The representation theory of the symmetric groups

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Representations of Symmetric groups

The symmetric group $\mathfrak{S}_n$ is the group of all permutations of $\{1, 2, \dots, n\}$

For $1 \le r \le n$ let $s_r = (r.r+1)$ be a simple transposition in $\mathfrak{S}_n$

Then $\{s_1, \dots, s_{n-1}\}$ is a set of Coxeter generators for $\mathfrak{S}_n$.
More precisely, $\mathfrak{S}_n$ is generated by $\{s_1, s_2, \dots, s_{n-1}\}$ with relations

$$s_r^2 = 1, s_r s_t = s_t s_r \text{ if } |r - t| > 1, \text{ and } s_r s_{r+1} s_r = s_{r+1} s_r s_{r+1}$$

These relations are neatly encoded by the Dynkin diagram for $\mathfrak{S}_n$:

In this talk I will describe some of highlights the representation theory of $\mathfrak{S}_n$ over an arbitrary field F

A representation of $\mathfrak{S}_n$ is a F-vector space V together with an action of $\mathfrak{S}_n$ or, equivalently, a F-algebra homomorphism $F\mathfrak{S}_n \to \mathrm{End}_F(V)$

A representation V is irreducible, or simple, if it contains no non-zero proper invariant submodules

Our aim

Understand the irreducible representations of $\mathfrak{S}_n$ over F

Composition factors

If $F = \mathbb{C}$ every $\mathbb{C}\mathfrak{S}_n$ can be written as a direct sum of irreducible $\mathbb{C}\mathfrak{S}_n$ modules. This is not always true for $F\mathfrak{S}_n$ modules if $\operatorname{char} F | n! = \#\mathfrak{S}_n$

A composition series for a (finite dimensional) module V is a filtration

$$V = V_0 \supset V_1 \supset V_2 \cdots \supset V_z = 0$$

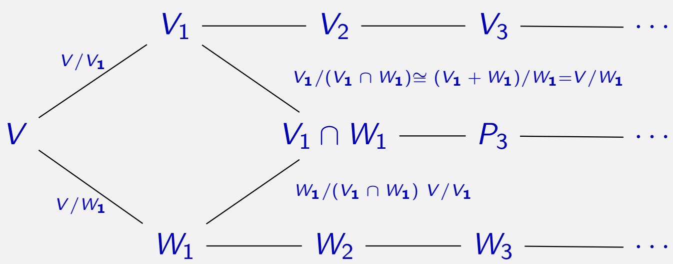
such that V_k / V_{k+1} is irreducible for all k

Jordan-Hölder Theorem

Let $V = V_0 \supset V_1 \cdots \supset V_y = 0$ and $V = W_0 \supset W_1 \cdots \supset W_z = 0$ be two composition series for V. Then $y = z$ and if S is simple then

$$[V{:}S] := \#\{\, k \mid S \cong V_k / V_{k+1} \,\} = \#\{\, k \mid S \cong W_k / W_{k-1} \,\}$$

Proof



An illustrative example of $\mathfrak{S}_n$ representations

Let F be a field and let $V = F^n = \{\, (\lambda_1, \dots, \lambda_n) \mid \lambda_r \in F \,\}$

The symmetric group $\mathfrak{S}_n$ acts on F by place permutations:

$$(\lambda_1, \dots, \lambda_n) \quad \overset{w}{\mapsto} \quad (\lambda_{w(1)}, \dots, \lambda_{w(n)})$$

Clearly $I = \{\, (\lambda, \lambda, \dots, \lambda) \mid \lambda \in F \,\}$ is an $\mathfrak{S}_n$-submodule

$\implies$ V is not irreducible

In fact, $V' = \{\, (\lambda_1, \dots, \lambda_n) \mid \sum_{r=1}^n \lambda_r = 0 \,\}$ is another $\mathfrak{S}_n$-submodule

$\implies$ $I \subseteq V'$ if and only if $\operatorname{char} F | n$ (i.e. $\operatorname{char} F$ divides n)

Now, V' has basis $\{(1, -1, 0, \dots, 0), (0, 1, -1, \dots, 0), \dots (0, \dots, 0, 1, -1)\}$

Fun exercise

Show that if $\operatorname{char} F \nmid n$ then $V = I \oplus V'$ and that these are the only irreducible submodules of V

On the other hand, if $n > 2$ and $\operatorname{char} F | n$ then V has composition series

$$V \supset V' \supset I \supset 0$$

with $V/V' \cong I$ and V'/I the irreducible composition factors of V

Conjugacy classes and partitions

Theorem

Let G be a finite group. Then the irreducible $\mathbb{C}G$-modules are in bijection with the conjugacy classes of G.

Two elements $g, h \in G$ are conjugate if $g = xhx^{-1}$ for some $x \in G$.

The conjugacy classes of $\mathfrak{S}_n$ are naturally indexed by cycle type

Example The decomposition of $\mathfrak{S}_3$ into conjugacy classes is given by

Cycle type	Conjugacy class
$(*)(*)(*)$	$\{\ 1\ \}$
$(*,*)(*)$	$\{\ (1,2),\ (2,3),\ (1,3)\ \}$
$(*,*,*)$	$\{\ (1,2,3),\ (1,3,2)\ \}$

$\Longrightarrow$ The conjugacy classes of $\mathfrak{S}_n$ have the form

$$\underbrace{(*,*,*,*,*,*,*)}_{\lambda_1}\underbrace{(*,*,*,*,*)}_{\lambda_2}\dots\underbrace{(\ *\ )}_{\lambda_k},$$

$\Longrightarrow$ The conjugacy classes of $\mathfrak{S}_n$, and irreducible $\mathbb{C}\mathfrak{S}_n$-modules, are indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq 0)$ with $\sum_r \lambda_r = n$.

Specht modules and Young diagrams

By the last slide, for each partition λ of n there should exist an irreducible $\mathbb{C}\mathfrak{S}_n$-module S^λ

The module S^λ is a Specht module for $\mathfrak{S}_n$.

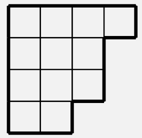
The best way to understand and to construct the Specht modules is by induction. More explicitly, we first describe

$$\operatorname{Res} S^\lambda \quad \text{and} \quad \operatorname{Ind} S^\lambda = S^\lambda \otimes_{\mathbb{C}\mathfrak{S}_n} \mathbb{C}\mathfrak{S}_{n+1},$$

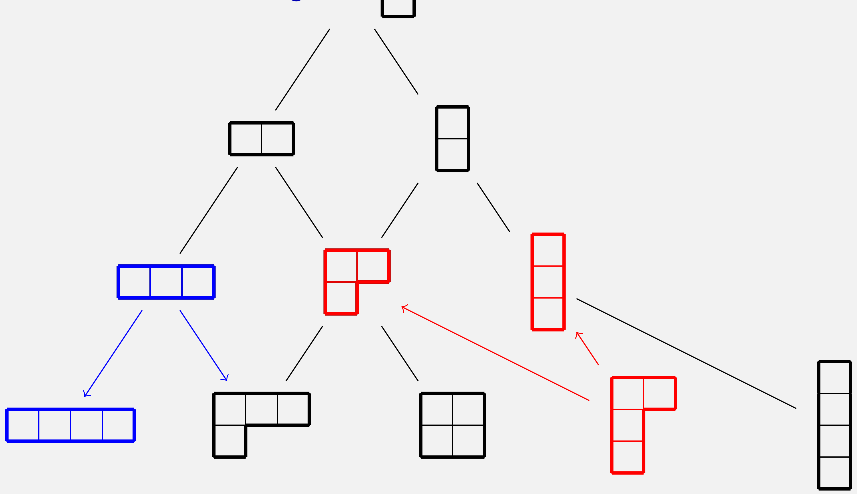
which are the restriction of S^λ to a $\mathbb{C}\mathfrak{S}_{n-1}$-module and the corresponding induced module for $\mathbb{C}\mathfrak{S}_{n+1}$, respectively

To do this, identify S^λ with diagram of λ: $\{\,(r,c)\,|\,1 \leq c \leq \lambda_r\,\}$, which we think of as an array of boxes in the plane.

Example The diagram of $(4,3,3,2)$ is

$S^\lambda \longleftrightarrow$ 

Branching rules and tableaux



Induction and restriction is described by the edges

$$\operatorname{Ind} S^{(3)} = S^{(4)} \oplus S^{(3,1)} \quad \text{and} \quad \operatorname{Res} S^{(3,1)} = S^{(3)} \oplus S^{(2,1)}$$

Multiplicity free restriction $\Longrightarrow$ Paths from $\square$ to λ index a basis of $\mathbb{S}^\lambda$

The Hook length formula

The paths in the branching graph are indexed by standard λ-tableaux:

$$\operatorname{Std}(3,1) = \left\{ \begin{array}{ccc}1&2&3\\4\end{array},\ \begin{array}{ccc}1&2&4\\3\end{array},\ \begin{array}{ccc}1&3&4\\2\end{array} \right\}$$

These are bijective labellings of the diagram of λ by $\{1,2,\dots,n\}$, which are strictly increasing along rows and down columns.

The (a,b)-hook length, h^λ_{ab}, the number of boxes to the left or right of the node (a,b) in the diagram of λ

Theorem (Frame, Robinson and Thrall, 1954)

Let λ be a partition. Then

$$\dim S^\lambda = \#\operatorname{Std}(\lambda) = \frac{n!}{\prod_{(a,b)\in[\lambda]} h^\lambda_{ab}}$$

Example The hook lengths in $\lambda = (4,3,2)$ are given by:

$$\begin{array}{cccc}6&5&3&1\\4&3&1\\2&1\end{array} \quad \Longrightarrow \quad \dim S^\lambda = \frac{9!}{6\cdot5\cdot3\cdot1\cdot4\cdot3\cdot1\cdot\cdot2\cdot1} = 168$$

The irreducible representations in characteristic zero

The content of the node $(a,b) \in [\lambda]$ is $c(a,b) = b - a$

Example If $\lambda = (4,3,2)$ then the contents in $[\lambda]$ are

0	1	2	3
−1	0	1	
−2	0		

If t is a standard tableau and $1 \le r \le n$ then the content of r in t is $c_r(t) = c(a,b)$, where r is in row a and column b of t

Theorem (Young's seminormal form)

Let λ be a partition. The Specht module S^λ has basis $\{ v_t \mid t \in \mathrm{Std}(\lambda) \}$ and where the $\mathfrak{S}_n$-action is determined by

$$s_r v_t = \frac{1}{\rho_r(t)} v_t + \frac{\rho_r(t)+1}{\rho_r} v_{s_r t},$$

where $\rho_r(t) = c_{r+1}(t) - c_r(t)$ and $v_{s_r t} = 0$ if $s_r t \notin \mathrm{Std}(\lambda)$

Proof Direct verification — almost!

Note The seminormal form is defined only if $\mathrm{char}\, F > n$

The Gelfand-Zetlin subalgebra

Define Jucys-Murphy elements, $L_k = \sum_{j=1}^{k} (1,j) \in \mathbb{Q}\mathfrak{S}_n$, for $1 \le k \le n$

$\Longrightarrow$ $L_k L_m = L_m L_k$, for all $1 \le k, m \le n$

The Gelfand-Zetlin subalgebra of $\mathbb{Q}\mathfrak{S}_n$ is $\mathrm{GZ}_n = \langle L_1, L_2, \dots, L_n \rangle$.

The content of k in t is $c_k(t) = c - r$ if k is in row r and column c.

The basis $\{ v_t \mid t \in \mathrm{Std}(\lambda) \}$ of S^λ, and the action of $\mathfrak{S}_n$, is (almost) uniquely determined by being a basis of simultaneous eigenvectors for GZ_n:

$$L_r v_t = c_r(t) v_t, \qquad \text{for } 1 \le r \le n,\ t \in \mathrm{Std}(\lambda)$$

Define $$F_t = \prod_{r=1}^{n} \prod_{c_r(s) \ne c_r(t)} \frac{L_k - c_r(s)}{c_r(t) - c_r(s)} \quad \in \mathrm{GZ}_n \subset \mathbb{Q}\mathfrak{S}_n$$

$\Longrightarrow$ $\{ F_t \mid t \text{ standard} \}$ is a complete set of pairwise orthogonal primitive idempotents and $S^\lambda \cong \mathbb{Q}\mathfrak{S}_n F_t$, for $t \in \mathrm{Std}(\lambda)$

$\Longrightarrow$ $F_s \mathbb{Q}\mathfrak{S}_n F_t = \mathbb{Q} f_{st}$ is a one dimensional $(\mathrm{GZ}_n, \mathrm{GZ}_n)$-bimodule with GZ_n-action $L_r f_{st} = c_r(s) f_{st}$ and $f_{st} l_r = c_r(t) f_{st}$

Note The idempotent F_t is defined only if $\mathrm{char}\, F > n$

Wedderburn decomposition

Let $\mathcal{P}_n^\Lambda = \{\text{partitions of } n\}$ and $d_\lambda = \#\,\mathrm{Std}(\lambda)$

Let $e_{st} \in \mathrm{End}_{\mathbb{Q}}(S^\lambda) \cong \mathrm{Mat}_{d_\lambda}(\mathbb{Q})$ be the matrix unit given by

$$e_{st}(v_u) = \delta_{tu} e_s, \quad \text{for } s, t \in \mathrm{Std}(\lambda)$$

Theorem

There is an algebra isomorphism,

$$\mathbb{Q}\mathfrak{S}_n \cong \bigoplus_{\lambda \in \mathcal{P}_n^\Lambda} \mathrm{Mat}_{d_\lambda}(\mathbb{Q});\ f_{st} \mapsto \gamma_t e_{st},$$

for (known) scalars $\gamma_t \in \mathbb{Q}$.

In particular, GZ_n maps to the subalgebra of diagonal matrices, so it is a maximal commutative algebra

Proof Follows directly from the seminormal form

It is easy to derive a recursive formula, the Murnaghan-Nakayama rule, for the characters of the Specht modules from the seminormal form

Representation theory in positive characteristic

The Specht module has a $\mathbb{Z}$-lattice $S^\lambda_{\mathbb{Z}}$ such that $S^\lambda_{\mathbb{Q}} = S^\lambda_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let F be a field of characteristic p.

Define the Specht module $S^\lambda_F = F \otimes_{\mathbb{Z}} S^\lambda_{\mathbb{Z}}$, an $F\mathfrak{S}_n$-module

In general, S^λ_F is not irreducible as S^λ_F carries a bilinear form $\langle\ ,\ \rangle$ and

$$\mathrm{rad}\, S^\lambda_F = \{ x \in S^\lambda_F \mid \langle x, y \rangle = 0 \text{ for all } y \in S^\lambda_F \}$$

is a $F\mathfrak{S}_n$-submodule which is often proper. Define $D^\mu_F = S^\mu / \mathrm{rad}\, S^\mu_F$.

A partition μ is p-restricted if $\mu_r - \mu_{r+1} < p$ for all r.

Let $\mathcal{K}^p_n$ be the set of p-restricted partitions of n

Theorem (James, 1978)

Suppose that μ is a p-restricted partition of n. Then D^μ_F is an absolutely irreducible $F\mathfrak{S}_n$-module. Moreover, $\{ D^\mu_F \mid \mu \in \mathcal{K}^p_n \}$ is a complete set of pairwise non-isomorphic irreducible $F\mathfrak{S}_n$-modules.

Main open questions

1. What is the dimension of D^μ_F?
2. What is the decomposition multiplicity $[S^\lambda : D^\mu]$?

Cyclotomic quiver Hecke algebras

As an algebra, $F\mathfrak{S}_n$ is generated by $s_1,\dots,s_{n-1}$ subject to the relations

$$s_r^2=1,\ s_rs_{r+1}s_r=s_{r+1}s_rs_{r+1},\ s_rs_t=s_ts_r \text{ if } |r-t|>1$$

Let $I=\mathbb{Z}/p\mathbb{Z}$ and let $\{\,\alpha_i\,|\,i\in I\,\}$ and $\{\,\Lambda_i\,|\,i\in I\,\}$ be the simple and fundamental roots for the Kac-Moody algebra $U(\widehat{\mathfrak{sl}}_p)$.

Definition (Khovanov-Lauda,Rouquier,Brundan-Kleshchev)

The cyclotomic quiver Hecke algebra $\mathcal{R}_n^{\Lambda_0}$ is the associative algebra with generators $\psi_1,\dots,\psi_{n-1},y_1,\dots,y_n,1_{\mathbf{i}}$, for $\mathbf{i}\in I^n$, and relations

$$y_1^{(\Lambda,\alpha_{i_1})}1_{\mathbf{i}}=0,\quad 1_{\mathbf{i}}1_{\mathbf{j}}=\delta_{\mathbf{i},\mathbf{j}}1_{\mathbf{i}}\quad \textstyle\sum_{\mathbf{i}\in I^n}1_{\mathbf{i}}=1\quad y_r1_{\mathbf{i}}=1_{\mathbf{i}}y_r$$
$$\psi_r1_{\mathbf{i}}=1_{s_r\cdot\mathbf{i}}\psi_r\qquad y_ry_s=y_sy_r\qquad \psi_ry_s=y_s\psi_r \text{ if } s\neq r,r+1,$$
$$\psi_ry_{r+1}1_{\mathbf{i}}=(y_r\psi_r+\delta_{i_r,i_{r+1}})1_{\mathbf{i}}\qquad y_{r+1}\psi_r1_{\mathbf{i}}=(\psi_ry_r+\delta_{i_r,i_{r+1}})1_{\mathbf{i}}$$
$$\psi_r^21_{\mathbf{i}}=Q_{i_r,i_{r+1}}(y_r,y_{r+1})\qquad \psi_r\psi_s=\psi_s\psi_r\quad \text{if } |r-s|>1,$$
$$(\psi_r\psi_{r+1}\psi_r-\psi_{r+1}\psi_r\psi_{r+1})1_{\mathbf{i}}=\delta_{i_r,i_{r+2}}\frac{Q_{i_r,i_{r+1}}(y_{r+2},y_{r+1})-Q_{i_r,i_{r+1}}(y_r,y_{r+1})}{y_r-y_{r+2}}1_{\mathbf{i}}$$

One nice feature of these relations is that $\mathcal{R}_n^{\Lambda_0}$ admits a $\mathbb{Z}$-grading via

$$\deg 1_{\mathbf{i}}=0,\qquad \deg y_r=2\qquad\text{and}\qquad \deg\psi_r1_{\mathbf{i}}=(\alpha_{i_r},\alpha_{i_{r+1}})$$

The Brundan-Kleshchev isomorphism theorem

Graded Isomorphism Theorem (Brundan-Kleshchev, 2008)

Let F be a field of characteristic p. Then $F\mathfrak{S}_n\cong\mathcal{R}_n^{\Lambda_0}$.

Corollary

The algebra $F\mathfrak{S}_n$ is $\mathbb{Z}$-graded.

Idea of proof: Construct explicit maps $\mathcal{R}_n^{\Lambda_0}\longrightarrow F\mathfrak{S}_n$ and $F\mathfrak{S}_n\longrightarrow\mathcal{R}_n^{\Lambda_0}$ and then check all of the relations. This is ugly!

- L_r acts on the Specht modules, and on $F\mathfrak{S}_n$, via matrices

$$L_r=\begin{pmatrix}c_r(\mathsf{t}_1)&*&\cdots&*\\0&c_r(\mathsf{t}_2)&\cdots&*\\\vdots&\ddots&\ddots&\vdots\\0&\cdots&0&c_r(\mathsf{t}_z)\end{pmatrix}\text{ and } y_r\mapsto\begin{pmatrix}0&*&\cdots&*\\0&0&\cdots&*\\\vdots&\ddots&\ddots&\vdots\\0&\cdots&0&0\end{pmatrix}$$

- Define $\mathrm{res}(\mathsf{t})=\mathbf{i}\in I^n$, where $i_r=c_r(\mathsf{t})+p\mathbb{Z}$. Then
  $1_{\mathbf{i}}\mapsto\sum_{\mathsf{t}}F_{\mathsf{t}}$, where the sum is over standard t with $\mathrm{res}(\mathsf{t})=\mathbf{i}$
- ψ_r is an intertwiner for these idempotents: $\psi_r1_{\mathbf{i}}=1_{s_r\cdot\mathbf{i}}\psi_r$.

A graded cellular basis of $F\mathfrak{S}_n$

The KLR generators of $\mathcal{R}_n^{\Lambda_0}$, which induce its grading, are

$$\psi_1,\dots,\psi_{n-1},\quad y_1,\dots,y_n,\quad 1_{\mathbf{i}},\qquad\text{for } \mathbf{i}\in I^n=(\mathbb{Z}/e\mathbb{Z})^n.$$

Theorem (Hu-M.)

Suppose that F is a field, Then $F\mathfrak{S}_n$ is a graded cellular algebra with graded cellular basis $\{\,\psi_{\mathsf{st}}\,|\,\mathsf{s},\mathsf{t}\in\mathrm{Std}(\lambda)$ and $\lambda\in\mathcal{P}_n^\Lambda\,\}$.

Example Take $e=3$ and $\lambda=(7,5,3)$.
The initial tableau t^λ and the residues in λ are:

$\mathsf{t}^\lambda=$

1	2	3	4	5	6	7
8	9	10	11	12		
13	14	15				

and

0	1	2	0	1	2	0
2	0	1	2	0		
1	2	0				

Then $\psi_{\mathsf{t}^\lambda\mathsf{t}^\lambda}=1_{\mathbf{i}^\lambda}y^\lambda$, where

$\mathbf{i}^\lambda=\mathrm{res}(\mathsf{t}^\lambda)=(0,1,2,0,1,2,0,2,0,1,2,0,1,2,0)$ and

$y_\lambda=y_3y_6y_{10}y_{15}$

In general, $\psi_{\mathsf{st}}=\psi_{d(\mathsf{s})^{-1}}1_{\mathbf{i}^\lambda}y^\lambda\psi_{d(\mathsf{t})}$, where $\mathsf{s}=\mathsf{t}^\lambda d(\mathsf{s})$ and $\mathsf{t}=\mathsf{t}^\lambda d(\mathsf{t})$.

Graded Specht modules – cellular algebras

One of the main properties of a cellular basis is that

$$h\psi_{\mathsf{sv}}=\sum_{\mathsf{a}\in\mathrm{Std}(\lambda)}r_{\mathsf{sa}}(h)\psi_{\mathsf{av}}\qquad(\text{mod higher shapes})$$

The graded Specht module $\mathbb{S}^\lambda$ has basis $\{\,\psi_{\mathsf{t}}\,|\,\mathsf{t}\in\mathrm{Std}(\lambda)\,\}$ and $\mathcal{R}_n^{\Lambda_0}$-action

$$h\psi_{\mathsf{s}}=\sum_{\mathsf{a}\in\mathrm{Std}(\lambda)}r_{\mathsf{sa}}(h)\psi_{\mathsf{a}}$$

Importantly, $\mathbb{S}^\lambda$ has a natural homogeneous bilinear form $\langle\ ,\ \rangle$

Consider: $\psi_{\mathsf{st}}\psi_{\mathsf{uv}}=\langle\psi_{\mathsf{t}},\psi_{\mathsf{u}}\rangle\psi_{\mathsf{sv}}$

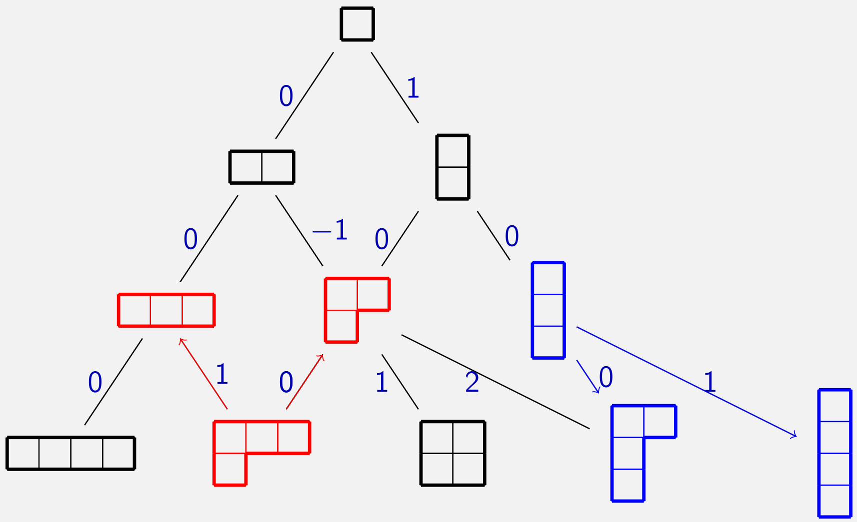
$\Longrightarrow$ $\mathrm{rad}\,\mathbb{S}^\lambda=\{\,x\in\mathbb{S}^\lambda\,|\,\langle x,y\rangle=0 \text{ for all } y\in\mathbb{S}^\lambda\,\}$ is a graded submodule of $\mathbb{S}^\lambda$ as $\langle xh,y\rangle=\langle x,yh^*\rangle$ is homogeneous

Define $\mathbb{D}^\mu=\mathbb{S}^\mu/\mathrm{rad}\,\mathbb{S}^\mu$, a graded quotient of $\mathbb{S}^\lambda$

Theorem (Brundan-Kleshchev, Hu-M.)

Over a field, $\{\,\mathbb{D}^\mu\langle k\rangle\,|\,\mu\in\mathcal{K}_n^p$ and $k\in\mathbb{Z}\,\}$ is a complete set of pairwise non-isomorphic irreducible $F\mathfrak{S}_n$-modules. Moreover, $(\mathbb{D}^\mu)^\circledast\cong\mathbb{D}^\mu$.

Graded branching rules and tableaux degrees



$\Longrightarrow$ $[\mathrm{Res}\,\mathbb{S}^{(3,1)}] = q[\mathbb{S}^{(3)}] + [\mathbb{S}^{(2,1)}]$ and $[\mathrm{Ind}\,\mathbb{S}_{(1^3)}] = [\mathbb{S}_{(2,1^2)}] + q[\mathbb{S}_{(1^4)}]$

Paths still index a basis $\Longrightarrow$ $\dim_q \mathbb{S}^{(3,1)} = q + q^{-1} + q$

$\dim_q \mathbb{D}^{(3,1)} = q + q^{-1}$

Graded branching rules and Fock spaces

Let $Rep(\mathcal{R}_n^{\Lambda_0})$ be the Grothendieck group of finitely generated graded $\mathcal{R}_n^{\Lambda_0}$-modules and let $Proj(\mathcal{R}_n^{\Lambda_0})$ be the split Grothendieck group. These are both naturally free $\mathbb{Z}[q, q^{-1}]$-modules with $[M\langle 1\rangle] = q[M]$

Both Grothendieck groups come with natural bases:

- $Rep(\mathcal{R}_n^{\Lambda_0})$: $\{\,[\mathbb{D}^\mu] \mid \mu \in \mathcal{K}_n^p\,\}$ and $\{\,[\mathbb{S}^\lambda] \mid \lambda \in \mathcal{K}_n^p\,\}$
- $Proj(\mathcal{R}_n^{\Lambda_0})$: $\{\,[\mathbb{P}^\mu] \mid \mu \in \mathcal{K}_n^p\,\}$ and $\{\,[\mathbb{S}^\lambda] \mid \lambda \in \mathcal{K}_n^p\,\}$

Induction, restriction induce maps on $\bigoplus_n Rep(\mathcal{R}_n^{\Lambda_0})$ and $\bigoplus_n Proj(\mathcal{R}_n^{\Lambda_0})$:

$$E_i[S^\lambda] = \textstyle\sum_A q^{d_A(\lambda)}[S^{\lambda-\{A\}}] \quad \text{and} \quad F_i[S^\lambda] = \textstyle\sum_A q^{-d^A(\lambda)}[S^{\lambda\cup\{A\}}]$$

Theorem (Hayashi, Miswa-Miwa, Lascoux-Leclerc-Thibon, ...)

For any field F, there is an isomorphism of integrable highest weight modules for the quantum group $U_q(\widehat{\mathfrak{sl}}_p)$:

$$L(\Lambda_0) \cong \bigoplus_n Proj(\mathcal{R}_n^{\Lambda_0}) \quad \text{and} \quad L(\Lambda_0)^* \cong \bigoplus_n Rep(\mathcal{R}_n^{\Lambda_0})$$

This is more tedious than difficult: it follows by carefully checking that graded induction and restriction satisfy the $U_q(\widehat{\mathfrak{sl}}_p)$-relations

Graded decomposition numbers

We now have graded Specht modules $\mathbb{S}^\lambda$ and graded simple modules $\mathbb{D}^\mu$

The corresponding graded decomposition number is

$$[\mathbb{S}^\lambda : \mathbb{D}^\mu]_q = \textstyle\sum_{k\in\mathbb{Z}} [\mathbb{S}^\lambda : \mathbb{D}^\mu\langle k\rangle]\, q^k \in \mathbb{N}[q, q^{-1}]$$

Theorem (Ariki (1996) + Brundan-Kleshchev (2009))

Suppose that $F = \mathbb{C}$. Then, in the categorification of the $U_q(\widehat{\mathfrak{sl}}_p)$-modules $L(\Lambda_0)$ and $L(\Lambda_0)^*$ by cyclotomic quiver Hecke algebras,

- $[\mathbb{P}^\mu]$ *corresponds to the Lusztig-Kashiwara canonical basis*
- $[\mathbb{D}^\mu]$ *corresponds to the Lusztig-Kashiwara dual canonical basis.*

$\Longrightarrow$ $[\mathbb{S}^\lambda : \mathbb{D}^\mu]_q = [\mathbb{P}^\mu : \mathbb{S}^\lambda]_q \in \delta_{\lambda\mu} + q\mathbb{N}[q]$ is a parabolic Kazhdan-Lusztig polynomial

This gives an approximation to the decomposition matrices of $\mathfrak{S}_n$ because

$$\Big([S^\lambda : D^\mu]\Big) = \Big([\mathbb{S}^\lambda : \mathbb{D}^\mu]_{q=1}\Big)\begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ * & \cdots & * & 1 \end{pmatrix}$$

The parabolic Kazhdan-Lusztig polynomials are, in principle, easy to calculate, however, we do not have a good understanding of the adjustment matrices on the right-hand side

Categorification of highest weight modules

The categorification of $L(\Lambda_0)^*$ and $L(\Lambda_0)$ by the algebras $\mathcal{R}_n^{\Lambda_0}$ is extensive:

- Multiplication by q corresponds to the grading shift functor
- $E_i \leftrightarrow i$-Res and $F_i \leftrightarrow i$-Ind
- The weight spaces of $L(\Lambda)$ are the blocks of $\mathcal{R}_n^{\Lambda_0}$
- The Shapovalov form on $L(\Lambda)$ is the Cartan pairing on $Rep(\mathcal{R}_n^{\Lambda_0})$
- The standard basis of $L(\Lambda)$ corresponds to the graded Specht modules
- The costandard basis of $L(\Lambda)$ corresponds to the dual graded Specht modules
- The crystal graph gives the modular branching rules
- The action of the affine Weyl group corresponds to the derived equivalences of Chuang and Rouquier
- If $F = \mathbb{C}$ the dual canonical basis is the basis of irreducible modules
- If $F = \mathbb{C}$ the canonical basis is the basis of projective indecomposable modules