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Theory and Applications of Numerical Schemes for Nonlinear Convection–Diffusion Problems and Compressible Navier–Stokes Equations

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Introduction

Numerical simulation of viscous compressible high-speed flow is one of the most difficult areas of Computational Fluid Dynamics because of the facts that convection effects dominate those of diffusion, that shock waves and boundary layers are present and that there is interaction between shocks and boundary layers. In addition there is a significant lack of theory for problems of this type. Our goal is to develop a robust theoretically based numerical method for the solution of viscous compressible flow applied on unstructured meshes.

Since the viscosity and heat conduction coefficients of gases are small, the viscous dissipative terms are often considered as perturbations of the inviscid Euler system. This leads us to the conclusion that any efficient method for the solution of viscous gas flow should be based on a numerical method which is effective for inviscid flow. The most popular technique for the solution of inviscid compressible flow at present is the finite volume (FV) method (cf., e. g., [Fei93, Par. 7.3]).

Some authors ([FHKV95], [HHK92], [VH92]) carry out the FV discretization for the complete viscous system. However, the character of second order viscous terms also offers the possibility of discretization by the finite element (FE) method. In this way we arrive at a combined finite volume – finite element method.

We have developed several combined FV – FE procedures for the numerical solution of nonlinear convection – diffusion problems and compressible viscous flow based on a general class of cell centred flux vector splitting FV schemes for the discretization of inviscid terms together with the discretization of viscous terms by the FE method over a triangular grid. Substantial attention has been paid to the theoretical analysis of the resulting combined FV – FE schemes applied to a simplified scalar nonlinear convection – diffusion conservation law equation. In the present paper we discuss the convergence and error estimates of the method and present some applications to the solution of viscous compressible transonic flow. The comparison of numerical results with experimental data shows the applicability and robustness of the method.

1.1 Continuous Model Problem

We will denote by $I\!R^n$ the n-dimensional Euclidean space equipped with the norm $|\cdot|$. By x_1, x_2 and t we will denote the Cartesian coordinates of points $x \in I\!R^2$ and time, respectively. Let $\Omega \subset I\!R^2$ be a bounded polygonal domain. In the space-time cylinder $Q_T = \Omega \times (0,T)$ $(0 < T < \infty)$ we consider the following *initial-boundary value problem*: Find $u : \overline{Q}_T \to I\!R,\ u = u(x,t),\ x \in \overline{\Omega},\ t \in [0,T]$, such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} - \nu\,\Delta\,u = g \quad \text{in } Q_T, \tag{1.1}$$

$$u|_{\partial\Omega \times (0,T)} = 0, \tag{1.2}$$

$$u(x,0) = u^0(x), \quad x \in \Omega, \tag{1.3}$$

where $\nu > 0$ is a given constant and $f_s : I\!R \to I\!R,\ s = 1,2,\ g : Q_T \to I\!R,\ u^0 : \Omega \to I\!R$ are given functions.

We assume that the following *assumptions on data* are satisfied: $f_s \in C^1(I\!R), g \in C([0,T]; W^{1,q}(\Omega))$ for some $q > 2$ and $u^0 \in W_0^{1,p}(\Omega)$ for some $p > 2$. (For the definition of these function spaces, see, e.g., [KJF77].)

Let us set $V = H_0^1(\Omega)$,

$$(u,v) \equiv \int_\Omega uv\,\mathrm{d}x, \quad u,\, v \in L^2(\Omega), \tag{1.4}$$

$$a(u,v) \equiv \nu \int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x, \quad u,\, v \in H^1(\Omega), \tag{1.5}$$

$$b(\varphi,v) \equiv -\int_\Omega \sum_{s=1}^{2} f_s(\varphi)\,\frac{\partial v}{\partial x_s}\,\mathrm{d}x \quad \text{for } \varphi \in L^\infty(\Omega), \quad v \in V. \tag{1.6}$$

We say that a function u is a *weak solution* of problem (1.1) – (1.3), if it satisfies the following conditions

a) $u \in L^2(0,T;V) \cap L^\infty(Q_T)$,

b) $\frac{\mathrm{d}}{\mathrm{d}t}(u(t),v) + b(u(t),v) + a(u(t),v) = (g(t),v) \quad \forall\, v \in V,$ (1.7) in the sense of distributions on $(0,T)$,

c) $u(0) = u^0$.

This problem has a unique solution.

Equation (1.1) is a strongly simplified prototype of equations describing viscous gas flow. It also appears in such other areas as hydrology and environmental protection. Convection-diffusion problems have been extensively investigated, mainly in a linear case. Let us mention the papers [Joh81], [AFRS89], [AR93], [Zho95], [ZR96], monographs [Ike83], [Mor96], [RST96] and the references therein.

In what follows we will be concerned with numerical methods for the solution of problem (1.1) – (1.3) based on the combination of the finite volume method with the finite element method. The numerical schemes so developed will be applied to solve problems of viscous gas flow.

1.2 Discrete Problem

The discretization of problem (1.7) is carried out with the aid of a generally unstructured triangular mesh. By $\mathcal{T}_h$ we denote a triangulation of Ω with standard properties ([Cia79]). The triangulation $\mathcal{T}_h$ is called a *basic mesh*. Let $\mathcal{P}_h = \{P_i; i \in J\}$ be the set of all vertices of all $T \in \mathcal{T}_h$. (J is a suitable index set.)

Now, over the basic mesh $\mathcal{T}_h$ we construct the *dual mesh* $\mathcal{D}_h = \{D_i; i \in J\}$ covering $\overline{\Omega}$. The *dual finite volume* D_i associated with a vertex $P_i \in \mathcal{P}_h$ is a closed polygon obtained in the following way (see Fig. 1): We join the centre of gravity of every triangle $T \in \mathcal{T}_h$ that contains the vertex P_i with the centre of every side of T containing P_i. If $P_i \in \mathcal{P}_h \cap \partial\Omega$, then we complete the contour so obtained with the additional straight segments joining P_i to the centres of boundary sides that contain P_i. In this way we get the boundary ∂D_i of the finite volume D_i.

Two different finite volumes D_i and D_j are called *neighbours*, if their boundaries contain a common straight segment. In this case we write

$$\Gamma_{ij} = \bigcup_{\alpha=1}^{\beta_{ij}} \Gamma_{ij}^{\alpha} = \partial D_i \cap \partial D_j = \Gamma_{ji}, \tag{1.8}$$

where $\Gamma_{ij}^{\alpha} = \Gamma_{ji}^{\alpha}$ are straight segments. We have $\beta_{ij} = 2$ for D_i or $D_j \subset \Omega$. If both D_i and D_j are adjacent to $\partial\Omega$, then $\beta_{ij} = 1$.

For $i \in J$, let $s(i) = \{j \in J; D_j$ is a neighbour of $D_i\}$. If $P_i \in \mathcal{P}_h \cap \partial\Omega$, then we denote by $\Gamma_{i,-1}^{\alpha}$, $\alpha = 1, 2 =: \beta_{i,-1}$, the segments that form $\partial D_i \cap \partial\Omega$. In this case we set $S(i) = s(i) \cup \{-1\}$, otherwise (for $P_i \in \mathcal{P}_h \cap \Omega$) we put $S(i) = s(i)$. Obviously, for every $D_i \in \mathcal{D}_h$ we have

$$\partial D_i = \bigcup_{j \in S(i)} \Gamma_{ij} = \bigcup_{j \in S(i)} \bigcup_{\alpha=1}^{\beta_{ij}} \Gamma_{ij}^{\alpha}. \tag{1.9}$$

(Dual finite volumes described above were applied to the numerical solution of inviscid gas flow in [Vij86], [ADF+89], [SS94], [FLWW95], [FFL95].)

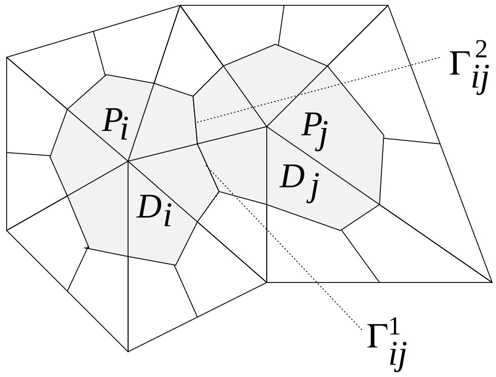

Figure 1 Dual finite volumes over a triangular mesh

Further, we introduce the following *notation*: $|D_i|$ = area of $D_i \in \mathcal{D}_h$, $|T|$ = area of $T \in \mathcal{T}_h$, $\mathbf{n}_{ij}^\alpha = (n_{1ij}^\alpha, n_{2ij}^\alpha)$ = unit outward normal to ∂D_i on Γ_{ij}^α, ℓ_{ij}^α = length of Γ_{ij}^α, $|\partial D_i|$ = the length of ∂D_i. Moreover, we consider a partition $0 = t_0 < t_1 < \ldots$ of the time interval $[0, T]$ and set $\tau_k = t_{k+1} - t_k$ for $k = 0, 1, \ldots$.

Let us define the following spaces:

$$\begin{aligned} X_h &= \{v_h \in C(\overline{\Omega}); v_h|_T \text{ is linear for each } T \in \mathcal{T}_h\} \subset H^1(\Omega) \qquad (1.10)\\ &\quad \text{(the space of linear conforming finite elements)},\\ V_h &= \{v_h \in X_h; v_h = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

By r_h we denote the operator of the Lagrange X_h- interpolation. Further, by L_h we denote the so-called *lumping operator*:

$$v \in C(\overline{\Omega}) \to L_h v : L_h v|_{D_i} = v(P_i), \ i \in J. \qquad (1.11)$$

In order to derive the discrete form of problem (1.7), we put

$$(u, v)_h = \int_\Omega r_h(uv)\,\mathrm{d}x, \qquad u, v \in C(\overline{\Omega}), \qquad (1.12)$$

and for $u, v \in V_h$ we write

$$\begin{aligned} b(u, v) &= \int_\Omega \sum_{s=1}^2 \frac{\partial f_s(u)}{\partial x_s}\, v\,\mathrm{d}x \approx \int_\Omega \sum_{s=1}^2 \frac{\partial f_s(u)}{\partial x_s}\, L_h\, v\,\mathrm{d}x = \qquad (1.13)\\ &= \sum_{i\in J} v(P_i) \int_{D_i} \sum_{s=1}^2 \frac{\partial f_s(u)}{\partial x_s}\,\mathrm{d}x =\\ &= \sum_{i\in J} v(P_i) \int_{\partial D_i} \sum_{s=1}^2 f_s(u)\, n_s\,\mathrm{d}S \ = \end{aligned}$$

$$= \sum_{i\in J} v(P_i) \sum_{j\in s(i)} \sum_{\alpha=1}^{\beta_{ij}} \int_{\Gamma_{ij}^\alpha} \sum_{s=1}^{2} f_s(u)\, n_s \,\mathrm{d}S \;\approx$$

$$\approx\;\; b_h(u,v) = \sum_{i\in J} v(P_i) \sum_{j\in s(i)} \sum_{\alpha=1}^{\beta_{ij}} H(u(P_i), u(P_j), \mathbf{n}_{ij}^\alpha)\, \ell_{ij}^\alpha .$$

Here $\mathbf{n} = (n_1, n_2)$ denotes the unit outward normal to ∂D_i. The function H defined on $I\!R^2 \times \mathcal{S}$, where $\mathcal{S} = \{\mathbf{n} \in I\!R^2; |\mathbf{n}| = 1\}$, is called a *numerical flux*, and has the following properties:

a) $H = H(y, z, \mathbf{n})$ is *Lipschitz-continuous* with respect to $y,\, z \in [-M, M]$ with a constant $c(M)$.

b) H is *consistent*: $H(u, u, \mathbf{n}) = \sum_{s=1}^{2} f_s(u)\, n_s \quad \forall\, u \in I\!R,\ \forall\, \mathbf{n} = (n_1, n_2), |\mathbf{n}| = 1.$

c) H is *conservative*: $H(y, z, \mathbf{n}) = -H(z, y, -\mathbf{n}) \quad \forall\, y,\, z \in I\!R, \forall\, \mathbf{n}$ with $|\mathbf{n}| = 1$.

d) H is *monotone*, i. e., for a given fixed number $M > 0$ the function $H(y, z, \mathbf{n})$ is nonincreasing with respect to the second variable z on the set $\{(y, z, \mathbf{n}); y,\, z \in [-M, M], |\mathbf{n}| = 1\}$.

We consider the following *numerical schemes*: We set $u_h^0 = r_h u^0, g^k = g(\cdot, t_k)$. Provided u^k has beeen computed, we determine u^{k+1} such that either

$$\frac{1}{\tau_k}(u_h^{k+1} - u_h^k, v_h)_h + b_h(u_h^k, v_h) + a(u_h^{k+1}, v_h) = (g^{k+1}, v_h)_h \qquad \forall\, v_h \in V_h,\ t_k \in [0, T), \tag{1.14}$$

(semiimplicit scheme) or

$$\frac{1}{\tau_k}(u_h^{k+1} - u_h^k, v_h)_h + b_h(u_h^k, v_h) + a(u_h^k, v_h) = (g^k, v_h)_h \forall\, v_h \in V_h,\ t_k \in [0, T), \tag{1.15}$$

(purely explicit scheme). The function u_h^k is the approximate solution at time t_k.

In view of [Hac89], the above schemes can be completely derived via the FV approach. We speak however about a combined FV–FE method, because the convergence analysis and the discretization of viscous flow are based on the FE methodology.

1.3 Convergence

Let us consider a system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ $(h_0 > 0)$ of triangulations of the domain Ω, set $\tau = T/r$ for an integer $r > 1$ and define the partition of the interval $[0, T]$ formed by $t_k = k\tau,\ k = 0, \ldots, r$. Hence, $\tau_k = \tau$ for $k = 0, \ldots, r-1$.

We denote by h_T and σ_T the length of the longest side of the triangle $T \in \mathcal{T}_h$ and the radius of the largest circle inscribed in T, respectively, and set $h = \max_{T\in\mathcal{T}_h} h_T$. We introduce the functions $u_{h\tau},\ w_{h\tau} : (-\infty, +\infty) \to V_h$ associated with an approximate solution $\{u_h^k\}_{k=0}^r$:

$$u_{h\tau}(t) = u_h^0, \quad t \le 0, \quad u_{h\tau}(t) = u_h^k, \quad t \in (t_{k-1}, t_k], \quad k = 1, \ldots, r, \tag{1.16}$$
$$u_{h\tau}(t) = u_h^r, \quad t \ge T;$$

$$\begin{aligned} &w_{h\tau} \text{ is a continuous, piecewise linear mapping of } [0,T] \text{ into } V_h, \\ &w_{h\tau}(t_k) = u_h^k,\ k = 0,\dots,r, \quad w_{h\tau}(t) = 0 \quad \text{for} \quad t < 0 \text{ or } t > T. \end{aligned} \tag{1.17}$$

Our goal is to prove the convergence (in suitable spaces) of the mappings $u_{h\tau}$ and $w_{h\tau}$ (considered on $[0,T]$) to the exact solution u, if $h,\tau \to 0$ in an appropriate way. Here we summarize results from [FFL96] and [FS96].

By $c, c_1, ..., \hat{c}_1, \hat{c}_2, ..., c^*$ we denote constants independent of h, τ, ν, whilst $C, C_1, ...$ will denote constants that are independent of h, τ, but depend on ν.

Assumptions (A1). a) The system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ is *regular*, i.e., all angles of all $T \in \mathcal{T}_h$ are bounded from below by a positive constant independent of T and h.
b) The triangulations $\mathcal{T}_h$ are of *weakly acute type*. This means that the magnitude of all angles of all $T \in \mathcal{T}_h,\ h \in (0,h_0)$, is less than or equal to $\pi/2$.

In the convergence analysis we proceed in several steps.

I. L^∞-stability. With the aid of the above assumptions and discrete maximum principle ([Ike83]) we find that the inequalities

$$\|u^0\|_{L^\infty(\Omega)} \le \tilde{M}, \quad \|g\|_{L^\infty(Q_T)} \le \tilde{K}. \tag{1.18}$$

imply that

$$\|u_{h\tau}\|_{L^\infty(Q_T)}, \quad \|w_{h\tau}\|_{L^\infty(Q_T)} \le M = \tilde{M} + T\tilde{K}, \tag{1.19}$$

provided the following *stability conditions* are satisfied:

$$\tau c(M)|\partial D_i| \le |D_i|,\ i \in J, \tag{1.20}$$

in the case of the semiimplicit scheme (hence, $\tau = O(h)$);

$$\begin{aligned} \frac{1}{3}|T| - \tau c(M)|\partial D_i \cap T| - \frac{1}{4}\tau\nu h_T/\sigma_T \ge 0&, \\ T \in \mathcal{T}_h, P_i \in \mathcal{P}_h \cap T&, \end{aligned} \tag{1.21}$$

in the case of the explicit scheme (hence, $\tau = O(h^2/(h+\nu))$).

II. Consistency is represented by the estimates

$$\begin{aligned} &\text{a)} \quad \hat{c}_1\|v\|_{L^2(\Omega)} \le \|L_h v\|_{L^2(\Omega)} \le \hat{c}_2\|v\|_{L^2(\Omega)}, \quad v \in X_h, \\ &\text{b)} \quad \|v - L_h v\|_{L^2(\Omega)} \le c\,h\|v\|_{H^1(\Omega)}, \quad v \in X_h, \\ &\text{c)} \quad |(u,v) - (u,v)_h| \le c\,h^2\|u\|_{H^1(\Omega)}\,\|v\|_{H^1(\Omega)}, \quad u, v \in X_h, \\ &\text{d)} \quad |(g^k,v) - (g^k,v)_h| \le c\,h\|g^k\|_{W^{1,q}(\Omega)}\|v\|_{H^1(\Omega)}, \quad v \in V_h. \\ &\text{e)} \quad \textit{If } M > 0, \textit{ then there exist constants } \tilde{c} = \tilde{c}(M), \hat{c} = \hat{c}(M) \textit{ such that} \\ &\qquad |b(u,v) - b_h(u,v)| \le \tilde{c}\,h\|u\|_{H^1(\Omega)}\,\|v\|_{H^1(\Omega)}, \\ &\text{f)} \quad |b_h(u,v)| \le \hat{c}M|v|_{H^1(\Omega)}, \\ &\qquad\quad u \in V_h \cap L^\infty(\Omega),\ \|u\|_{L^\infty(\Omega)} \le M,\ v \in V_h,\ h \in (0,h_0). \end{aligned} \tag{1.22}$$

III. A priori estimates :

$$\|u_{h\tau}\|_{L^2(-1,T;V)}, \quad \|w_{h\tau}\|_{L^2(0,T;V)} \quad \le C, \quad \|u_{h\tau} - w_{h\tau}\|_{L^2(Q_T)} \le C\tau, \tag{1.23}$$

provided condition (1.20) is satisfied in the case of scheme (1.14), whereas in the case of scheme (1.15) in addition to condition (1.21), the *inverse assumption*

$$h \le c_1\, h_T, \quad T \in \mathcal{T}_h,\ h \in (0, h_0), \tag{1.24}$$

is satisfied and the *second stability condition* holds:

$$0 < \tau \le \frac{c_2 h^2}{\nu + h^2 + h}. \tag{1.25}$$

IV. Limit process. The above results, the application of the Fourier transform with respect to time combined with suitable compactness arguments and uniqueness of the exact solution imply the following result:

Theorem. *If the assumptions introduced above are satisfied, then*

$$\begin{array}{lll} u_{h\tau},\, w_{h\tau} \to u & \textit{weakly in} & L^2(0,T;V), \\ u_{h\tau},\, w_{h\tau} \to u & \textit{weak-}{*}\textit{ in} & L^\infty(Q_T), \\ u_{h\tau},\, w_{h\tau} \to u & \textit{strongly in} & L^2(Q_T), \end{array}$$

as $h,\ \tau \to 0$ *so that the stability conditions* (1.20) *or* (1.21) *together with* (1.24) *and* (1.25) *are satisfied in the case of schemes* (1.14) *or* (1.15) , *respectively, where* u *is the unique weak solution of problem* (1.1)–(1.3) (*i. e.,* u *satisfies* (1.7)).

1.4 Error Estimates

Let us assume that the exact solution of the continuous problem (1.7) is regular in the following sense:

$$\begin{aligned} & u \in L^\infty(0,\, T; H^{1+\tilde\varepsilon}(\Omega)), \\ & u' = \frac{\partial u}{\partial t} \in L^\infty(0,\, T; L^2(\Omega)), \\ & u'' = \frac{\partial^2 u}{\partial t^2} \in L^\infty(0,\, T; V^*) \end{aligned} \tag{1.26}$$

for some $\tilde\varepsilon \in (0,1]$. By V^* we denote the dual of V. For $\tilde\varepsilon \in (0,1)$, $H^{1+\tilde\varepsilon}(\Omega)$ is the Sobolev–Slobodetskii space of functions with "noninteger derivatives of order $1+\tilde\varepsilon$" (see [KJF77]). For piecewise polynomial approximations of such functions, see, e. g. [Fei89]. Assumptions (1.26) imply that

$$u \in C([0,T]; L^2(\Omega)), \quad u' \in C([0,T]; V^*). \tag{1.27}$$

In a similar manner to that of the previous Section we consider a system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ of triangulations of the domain Ω satisfying Assumptions (A1,a–b). Moreover, we use the inverse assumption (1.24).

If $\varphi \in V$, then by $P_h\,\varphi$ we denote the *Ritz projection* of φ:

$$\text{a)} \quad P_h\,\varphi \in V_h, \qquad \text{b)} \quad a(P_h\,\varphi - \varphi, v_h) = 0 \quad \forall\, v_h \in V_h. \tag{1.28}$$

Since Ω is a polygonal domain, the regularity results and duality technique imply the existence of such an $\varepsilon^* \in (0,1]$ that for $\varepsilon \in (0,\varepsilon^*]$ and $\varphi \in H^{1+\varepsilon}(\Omega) \cap V$ we have

$$\begin{aligned} |P_h\,\varphi|_{H^1(\Omega)} &\leq c|\varphi|_{H^1(\Omega)}, \\ \|\varphi - P_h\,\varphi\|_{L^2(\Omega)} &\leq c\,h^{1+\varepsilon}\|\varphi\|_{H^{1+\varepsilon}(\Omega)}, \\ \|\varphi - P_h\,\varphi\|_{H^1(\Omega)} &\leq c\,h^{\varepsilon}\|\varphi\|_{H^{1+\varepsilon}(\Omega)}, \qquad h \in (0,h_0). \end{aligned} \tag{1.29}$$

The error estimates for the combined finite volume – finite element method have been derived in several steps. We demonstrate the process for the semiimplicit scheme (1.14). The detailed analysis is carried out in [FFLW96].

I. Additional a priori estimate: There exists a constant $C > 0$ $(C = O(\nu^{-1}))$ such that

$$\begin{aligned} &\|u_h^k\|_{H^1(\Omega)} \leq C, \quad t_k \in [0,T], \\ &h \in (0,h_0), \tau > 0 \text{ satisfy the stability condition (1.20).} \end{aligned} \tag{1.30}$$

II. Truncation error. Let us set $u^k = u(\cdot,t_k), \hat{M} = \max\{\|u\|_{L^\infty(Q_T)}, M\}$. Assumptions (1.26) imply the following estimates:

$$\text{a)} \quad \left|(u^{k+1} - u^k, v) - \tau(u'(t_{k+1}), v)\right| \leq \tau^2\|v\|_{L^2(\Omega)}\|u''\|_{L^\infty(0,T;V^*)}, \quad v \in V, \tag{1.31}$$

$$\text{b)} \quad |b(u^{k+1}, v) - b(u^k, v)| \leq 2 \max_{\substack{\xi \in [-\hat{M},\hat{M}] \\ s=1,2}} |f_s'(\xi)|\, \|u^{k+1} - u^k\|_{L^2(\Omega)}\, |v|_{H^1(\Omega)}, \quad v \in V,$$

$$\text{c)} \quad \|u^{k+1} - u^k\|_{L^2(\Omega)} \leq \tau\|u'\|_{L^\infty(0,T;L^2(\Omega))},$$

$$\text{d)} \quad |b(u^{k+1}, v) - b(u^k, v)| \leq c\,\tau|v|_{H^1(\Omega)}, \quad v \in V.$$

Using estimates (1.22) and (1.30), we derive the following relation for the error of the method defined as $e^k = u_h^k - u^k$ for $t_k \in [0,T]$:

$$\begin{aligned} &(e^{k+1}, v_h) - (e^k, v_h) + \tau a(e^{k+1}, v_h)) = \\ = \;& -\tau\left[b(u_h^k, v_h) - b(u^k, v_h)\right] + \varepsilon_1(\tau; u, v_h) + \varepsilon_2(\tau, h; u_h^k, u_h^{k+1}, v_h), \\ & v_h \in V_h,\ t_k \in [0,T), \end{aligned} \tag{1.32}$$

where

$$\begin{aligned} |\varepsilon_1(\tau; u, v_h)| &\leq c\,\tau^2\|v_h\|_{H^1(\Omega)} \\ |\varepsilon_2(\tau, h; u_h^k, u_h^{k+1}, v_h)| &\leq c\|v_h\|_{H^1(\Omega)}\,[h^2(\|u_h^k\|_{H^1(\Omega)} + \|u_h^{k+1}\|_{H^1(\Omega)}) + \tau\,h]. \end{aligned} \tag{1.33}$$

Moreover, let us note that

$$\begin{aligned} &|b(\varphi_1, v) - b(\varphi_2, s)| \leq \tilde{c}(\hat{M})\,\|\varphi_1 - \varphi_2\|_{L^2(\Omega)}\,|v|_{H^1(\Omega)} \\ &\qquad \text{for } \varphi_i,\ v \in V,\ \|\varphi_i\|_{L^\infty(\Omega)} \leq \hat{M},\ i = 1,2. \end{aligned} \tag{1.34}$$

III. Error estimates. Substituting $v_h := P_h\, e^{k+1}$ in (1.32), assuming that $h = 0(\tau)$ and using (1.33) – (1.34) and Young's inequality, we find that

$$\begin{aligned} & \|e^{k+1}\|^2_{L^2(\Omega)} + \tau\,\nu |e^{k+1}|^2_{H^1(\Omega)} \le \\ \le\ & \left(1 + \frac{c\tau}{\nu}\right) \|e^k\|^2_{L^2(\Omega)} + \tau\,(\hat{C}_1 h^2 + \hat{C}_2 h^{2\varepsilon} + \hat{C}_3 h^{1+2\varepsilon}), \\ & t_k \in [0,T),\ h \in (0,h_0),\ \tau > 0 \text{ satisfy (1.20)}, \\ & \hat{C}_1 = O(\nu^{-3}),\ \hat{C}_2 = O(\nu),\ \hat{C}_3 = O(1). \end{aligned} \tag{1.35}$$

From this we obtain:

Theorem. *If $h \in (0,h_0)$ and $\tau > 0$ satisfy the conditions* (1.20) *and* $h = 0(\tau)$ *and we set* $\varepsilon = \min(\tilde{\varepsilon}, \varepsilon^*)$, *then*

$$\begin{aligned} & \max_{t_k\in[0,T]} \|e^k\|_{L^2(\Omega)} \ \le \\ \le\ & \left[C_1 h^{1+\varepsilon} + C_2 h + C_3 h^{\varepsilon} + C_4 h^{1/2+\varepsilon}\right] exp\left(\frac{cT}{\nu}\right), \end{aligned} \tag{1.36}$$

$$\begin{aligned} & \left\{\nu\tau \sum_{t_k\in[0,T]} |e^k|^2_{H^1(\Omega)}\right\}^{1/2} \ \le \\ \le\ & \left[C_1 h^{1+\varepsilon} + C_2 h + C_3 h^{\varepsilon} + C_4 h^{1/2+\varepsilon}\right] exp\left(\frac{cT}{\nu}\right) \nu^{-1/2}, \\ & C_1 = O(1),\ C_2 = O(\nu^{-1}),\ C_3 = O(\nu),\ C_4 = O(\nu^{1/2}). \end{aligned}$$

Remarks. a) The expressions $\max_{t_k\in[0,T]} \|e^k\|_{L^2(\Omega)}$ and $\left\{\nu\,\tau \sum_{t_k\in[0,T]} |e^k|^2_{H^1(\Omega)}\right\}^{1/2}$ are discrete analogues of the $L^\infty(0,T;L^2(\Omega))$–norm and of the energy norm, respectively, of the error. The above asymptotic error estimates are rather pessimistic for a very small diffusion parameter ν.
b) If the polygonal domain Ω is convex, then it is possible to set $\varepsilon^* = 1$. Assuming also $\tilde{\varepsilon} = 1$, we have $\varepsilon = 1$ in (1.36).
c) The above convergence results and error estimates can be extended to the case when $\Omega \subset I\!R^3$ is a bounded polyhedral domain and $p,\ q > 3$.
d) With the aid of the theory of finite element variational crimes developed by Feistauer and Ženíšek ([FŽ87], [FŽ88]), the theoretical analysis presented here can be generalized to the case of nonhomogeneous mixed Dirichlet–Neumann boundary conditions on a piecewise–smooth boundary $\partial\Omega$. Moreover, it is possible to consider a nonlinear diffusion term on the right hand side of equation (1.1) provided some assumptions on monotonicity or pseudomonotonicity are satisfied.

1.5 Schemes over Other Meshes

The combined FV – FE schemes for the discretization of the convection – diffusion problem (1.1) – (1.3) can also be applied on other types of meshes. Let us mention two possibilities.

I) *Triangular finite volumes – triangular finite elements.* The convection terms represented by the form b are discretized by the FV method on a mesh $\mathcal{D}_h = \{D_i\}_{i \in J}$, where D_i are triangles and $\mathcal{D}_h$ is a triangulation of the domain Ω with standard properties from the FE method. The same notation as in Section 2, where $\beta_{ij} := 1$, can be used. We consider $\mathcal{D}_h$ as a *primary mesh.* (For the application of triangular finite volumes to the solution of inviscid transonic flow, see, e.g., [Fel92], [FDF94], [FFL95], [FLWW95], [FD96].)

The diffusion term is discretized using conforming piecewise linear finite elements on a triangulation $\mathcal{T}_h$ of Ω, compatible with the primary grid $\mathcal{D}_h$ in the sense that the set of all vertices of the triangles $T \in \mathcal{T}_h$ consists of the barycentres P_i, $i \in J$, of all $D_i \in \mathcal{D}_h$ and the vertices of $D_i \in \mathcal{D}_h$ lying on $\partial\Omega$. We call the mesh $\mathcal{T}_h$ *adjoint* to the grid $\mathcal{D}_h$. See Fig. 2.

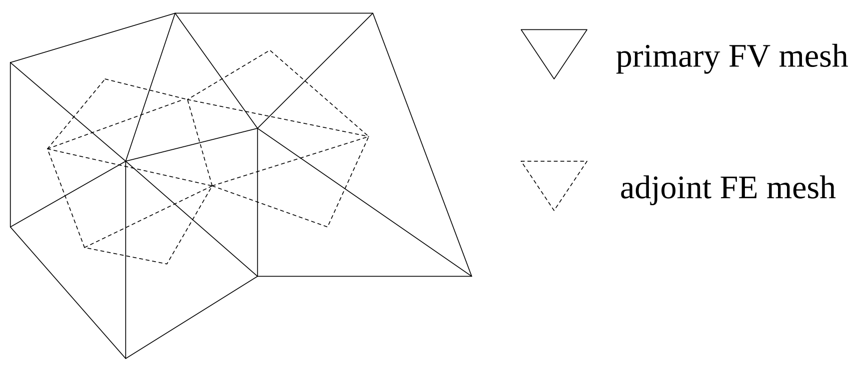

Figure 2 Adjoint finite elements $T \in \mathcal{T}_h$ to the primary finite volume mesh $\mathcal{D}_h$

In this case the spaces X_h, V_h and the lumping operator L_h are defined in the same way as in Section 2.

The convergence of this scheme remains open.

II) *Barycentric finite volumes – nonconforming finite elements.* Let $\mathcal{T}_h$ be a triangulation of the domain Ω. By $\{S_i; i \in J\}$ we denote the set of all sides of all $T \in \mathcal{T}_h$. (J is a suitable index set.) We construct the *barycentric finite volume mesh* $\mathcal{D}_h = \{D_i; i \in J\}$ over the primary mesh $\mathcal{T}_h$: Each side $S_i \not\subset \partial\Omega$ is associated with the quadrilateral D_i for which the boundary consists of segments connecting the endpoints of S_i with the barycentres of the triangles adjacent to the side S_i. For $S_i \subset \partial\Omega$, D_i is defined as the triangle with sides formed by the above segments and S_i. See Fig. 3.

The finite volume approximation of the exact solution u at time $t = t_k$ is sought as a piecewise constant function in Ω with constant values on D_i, $i \in J$, whereas the finite element approximation uses the Crouzeix–Raviart nonconforming finite elements (cf. [Fei93, Par. 8.9]), linear on each $T \in \mathcal{T}_h$ and continuous at midpoints of S_i, $i \in J$.

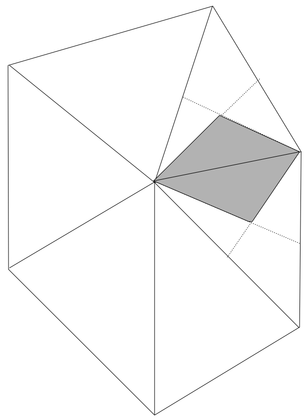

Figure 3 Barycentric finite volume over a triangular mesh

1.6 Numerical Solution of Compressible Transonic Viscous Flow

The complete system consisting of the continuity equation, the Navier-Stokes equations, the energy equation and the state equation, describing viscous compressible flow in a plane bounded domain Ω and time interval $(0,T)$ is written in the form

$$\frac{\partial w}{\partial t}+\sum_{s=1}^{2}\frac{\partial f_s(w)}{\partial x_s}=\sum_{s=1}^{2}\frac{\partial R_s(w,\nabla w)}{\partial x_s} \quad \text{in } Q_T=\Omega\times(0,T), \tag{1.37}$$

where

$$\begin{aligned}
&w=(\rho,\rho v_1,\rho v_2,e)^{\mathrm{T}}, \quad w=w(x,t), \quad x\in\Omega,\ t\in(0,T),\\
&f_s(w)=(\rho v_s,\rho v_s v_1+\delta_{s1}p,\rho v_s v_2+\delta_{s2}p,(e+p)\,v_s)^{\mathrm{T}},\\
&R_s(w)=(0,\tau_{s1},\tau_{s2},\tau_{s1}v_1+\tau_{s2}v_2+k\partial\theta/\partial x_s)^{\mathrm{T}},\\
&\tau_{ij}=\lambda\,\mathrm{div}\,\boldsymbol{v}\,\delta_{ij}+\mu(\partial v_i/\partial x_j+\partial v_j/\partial x_i),\\
&p=(\gamma-1)\,(e-\rho|\boldsymbol{v}|^2/2), \quad e=\rho(c_v\theta+|\boldsymbol{v}|^2/2).
\end{aligned} \tag{1.38}$$

We use the following notation: $t-$time, x_s $(s=1,2)-$Cartesian coordinates, $\rho-$density, $p-$pressure, $\theta-$absolute temperature, $e-$total energy, $\boldsymbol{v}=(v_1,v_2)-$velocity, $\delta_{ij}-$Kronecker delta, $\gamma>1$, c_v, k, $\mu>0-$given constants, $\lambda=-2\mu/3$.

The above system is equipped with *initial conditions* prescribing the state w on Ω at time $t=0$ and *boundary conditions*: At inlet we prescribe ρ, v_1, v_2 and set $\partial\theta/\partial n=0$, on fixed walls we assume that $v_1=v_2=0$, $\partial\theta/\partial n=0$ and at outlet we set $-pn_j+\sum_{i=1}^{2}\tau_{ij}n_i=0$, $j=1,2$, and $\partial\theta/\partial n=0$. (Here $\partial/\partial n$ is the derivative in the direction of the unit outer normal $\boldsymbol{n}=(n_1,n_2)$ to the boundary $\partial\Omega$.) In the solution of *flow through a cascade of profiles*, the domain Ω represents one period of the cascade. Then $\partial\Omega$ consists of the inlet, outlet, impermeable profile and artificial boundary formed by two arcs Γ^+ and Γ^- on which periodicity conditions are considered. (Cf., e.g., [FDF94], [FFL95].)

The above problem is discretized with the aid of the numerical schemes described in Sections 2 and 5. These schemes can be written in a similar form to (1.14) (semiimplicit scheme) or (1.15) (purely explicit scheme), where the form a is replaced by its approximation a_h obtained with the aid of numerical integration (cf. (1.42)). The form b_h (see (1.13)) is now defined with the aid of the numerical flux H of an *upwind flux vector splitting scheme of Godunov type* ([Fei93, Chap. 7]).

Another method, which has been applied with success, is the *inviscid–viscous operator splitting* ([FFD96]). In this case we split (1.37) into the inviscid system

$$\frac{\partial w}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(w)}{\partial x_s} = 0, \tag{1.39}$$

and the purely viscous system

$$\frac{\partial w}{\partial t} = \sum_{s=1}^{2} \frac{\partial R_s(w, \nabla w)}{\partial x_s}, \tag{1.40}$$

and discretize these separately.

The inviscid system (1.39) is discretized by the FV method on a mesh $\mathcal{D}_h = \{D_i\}_{i \in J}$, whereas the purely viscous system (1.40) is discretized using conforming piecewise linear finite elements on a triangulation $\mathcal{T}_h$ of Ω, as described above. One time step $t_k \to t_{k+1}$ is divided into two *fractional steps*:

Step I (*inviscid FV step on the mesh* $\mathcal{D}_h$): Assume that the values w_i^k, $i \in J$, approximating the solution on the finite volumes D_i at time t_k are known. Compute the values $w_i^{k+1/2}$, $i \in J$, from the FV formula

$$w_i^{k+1/2} = w_i^k - \frac{\tau_k}{|D_i|} \sum_{j \in S(i)} \sum_{\alpha=1}^{\beta_{ij}} H(w_i^k, w_j^k, \boldsymbol{n}_{ij}^{\alpha})\, \ell_{ij}^{\alpha} \tag{1.41}$$

equipped with inviscid boundary conditions. On $\Gamma^{\pm}$ periodicity conditions are considered. (Cf. [Fei93, Par. 7.3], [FFL95], [FDF94].)

Step II (*viscous FE step on the mesh* $\mathcal{T}_h$): Define the FE (i.e., continuous piecewise linear) approximation $w_h^{k+1/2}$ with values $w_h^{k+1/2}(P_i) = w_i^{k+1/2}$ at the vertices P_i, $i \in J$ of $\mathcal{T}_h$. On $\partial\Omega$, the viscous boundary conditions are used. Compute the FE approximation w_h^{k+1} as the solution of the following problem:

$$\begin{aligned} &\text{(i) } w_h^{k+1} \text{ satisfies the viscous Dirichlet boundary conditions,} \\ &\text{(ii)} (w_h^{k+1}, \varphi_h)_h = (w_h^{k+1/2}, \varphi_h)_h - \tau_k\, a_h(w_h^{k+1/2}, \varphi_h) \end{aligned} \tag{1.42}$$

for all test functions $\varphi_h = (\phi_1, \dots, \phi_4)$ such that ϕ_j $(j = 1, \dots, 4)$ is continuous in Ω, linear on each $T \in \mathcal{T}_h$ and vanishes on the part of $\partial\Omega$ where the j-th component w_j of the state vector w satisfies the Dirichlet boundary condition.

Now set $w_i^{k+1} := w_h^{k+1}(P_i)$ for $i \in J$, $k := k + 1$ and go to Step I.

In (1.42), $(w,\varphi)_h$ and $a_h(w,\varphi)$ denote the approximation of $\int_\Omega w\,\varphi\,\mathrm{d}x$ and $\int_\Omega \sum_{s=1}^2 R_s(w,\nabla w)\,\partial\varphi/\partial x_s\,\mathrm{d}x$, respectively.

The computation of viscous flow can be realized with the aid of a purely explicit scheme similar to (1.15) and the operator splitting scheme (1.41) – (1.42), because their algorithmization is simple. The following *stability conditions* are used:

$$\frac{\tau_k}{|D_i|}\,|\partial D_i| \max_{j\in S(i),\alpha=1,\dots,\beta_{ij}} \rho(\boldsymbol{P}(w_i^k,\boldsymbol{n}_{ij}^\alpha)) \le \mathrm{CFL} \approx 0.85, \quad i \in J, \tag{1.43}$$

$$\frac{3}{4}\,\frac{h_T}{\sigma_T}\,\frac{\tau_k}{|T|}\max(\mu,k) \le \mathrm{CFL}, \quad T \in \mathcal{T}_h, \tag{1.44}$$

where $\boldsymbol{P}(w,\boldsymbol{n}) = \sum_{s=1}^2 (D\,f_s(w)/Dw)\,n_s$, $\rho(\boldsymbol{P})$ = spectral radius of the matrix $\boldsymbol{P}$. Conditions (1.43) and (1.44) are obtained on the basis of linearization and in analogy with a scalar problem.

In order to increase the accuracy of the method, the term $H(w_i^k, w_j^k, \boldsymbol{n}_{ij}^\alpha)$ is replaced by $H(w_{ij}^{k,\alpha}, w_{ji}^{k,\alpha}, \boldsymbol{n}_{ij}^\alpha)$ where $w_{ij}^{k,\alpha}$ and $w_{ji}^{k,\alpha}$ are the values of the *"second order recovery"* of the piecewise constant FV solution combined with the use of a suitable flux limiter ([FFL95]).

The accuracy of the solution of transonic flow is also increased with the aid of *automatic adaptive mesh refinement* in the vicinity of shock waves, based on a shock indicator using divided differences of the density and taking into account the direction of the flow ([FDF94], [FFL95], [FD96]).

Computational results. The combined FV – FE method was applied to several test problems. The goal was to obtain steady viscous transonic flow via time stabilization for $t_k \to \infty$. The numerical fluxes by Steger – Warming ([SW81]), Van Leer ([Van75]), Vijayasundaram ([Vij86]) and Osher – Solomon ([OS82], [Spe88]) were used in our schemes applied on the types of meshes described above. The best results were obtained with aid of the method using the *Osher – Solomon numerical flux* (in the form described in [FFL95]) *applied on a primary triangular finite volume mesh* and combined with the *FE method on an adjoint triangulation.* Here we present two examples of the application of this method which were obtained in cooperation with V. Dolejší.

I) *Flow of air* ($\gamma = 1.4$) *through the GAMM channel* (10 % circular bump) with inlet Mach number = 0.67 and Reynolds number $\approx 10^5$. In Fig. 4 and Fig. 5, the primary FV and adjoint FE meshes are plotted. Fig. 6 shows the convergence history to the steady state measured in the L_1-norm and the density distribution on the walls. In Fig. 7, Mach number isolines are plotted. We obtained identical results with the aid of the purely explicit scheme and the inviscid-viscous operator splitting method.

II) *Flow past a cascade of profiles* was solved by the inviscid – viscous operator splitting method using the Osher – Solomon numerical flux, applied on a primary triangular FV mesh and an adjoint triangular FE mesh. The computational results are compared with a wind tunnel experiment (by courtesy of the Institute of Thermomechanics of the Czech Academy of Sciences in Prague, see [ŠŠ90]). The experiment and the computations were performed for the following data: angle of attack = $19^\circ\,18'$, inlet Mach number = 0.32, outlet Mach number = 1.18, $\gamma = 1.4$, Reynolds number $\approx 10^5$. (Inviscid computations were presented in [FD96], [FDF94], [FFL95].) Fig. 8 shows the primary FV mesh $\mathcal{D}_h$ and the adjoint FE mesh $\mathcal{T}_h$, refined a priori towards the profile and adaptively in the vicinity of the shock waves according to the algorithm described in [FDF94], [FFL95], [FD96]. In Fig. 9, the convergence history to the steady state is shown. Fig. 10, where density isolines are plotted and compared with an interferogram, indicates a very good agreement of computational and experimental results.

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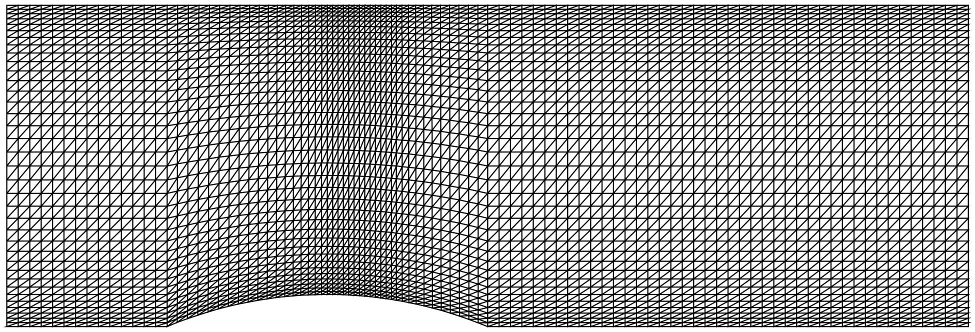

Figure 4 Primary FV mesh

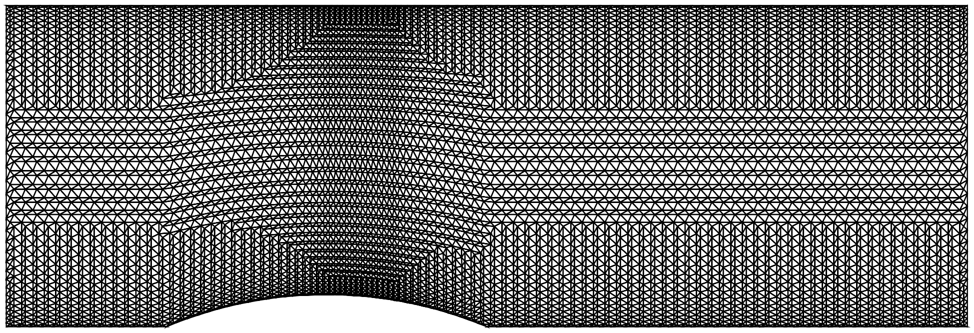

Figure 5 Adjoint FE mesh

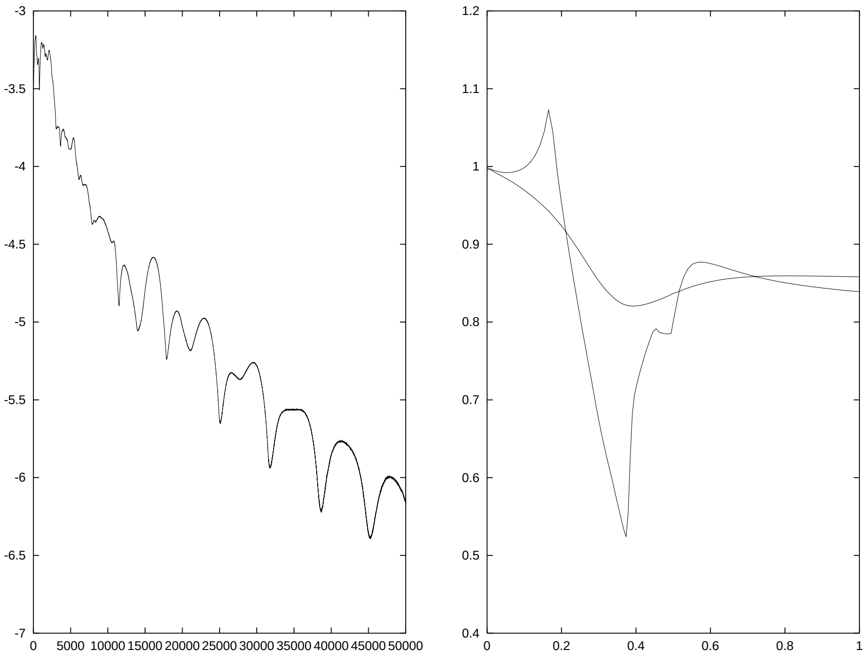

Figure 6 Convergence history ($\log\|(\rho^{k+1}-\rho^k)/\rho^k\|_{L^1(\Omega)}$ against the number of time steps); the density distribution on the walls against the relative arclength

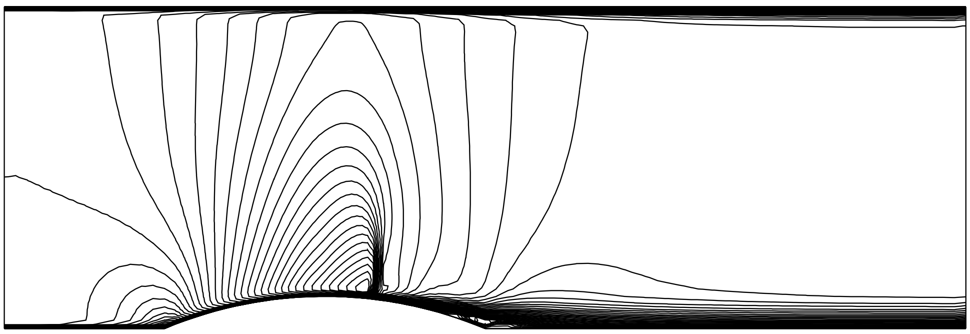

Figure 7 Mach number isolines

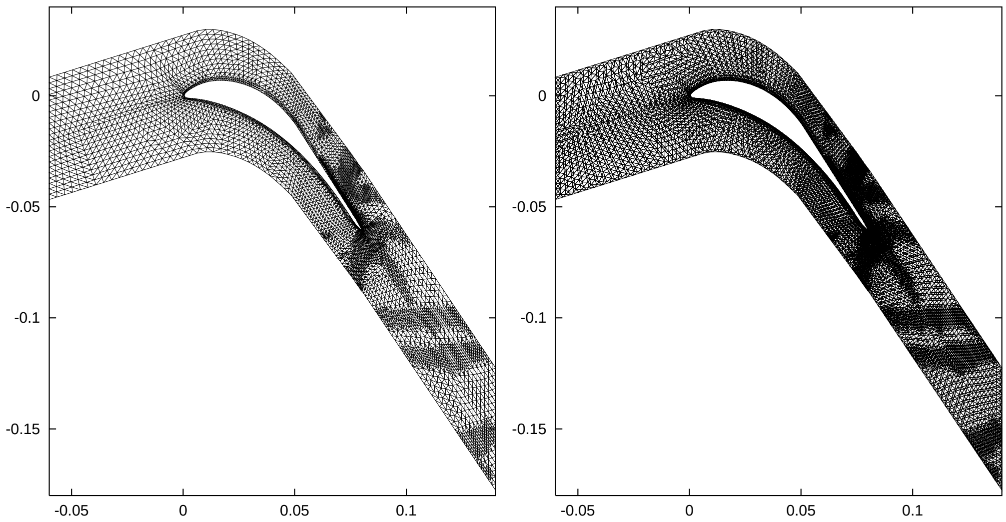

Figure 8 Basic and adjoint mesh

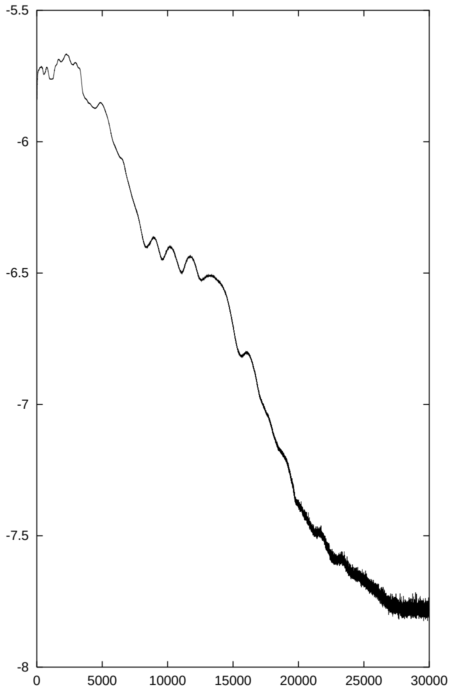

Figure 9 Convergence history

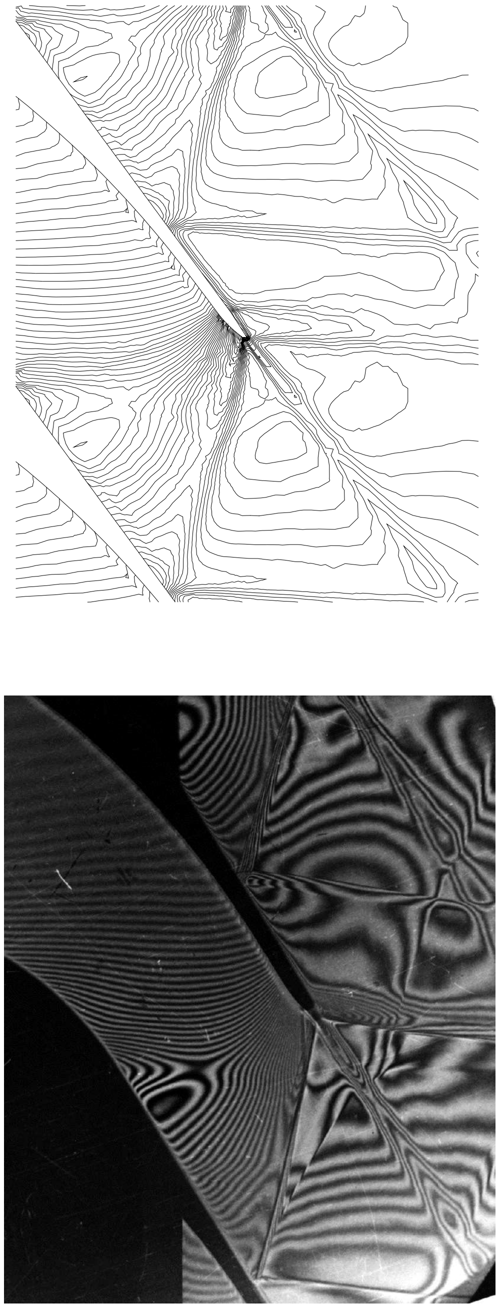

Figure 10 Density isolines compared with interferogram

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