Q_T = \Omega \times (0, T)$ and by $\partial\Omega$ denote the boundary of Ω. We consider the following *initial-boundary value problem*: Find $u : Q_T \to I\!R$ such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} = \sum_{s=1}^{d} \frac{\partial R_s(u, \nabla u)}{\partial x_s} + g \quad \text{in } Q_T, \tag{2.1}$$

$$u|_{\partial\Omega\times(0,T)} = u_D, \tag{2.2}$$

$$u(x, 0) = u^0(x), \quad x \in \Omega. \tag{2.3}$$

We suppose that $f_s \in C^1(I\!R)$, $R_s \in C^1(I\!R^{d+1})$, $R_s(u, \alpha)$ is linear with respect to $\alpha \in I\!R^d$ and the data are sufficiently regular so that we can assume the existence of a *strong solution* u satisfying (2.1) – (2.3) pointwise.

2.2 Discretization

Let T_h ($h > 0$) denote a partition of the closure $\overline{\Omega}$ of the domain Ω into a finite number of closed d-dimensional convex polyhedra K with mutually disjoint interiors. We call T_h a triangulation of $\overline{\Omega}$, but *do not* require the usual conforming properties from the finite element method. In 2D problems we choose usually $K \in T_h$ as triangles or quadrilaterals. In 3D, $K \in T_h$ can be, e. g., tetrahedra, pyramids or hexahedra. (We can allow even more general convex elements K.)

We set $h_K = \mathrm{diam}(K)$, $h = \max_{K\in T_h} h_K$. By $|K|$ we denote the d-dimensional Lebesgue measure of K. All elements of T_h will be numbered so that $T_h = \{K_i\}_{i\in I}$, where $I \subset Z^+ = \{1, 2, \ldots\}$ is a suitable index set. If two elements K_i, $K_j \in T_h$ contain a nonempty open face which is a part of a $(d-1)$-dimensional hyperplane, we call them *neighbours*. We set in this case $\Gamma_{ij} = \partial K_i \cap \partial K_j$ and assume that the whole set Γ_{ij} is a part of a $(d-1)$-dimensional hyperplane. For $i \in I$ we set $s(i) = \{j \in I; K_j \text{ is a neighbour of } K_i\}$.

The boundary $\partial\Omega$ is formed by a finite number of faces of elements K_i adjacent to $\partial\Omega$. We denote all these boundary faces by S_j, where $j \in I_b \subset Z^- = \{-1, -2, \ldots\}$ and set $\gamma(i) = \{j \in I_b; S_j \text{ is a face of } K_i\}, \Gamma_{ij} = S_j$ for $K_i \in T_h$ such that $S_j \subset \partial K_i,\ j \in I_b$. For K_i not containing any boundary face S_j

we set $\gamma(i) = \emptyset$. Obviously, $s(i) \cap \gamma(i) = \emptyset$ for all $i \in I$. Now, if we write $S(i) = s(i) \cup \gamma(i)$, we have

$$\partial K_i = \bigcup_{j \in S(i)} \Gamma_{ij}, \qquad \partial K_i \cap \partial\Omega = \bigcup_{j \in \gamma(i)} \Gamma_{ij}. \tag{2.4}$$

Furthermore, we use the following notation: $\boldsymbol{n}_{ij} = ((n_{ij})_1, ..., (n_{ij})_d) =$ unit outer normal to ∂K_i on the face Γ_{ij}, $d(\Gamma_{ij}) = \mathrm{diam}(\Gamma_{ij}), |\Gamma_{ij}| = (d-1)-$ dimensional Lebesgue measure of Γ_{ij}.

Over the triangulation T_h we define the *broken Sobolev space*

$$H^k(\Omega, T_h) = \{v; v|_K \in H^k(K)\ \forall\, K \in T_h\}. \tag{2.5}$$

For $v \in H^1(\Omega, T_h)$ we introduce the following notation $v|_{\Gamma_{ij}} =$ the trace of $v|_{K_i}$ on Γ_{ij}, $v|_{\Gamma_{ji}} =$ the trace of $v|_{K_j}$on $\Gamma_{ji} = \Gamma_{ij}$, $\langle v\rangle_{\Gamma_{ij}} = \frac{1}{2}\left(v|_{\Gamma_{ij}} + v|_{\Gamma_{ji}}\right)$, $\quad [v]_{\Gamma_{ij}} = v|_{\Gamma_{ij}} - v|_{\Gamma_{ji}}$. Obviously, $\langle v\rangle_{\Gamma_{ij}} = \langle v\rangle_{\Gamma_{ji}}$, but $[v]_{\Gamma_{ij}} = -[v]_{\Gamma_{ji}}$ and $[v]_{\Gamma_{ij}}\boldsymbol{n}_{ij} = [v]_{\Gamma_{ji}}\boldsymbol{n}_{ji}$.

The approximate solution of problem (2.1) – (2.3) is sought in the space of discontinuous piecewise polynomial functions

$$S_h = S^{p,-1}(\Omega, T_h) = \{v; v|_K \in P_p(K)\ \forall\, K \in T_h\}, \tag{2.6}$$

where $P_p(K)$ denotes the space of all polynomials on K of degree $\leq p$.

In order to derive the discrete problem, we multiply equation (2.1) by any $v \in S_h$, integrate over $K \in T_h$, apply Green's theorem and sum over all $K \in T_h$. Moreover, we use the relations $[u]|_{\Gamma_{ij}} = 0$, $\langle \nabla u\rangle_{\Gamma_{ij}} = \nabla u|_{\Gamma_{ij}} = \nabla u|_{\Gamma_{ji}}$, to the identity obtained we add some terms which mutually cancel and find that

$$\begin{aligned}
&\int_\Omega \frac{\partial u}{\partial t}\, v \,\mathrm{d}x + \sum_{i \in I}\sum_{j \in S(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{d} f_s(u)\,(n_{ij})_s\, v|_{\Gamma_{ij}} \,\mathrm{d}S \qquad (2.7)\\
&- \sum_{i \in I} \int_{K_i} \sum_{s=1}^{d} f_s(u)\, \frac{\partial v}{\partial x_s} \,\mathrm{d}x + \sum_{i \in I} \int_{K_i} \sum_{s=1}^{d} R_s(u, \nabla u))\, \frac{\partial v}{\partial x_s} \,\mathrm{d}x\\
&- \sum_{i \in I} \sum_{\substack{j \in S(i)\\ j<i}} \int_{\Gamma_{ij}} \sum_{s=1}^{d} \langle R_s(u, \nabla u)\rangle\,(n_{ij})_s\,[v] \,\mathrm{d}S\\
&+ \sum_{i \in I} \sum_{\substack{j \in S(i)\\ j<i}} \int_{\Gamma_{ij}} \sum_{s=1}^{d} \langle R_s(u, \nabla v)\rangle\,(n_{ij})_s\,[u] \,\mathrm{d}S
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i\in I}\sum_{j\in\gamma(i)}\int_{\Gamma_{ij}}\sum_{s=1}^{d} R_s(u,\nabla u)\,(n_{ij})_s\, v\,\mathrm{d}S \\
&+\sum_{i\in I}\sum_{j\in\gamma(i)}\int_{\Gamma_{ij}}\sum_{s=1}^{d} R_s(u,\nabla v)\,(n_{ij})_s\, u\,\mathrm{d}S \\
&+\sum_{i\in I}\sum_{j\in s(i)}\int_{\Gamma_{ij}}\sigma[u]\,[v]\,\mathrm{d}S+\sum_{i\in I}\sum_{j\in\gamma(i)}\int_{\Gamma_{ij}}\sigma\, u\, v\,\mathrm{d}S \\
=&\int_{\Omega} g\, v\,\mathrm{d}x+\sum_{i\in I}\sum_{j\in\gamma(i)}\int_{\Gamma_{ij}}\sum_{s=1}^{d} R_s(u,\nabla v)\,(n_{ij})_s\, u_D\,\mathrm{d}S \\
&+\sum_{i\in I}\sum_{j\in\gamma(i)}\int_{\Gamma_{ij}}\sigma\, u_D\, v\,\mathrm{d}S.
\end{aligned}
$$

Here σ is a weight function defined on each Γ_{ij} by $\sigma|_{\Gamma_{ij}} = \varepsilon/\mathrm{d}(\Gamma_{ij})$, where $\varepsilon \approx R_s$.

Finally, we approximate the flux $\int_{\Gamma_{ij}} f_s(u)\, n_s\, v\,\mathrm{d}S$ with the aid of the *numerical flux* $H = H(\alpha, \beta, \boldsymbol{n})$:

$$
\int_{\Gamma_{ij}}\sum_{s=1}^{d} f_s(u)\, n_s\, v|_{\Gamma_{ij}}\,\mathrm{d}S \approx \int_{\Gamma_{ij}} H\left(u|_{\Gamma_{ij}}, u|_{\Gamma_{ji}}, \boldsymbol{n}_{ij}\right)\, v|_{\Gamma_{ij}}\,\mathrm{d}S \tag{2.8}
$$

and define the approximate convective form

$$
\tilde{b}_h(u_h, v_h) = \sum_{i\in I}\sum_{j\in S(i)}\int_{\Gamma_{ij}} H\left(u_h|_{\Gamma_{ij}}, u_h|_{\Gamma_{ji}}, \boldsymbol{n}_{ij}\right)\, v_h|_{\Gamma_{ij}}\,\mathrm{d}S,\ \ u_h,\ v_h\in S_h. \tag{2.9}
$$

If $\Gamma_{ij} \subset \partial\Omega$, we use the Dirichlet boundary condition (2.2) in order to specify $u|_{\Gamma_{ji}}$.

If the diffusion terms R_s vanish, it is necessary to use boundary conditions suitable for hyperbolic equations. We do not treat this topic here. Some numerical experiments will be carried out for nonlinear conservation laws equipped with periodic boundary conditions. In this case we easily develop the numerical scheme in a similar way as here.

Now, for $u_h, v_h \in S_h$ we set

$$
a_h(u_h, v_h) = \sum_{i\in I}\int_{K_i}\sum_{s=1}^{d} R_s(u_h, \nabla u_h))\,\frac{\partial v_h}{\partial x_s}\,\mathrm{d}x \tag{2.10}
$$

$$
\begin{aligned}
&-\sum_{i\in I}\sum_{\substack{j\in S(i)\\ j<i}}\int_{\Gamma_{ij}}\sum_{s=1}^{d}\langle R_s(u_h,\nabla u_h)\rangle\,(n_{ij})_s\,[v_h]\,\mathrm{d}S\\
&+\sum_{i\in I}\sum_{\substack{j\in S(i)\\ j<i}}\int_{\Gamma_{ij}}\sum_{s=1}^{d}\langle R_s(u_h,\nabla v_h)\rangle\,(n_{ij})_s\,[u_h]\,\mathrm{d}S\\
&-\sum_{i\in I}\sum_{j\in\gamma(i)}\int_{\Gamma_{ij}}\sum_{s=1}^{d}R_s(u_h,\nabla u_h)\,(n_{ij})_s\,v_h\,\mathrm{d}S\\
&+\sum_{i\in I}\sum_{j\in\gamma(i)}\int_{\Gamma_{ij}}\sum_{s=1}^{d}R_s(u_h,\nabla v_h)\,(n_{ij})_s\,u_h\,\mathrm{d}S
\end{aligned}
$$

$$
J_h(u_h,v_h)=\sum_{i\in I}\sum_{j\in s(i)}\int_{\Gamma_{ij}}\sigma[u_h]\,[v_h]\,\mathrm{d}S+\sum_{i\in I}\sum_{j\in\gamma(i)}\int_{\Gamma_{ij}}\sigma\,u_h\,v_h\,\mathrm{d}S \tag{2.11}
$$

$$
\begin{aligned}
\ell_h(v_h)\,(t)=&\int_\Omega g(t)\,v_h\,\mathrm{d}x+\sum_{i\in I}\sum_{j\in\gamma(i)}\int_{\Gamma_{ij}}\sum_{s=1}^{d}R_s(u,\nabla v_h)\,(n_{ij})_s\,u_D(t)\,\mathrm{d}S\\
&+\sum_{i\in I}\sum_{j\in\gamma(i)}\int_{\Gamma_{ij}}\sigma\,u_D(t)\,v_h\,\mathrm{d}S,
\end{aligned}\tag{2.12}
$$

$$
(\alpha,\beta)=\int_\Omega \alpha\beta\,\mathrm{d}x, \tag{2.13}
$$

and formulate the *discrete problem*: We define an approximate solution as a function u_h satisfying the conditions

$$
\text{a) } u_h\in C^1([0,T],S_h), \tag{2.14}
$$

$$
\text{b) } \left(\frac{\partial u_h(t)}{\partial t},v_h\right)+\tilde{b}_h(u_h(t),v_h)+a_h(u_h(t),v_h)+J_h(u_h(t),v_h)=\ell_h(v_h)(t)
$$
$$
\forall\, v_h\in S_h\quad \forall\, t\in(0,T),
$$

$$
\text{c) } u_h(0)=u_h^0,
$$

where u_h^0 is an S_h-approximation of u^0 (e. g., L^2-projection).

We have carried out the semidiscretization in space (called the method of lines) leading to a system of ordinary differential equations. In practical computations, the full discretization is carried out. We can use, e. g. the explicit Euler or Runge–Kutta schemes. Semiimplicit or fully implicit time discretization lead to large nonlinear algebraic systems which must be solved iteratively. Moreover, the integrals are evaluated with the aid of quadrature formulae. Let us note that the form a_h is a variant of the DG FE approximation of the diffusion terms proposed in [11].

We assume that the numerical flux has the following properties:

1) $H(\alpha, \beta, \boldsymbol{n})$ is defined on $I\!R^2 \times B_1$, where $B_1 = \{\boldsymbol{n} \in I\!R^d; |\boldsymbol{n}| = 1\}$ is locally Lipschitz-continuous,
2) $H(\alpha, \beta, \boldsymbol{n})$ is *consistent*:

$$H(\alpha, \alpha, \boldsymbol{n}) = \sum_{s=1}^{d} f_s(\alpha)\, n_s, \quad \alpha \in I\!R,\ \boldsymbol{n} = (n_1, \ldots, n_d) \in B_1, \tag{2.15}$$

3) $H(\alpha, \beta, \boldsymbol{n})$ is *conservative*:

$$H(\alpha, \beta, \boldsymbol{n}) = -H(\beta, \alpha, -\boldsymbol{n}), \quad \alpha,\, \beta \in I\!R,\ \boldsymbol{n} \in B_1. \tag{2.16}$$

The described process yields a higher order scheme using only one (in general unstructured) mesh. Its disadvantage are spurious oscillations in approximate solutions which appear in areas with steep gradients in the case of small diffusion terms. We shall discuss two methods how to cure this disease.

3 Finite volume discontinuous Galerkin method

3.1 Modification of the convective form

The first modification of scheme (2.14) is an analogy of the FV method. Therefore, we call it the finite volume discontinuous Galerkin method (FV DG).

In (2.6) we put $p = 1$, i. e., we use piecewise linear elements, and introduce a modification b_h of the form $\tilde{b}_h$ defined in the following way. By π_0 we denote the L^2-projection of functions $v \in L^2(\Omega)$ on the space $S_h^{0,-1}(\Omega, T_h)$ of piecewise constant functions: $\pi_0 v|_K = \int_K v \,\mathrm{d}x / |K|$ for $K \in T_h$. Then, instead of $\tilde{b}_h$ we use in (2.14), b) the approximate convective form

$$\begin{aligned} b_h(u_h, v_h) = \sum_{i \in I} \pi_0 v_h|_{K_i} \Big\{ & \sum_{j \in s(i)} H\left(\pi_0 u_h|_{K_i}, \pi_0 u_h|_{K_j}, \boldsymbol{n}_{ij}\right) |\Gamma_{ij}| \\ & + \sum_{j \in \gamma(i)} H\left(\pi_0 u_h|_{K_i}, \pi_0 u_h|_{K_i}, \boldsymbol{n}_{ij}\right) |\Gamma_{ij}| \Big\}. \end{aligned} \tag{3.1}$$

In this case, the boundary values are realized in the form b_h by extrapolation. A detailed theoretical analysis of error estimates for this method was carried out in [4]. The use of the operator π_0 in the definition of the approximate convective form b_h causes that the method is of the first order, but in contrast to the combined FV - FE methods, only one mesh is used.

3.2 Example

In order to show that the use of the FV DG method strongly suppresses spurious oscillations, we consider the numerical solution of the viscous Burgers equation

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x_1} + u\,\frac{\partial u}{\partial x_2} = \varepsilon\,\Delta\, u \quad \text{in } Q_T = \Omega \times (0, T), \tag{3.2}$$

where $\Omega = (-1, 1)^2$, $T = 1$, $\varepsilon = 0.002$. Equation (3.2) is equipped with periodic boundary conditions and the initial condition

$$u^0(x_1, x_2) = 0.25x + 0.5\sin(\pi(x_1 + x_2)), \quad (x_1, x_2) \in \Omega. \tag{3.3}$$

Let us note that the periodic boundary conditions are easily treated using a simple modification of schemes described in Par. 2.2 and 3.1.

It is known that the inviscid Burgers equation ($\varepsilon = 0$) equipped with the initial condition (3.3) and periodic boundary conditions has a solution discontinuous for $t \geq 0.3$. The viscous case, when $\varepsilon = 0.002$, has a similar solution (the discontinuity is slightly smeared due to the diffusion). In our numerical experiments we use the numerical flux in the form

$$H(u_1, u_2, \boldsymbol{n}) = \begin{cases} \sum_{s=1}^{2} f_s(u_1) n_s, & \text{if } A > 0 \\ \sum_{s=1}^{2} f_s(u_2) n_s, & \text{if } A \leq 0 \end{cases}, \tag{3.4}$$

where $f_s(u) = u^2/2, \sum_{s=1}^{2} f_s'(\bar{u}) n_s,\ \ \bar{u} = \frac{1}{2}(u_1 + u_2)$.

In Figure 1 we see the numerical solution at $t = 0.45$ obtained with the aid of scheme (2.14), where the forward Euler time discretization is carried out with a rather small time step $\tau = 10^{-4}$, in order to guarantee the stability and sufficiently accurate resolution in time. (For more details see Par. 4.2.) The solution suffers from spurious oscillations. The use of the scheme with the modification b_h of the convective form leads to a conspicuous improvement as is shown in Figure 2.

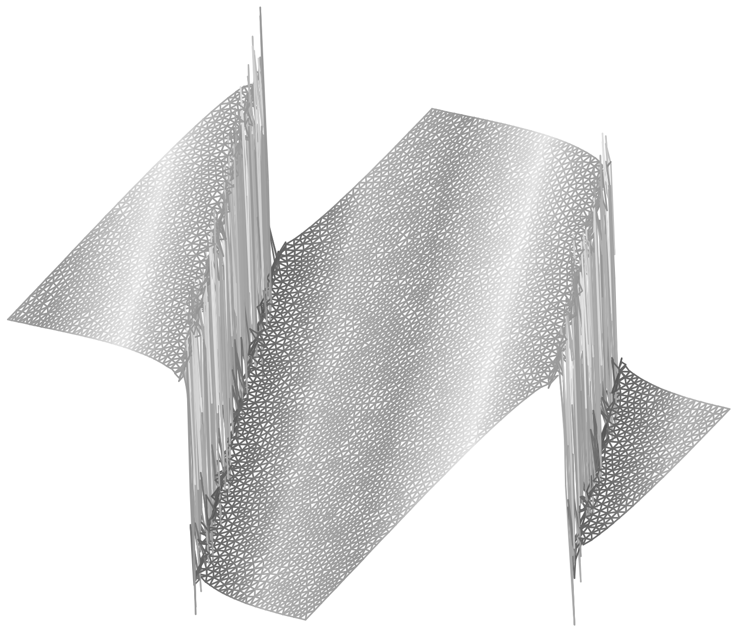

Fig. 1. The numerical solution of computed by DG FE method plotted at $t = 0.45$

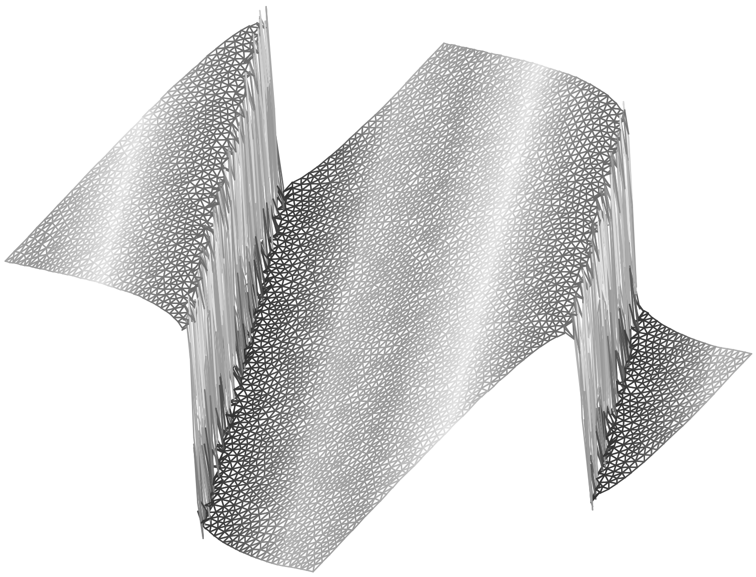

Fig. 2. The numerical solution of computed by FV DG method plotted at $t = 0.45$

4 Second-order DG FE method with a new limiting

4.1 General framework of higher-order schemes for conservation laws

In this section we develop a numerical technique avoiding disadvantages of both schemes discussed above: spurious oscillations in solutions obtained by the pure DG FE method (2.14) and a low order (=first order) of the FV DG scheme from Par. 3.1.

All methods for avoiding the Gibbs effect in higher-order schemes are based on a limiting procedure applied in the whole computational domain which should however suppress the accuracy to the first order in the vicinity of steep gradients or discontinuities in the solution. There exists a rigorous method using the concept of a flux limiter for finite difference or finite volume schemes applied to scalar conservation laws with one space variable over uniform meshes. (See, e. g., [12]). However, as numerical experiments show, the heuristic extension of this approach to FV schemes for systems, problems with more space variables and/or nonuniform unstructured meshes can lead to an undesired decrease of accuracy everywhere. The application of this methodology in the framework of the DG FE method seems to be inadequate.

Here we present a different type of limiting which definitely keeps the second order of accuracy in the region of regularity of the solution, because the original higher order scheme remains unchanged in this region and is modified to a first order scheme only in the vicinity of steep gradients or discontinuities.

4.2 Fully discrete scheme with adaptive limiting of order

Let us return to scheme (2.14), where *we suppose that* $d = 2$, T_h is formed by triangles and $p = 1$. We carry out the discretization in time by the forward Euler method. To this end, we consider a partition $0 = t_0 < t_1 < t_2 < \dots$ of the time interval $(0, T)$ and set $\tau_k = t_{k+1} - t_k$.

The *fully discrete problem* reads: for each $k \geq 0$ find u_h^{k+1} such that

$$\text{a) } u_h^{k+1} \in S_h = S_h^{1,-1}(\Omega, T_h), \tag{4.1}$$

$$\begin{aligned}\text{b) } (u_h^{k+1}, v_h) = {} & (u_h^k, v_h) - \tau_k\, a_h(u_h^k, v_h) \\ & - \tau_k\, \tilde{b}_h(u_h^k, v_h) - \tau_k\, J_h(u_h^k, v_h) + \tau_k\, \ell_h(v_h)\,(t_k) \quad \forall\, v_h \in S_h.\end{aligned}$$

The formal order of accuracy is $O(\tau + h^2)$. The stability of the scheme requires that $\tau_k \approx O(h^2/\varepsilon + h)$, if $\varepsilon \approx R_s, s = 1, 2$.

Let us go back to problem (3.2) – (3.3) with periodic boundary conditions. Our first task is to identify areas with very steep gradients (leading to discontinuities for $\varepsilon \to 0+$). Numerical experiments show (see Section 4.3) that the interelement jumps in the approximate solution are of the order $O(1)$ on discontinuities, whereas $O(h^2)$ in the areas, where the solution is regular. On general unstructured grids it appears to be suitable to measure the magnitude of interelement jumps in the integral form by

$$\int_{\partial K_i} [u_h^k]^2 \, \mathrm{d}S, \quad K_i \in T_h. \tag{4.2}$$

In the areas, where the solution is regular, we have

$$g^1(i) = \int_{\partial K_i} [u_h^k]^2 \, \mathrm{d}S/h^5 \approx \int_{\partial K_i} (O(h^2))^2 \, \mathrm{d}S/h^5 \approx O(1), \tag{4.3}$$

whereas

$$g^2(i) = \int_{\partial K_i} [u_h^k]^2 \, \mathrm{d}S/h \approx \int_{\partial K_i} (O(1))^2 \, \mathrm{d}S/h \approx O(1) \tag{4.4}$$

for very steep gradients (discontinuities). These results lead us to the idea that the switch between the higher order scheme and its first order modification should be tested with the aid of the indicator

$$g(i) = \int_{\partial K_i} [u_h^k]^2 \, \mathrm{d}S/h^\alpha, \quad K_i \in T_h, \tag{4.5}$$

where $\alpha \in (1, 5)$. We choose, e. g., $\alpha = 5/2$. However, in general, using unstructured grids, it is suitable to define the indicator $g(i)$ in terms of h_{K_i} and $|K_i|$. For $\alpha = 5/2$ we set

$$g(i) = \int_{\partial K_i} [u_h^k]^2 \, \mathrm{d}S/(h_{K_i}|K_i|^{3/4}), \quad K_i \in T_h. \tag{4.6}$$

Now we define an *adaptive strategy* for an *automatic limiting* of the order of accuracy of the scheme (4.1):

$$\text{a) } u_h^{k+1} \in S_h = S_h^{1,-1}(\Omega, T_h), \tag{4.7}$$

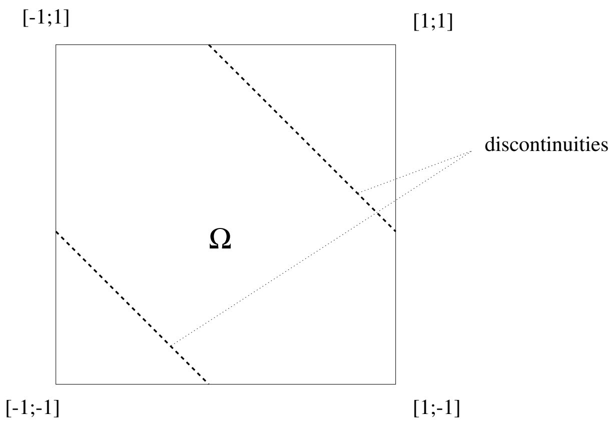

Fig. 3. Position of dicontinuities in the computational domain

$$\text{b) } (u_h^{k+1}, v_h) = (\tilde{u}_h^k, v_h) - \tau_k\, a_h(u_h^k, v_h)$$
$$-\tau_k\, \tilde{b}_h(\tilde{u}_h^k, v_h) - \tau_k J_h(u_h^k, v_h) + \tau_k\, \ell_h(v_h)\,(t_k) \quad \forall\, v_h \in S_h$$

where $\tilde{u}_h^k$ is the modification of u_h^k defined with the aid of our limiting strategy in the following way:

$$\text{a) Set} \tilde{u}_h^k|_{K_i} := u_h^k|_{K_i} \quad \forall i \in I. \tag{4.8}$$

b) If $g(i) > 1$ for some $i \in I$, then set $\tilde{u}_h^k|_{K_i} := \pi_0\, u_h^k|_{K_i}$.

This means that in (4.8) the limiting of the order of the scheme is applied on the elements lying on the discontinuity. In other areas the second order of accuracy is preserved. (The extension to the case $d = 3$ is straighforward.)

4.3 Verification of the adaptive limiting

At first we show that the interelement jumps in the approximate solution are of the order $O(1)$ on discontinuities, whereas $O(h^2)$ in the regions, where the solution is regular. To demonstrate it, we consider the inviscid case ($\varepsilon = 0$) of (3.2) – (3.3). This is the limit case of the considered problem with the worst scenario with respect to the behaviour of the solution.

We denote by $\overline{u}$ the exact solution of (3.2) – (3.3) with $\varepsilon = 0$ at time $t = 0.45$. It was mentioned above that the function $\overline{u}$ contains two line discontinuities expressed as $x_1 + x_2 = -0.78, x_1 + x_2 = 1.22, \quad (x_1, x_2) \in \Omega$, see Figure 3. We denote by Ω^d a union of triangles $K_i \in T_h$ lying in the vicinity of these lines. Further we put $\Omega^c \equiv \Omega \setminus \Omega^d$.

We denote by $\overline{u}_h$ the approximation of $\overline{u}$ computed by the numerical scheme

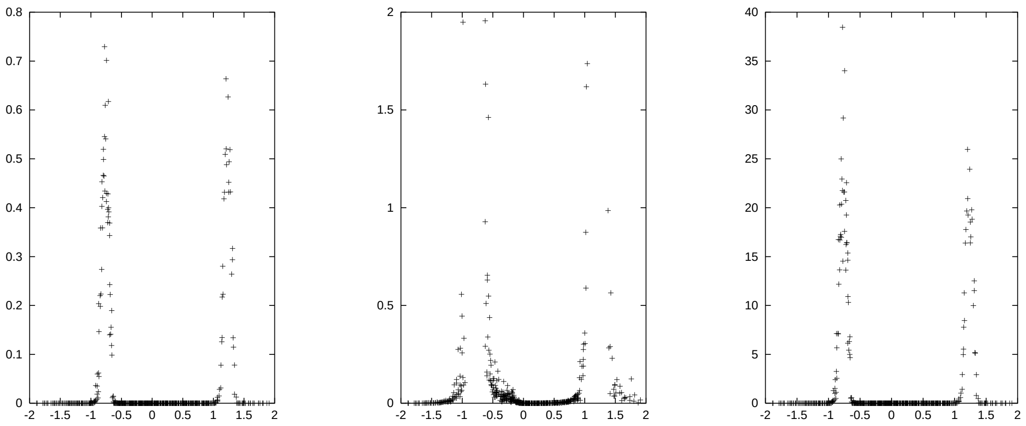

Fig. 4. Dependence of the quantities $g^1(i)$ (left), $g^2(i)$ (center) and $g(i)$ (right) on the position $x_1 + x_2$ (vertical axes) for triangulation mesh.500

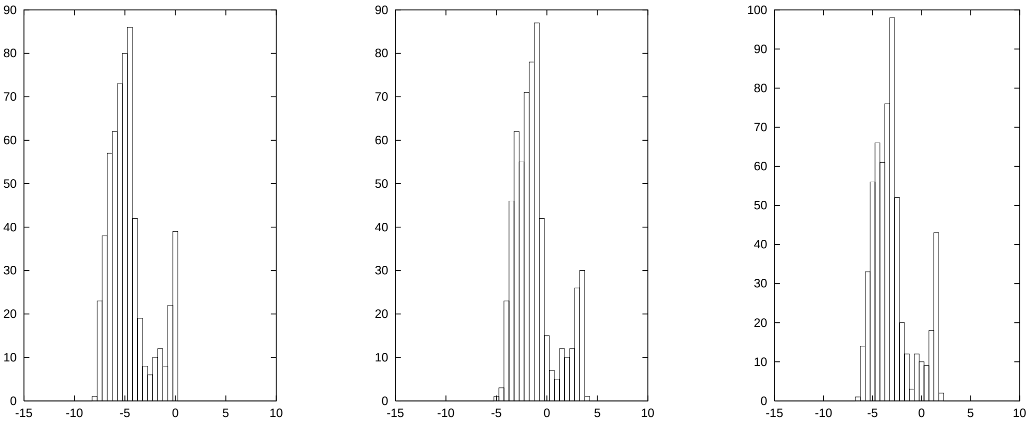

Fig. 5. Distribution of triangles: horizontal axis (in logarithmic scale) expresses the order of magnitude of $g^1(i)$ (left), $g^2(i)$ (center) and $g(i)$ (right), vertical axis determines the number of triangles, where g^1, g^2 or g have the corresponding order, for triangulation mesh.500

(4.7) – (4.8) on a mesh T_h. We investigate the magnitude of interelement jumps of $\overline{u}_h$ with the aid of indicators $g^1(i)$, $g^2(i)$ and $g(i)$ given for $K_i \in T_h$ by (4.3), (4.4) and (4.6), respectively. Figure 4 shows the distribution of magnitudes of g^1, g^2 and g computed on the triangulation mesh.500 from Figure 6. There is a point of this graph assigned to each $K_i \in T_h$. The horizontal axis represents the value of $x_1^i + x_2^i$, where (x_1^i, x_2^i) are the coordinates of the barycentre of the element K_i and the vertical axis corresponds to the values $g^1(i)$, $g^2(i)$ and $g(i)$ for $K_i \in T_h$. We see the increments of values of g^1, g^2 and g near the discontinuities. For $K_i \in \Omega^d$, the values of g^2 are from the interval $(0.1, 1)$, which confirms relation (4.4). On the other hand, for $K_i \in \Omega^c$ the values of g^1 satisfy the condition $g^1(i) \leq 1$ and this confirms (4.3). From the right graph in Figure 4 we can see that the limiting given by the condition $g(i) > 1$ indicates the region of steep gradient (discontinuity). Figure 5 shows the dis-

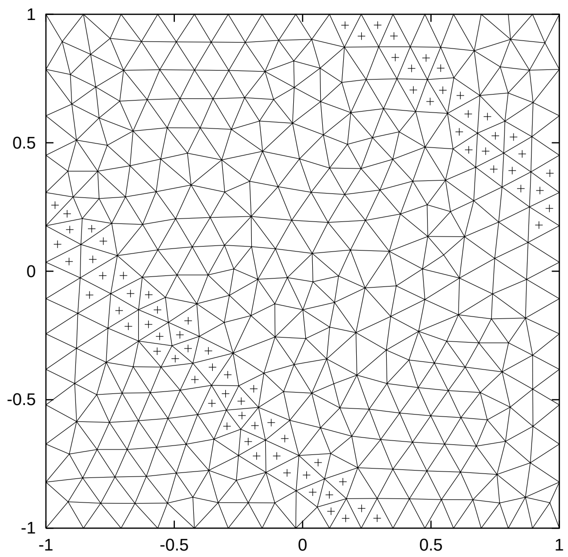

Fig. 6. Triangulation mesh.500 with marked triangles with $g(i) > 1$

tribution of triangles. The horizontal axis (in logarithmic scale) expresses the order of magnitude of the values $g^1(i)$, $g^2(i)$ or $g(i)$ and the vertical axis gives the number of triangles, where g^1, g^2 or g have the corresponding order. The Gaussian distribution of elements in Ω^c as well as Ω^d is observed. The graphs with g^1 and g^2 again verify the relations (4.3) and (4.4), respectively.

Further, Figure 6 shows the marked triangles $K_i \in T_h$ with $g(i) > 1$ where the limiting (4.8) should be applied. This proves the correctness of the choice $\alpha = 5/2$ in (4.5). Moreover, Figure 7 shows the marked triangles ($g(i) > 1$) for another unstructured anisotropic grids, which verifies the form of the denominator in (4.6) (i. e. $h_{K_i}|K_i|^{3/4}$).

The efficiency of the limiting is viewed in Figure 8, where we present the numerical solution of the inviscid Burgers equation. Figures show the graphs of the approximate solutions along the diagonal connecting the points $(-1,-1)$ and $(1,1)$. The centre figure is the exact solution and the left and right one are the numerical solutions computed without and with limiting, respectively. The elimination of spurious oscilations and satisfactory correspondance with the exact solution are observed.

The last Figure 9 shows the numerical solution of (3.2)–(3.3) from Par. 3.2 computated by the scheme (4.7) - (4.8).

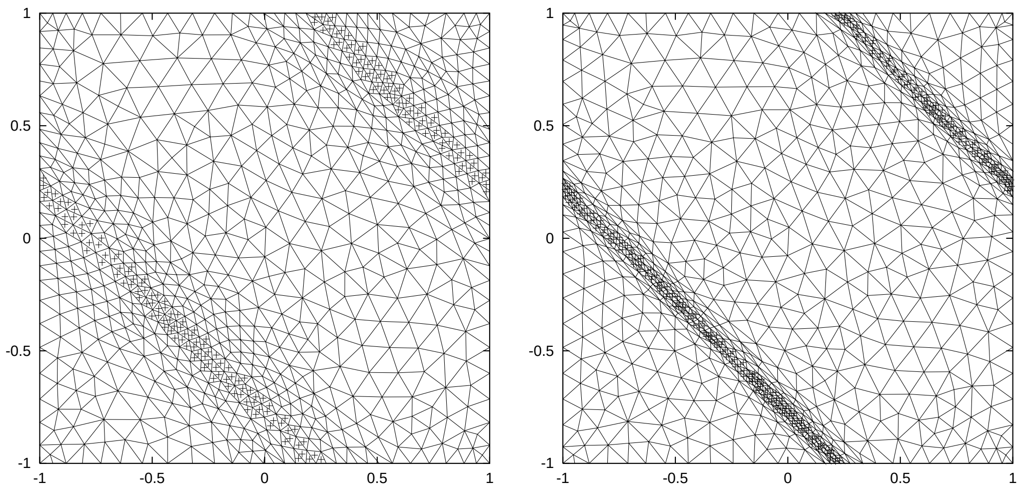

Fig. 7. Examples of anisotropic triangulations with marked triangles satisfying the condition $g(i) > 1$

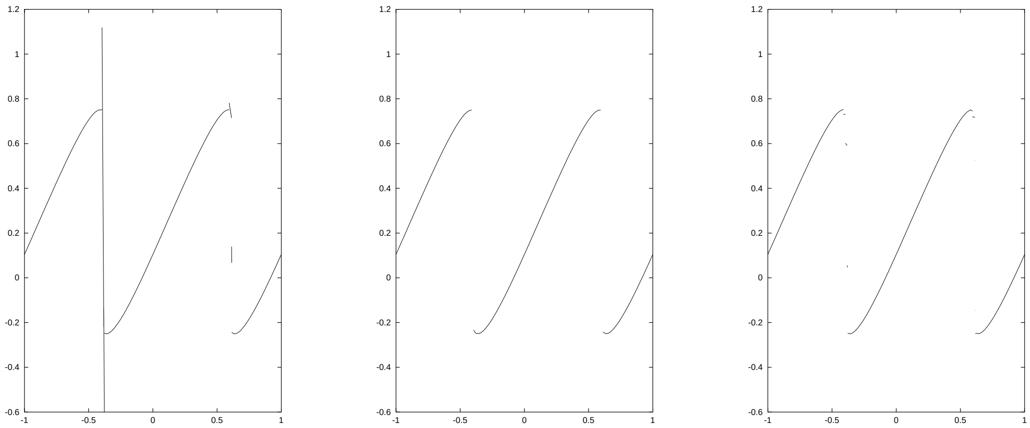

Fig. 8. Inviscid Burgers equation: graphs along the diagonal, DG FE(left), exact solution (centre), DG FE with limiting (right)

5 Conclusion

¿From the above analysis and numerical experiments we see that the discontinuous Galerkin finite element method and its modifications represent very challenging techniques for the solution of nonlinear conservation laws and convection diffusion problems. The most important achievement is the construction of adaptive limiting of the order of accuracy of the method avoiding spurious oscillations and preserving the higher order numerical scheme unchanged in regions, where the solution is regular. Of course, there is a number of unsolved questions and problems, as, e.g., analysis of error estimates of this method, theoretical justification of the proposed adaptive limiting and the extension of the presented results to compressible Euler and Navier-Stokes

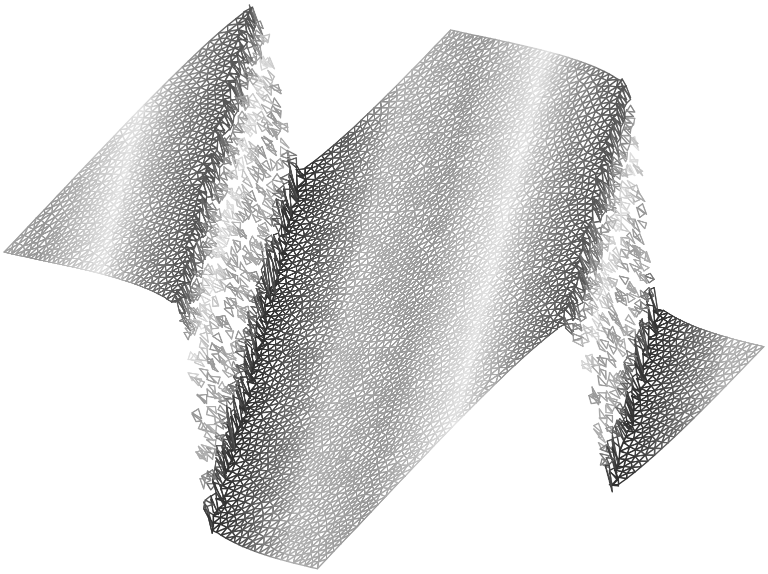

Fig. 9. Numerical solution of problem from Par 3.2 computed by DG FE method with limiting, plotted at $t = 0.45$

equations.

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