Numerical quadratures

Vít Dolejší

Charles University Prague
Faculty of Mathematics and Physics

Quiz # 1

Question #1

Let us consider numerical quadrature

- $I(f) := \int_a^b f(x)\,\mathrm{d}x \approx Q(f) := \sum_{i=1}^n w_i f(x_i)$,
- $x_i,\ i = 1,\dots,n$ are the nodes, $w_i \in \mathbb{R},\ i = 1,\dots,n$ are the weights.
- it is exact for polynomial functions of degree $\le p$, i.e.

$$I(x^q) = Q(x^q) \text{ for } q = 0, 1, \dots, p.$$

What is the order of error of the corresponding composite formula Q_h with the step h?

(A) $|I(f) - Q_h(f)| = O(h^{p-1})$

(B) $|I(f) - Q_h(f)| = O(h^p)$

(C) $|I(f) - Q_h(f)| = O(h^{p+1})$

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Idea of proof: Taylor expansion on each sub-interval

$f(x) = \phi_p(x) + O(h^{p+1})$

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- $x_i,\ i = 1, \dots, n$ are the nodes, $w_i \in \mathbb{R},\ i = 1, \dots, n$ are the weights.

This numerical quadrature has order ≥ 1. Which of the following conditions are necessary? (Multiple answers are possible)

(A) $w_1 + w_2 + \cdots + w_n = 1$

(B) $w_i \geq 0$ for $i = 1, \dots, n$

(C) $a \leq x_i \leq b$ for $i = 1, \dots, n$

(D) $w_1 x_1 + w_2 x_2 + \cdots + w_n x_n = \frac{1}{2}$

$Q(f)$ is exact for $f = 1$ (A) and $f = x$ (D),
weights can be negative, nodes can be outside of interval.

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How are defined the Newton-Cotes formulas for the given n?

(A) The nodes and weights are chosen in such a way that the order of $Q(f)$ is the maximal possible.

(B) The nodes are chosen equidistantly and the weights are chosen in such a way that the order of $Q(f)$ is the maximal possible.

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Question #4

We integrate $\int_0^1 \exp(2\sqrt{x})\,\mathrm{d}x$ numerically by the composite midpoint formula and the composite trapezoid formula. We obtain the results

- $M_h(f) = 4.21$
- $T_h(f) = 4.24$

- What is the estimate of the error (EST) of these results?
- What is the results obtained by the Simpson rule $(S_h(f))$?

Outputs are two numbers.

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Answer

- $EST = \frac{1}{3}(M_h(f) - T_h(f)) = 0.01$ (estimate of the error of $M_h(f)$)
- $S_h(f) = \frac{1}{3}(2M_h(f) + T_h(f)) = 4.22$

Question #5

We integrate $\int_0^1 f(x)\,\mathrm{d}x$ numerically by the composite Simpson formula. We obtain the following results:

- for $h = 0.2$, $S_h(f) = 2.220$
- for $h = 0.1$, $S_h(f) = 2.234$

- What is the estimate of the error of the result with $h = 0.1$?

Output is one number.

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Answer

- Simpson formula has order $= 3$
- estimate of the error by the half-step size method is

$$EST = \frac{Q_h - Q_{h/2}}{2^{p+1} - 1} = \frac{2.234 - 2.220}{2^4 - 1} = \frac{0.014}{15} \approx 10^{-3}$$

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Question #7

Fill the following table which contains the order of the Newton-Cotes and Gauss formulas for n integration nodes.

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$n = 2$		
$n = 3$		
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$n = 3$	3	5
$n = 4$	3	7
$n = 5$	5	9
$n = 6$	5	11
$n = 7$	7	13

Question #8

Why the half-step size method can not be used for the estimation of the error of the Gauss quadrature?

(A) this method is unstable since the nodes of a Gauss quadrature are not distributed equidistantly,

(B) this method significantly over-estimates the error (it is not sufficiently accurate) since the error of the Gauss quadrature is too small due to its high order,

(C) it can be used but it is less efficient (it requires too many evaluation in integration nodes).

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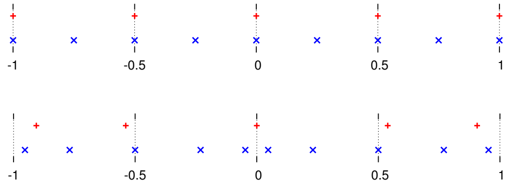
Newton-Cotes

Gauss



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- Which assertion about Gauss-Kronrod quadrature formulae is true?

(Multiple answers are possible)

(A) The pair of quadrature formulas where the Gauss quadrature G_n has order $2n-1$ and the Kronrod quadrature K_{2n+1} has order $3n+1$.

(B) The pair of quadrature formulas $G_n K_{2n+1}$ which is suitable for the estimation of the error of the Gauss quadrature.

(C) The quadrature formulas where the Gauss quadrature G_n is enhanced by additional $n+1$ nodes in such a way that the resulting formula has the maximal order of accuracy.

(D) The pair of quadrature formulas which are open (i.e., $a \neq x_i \neq b$, $i = 1, 2, 3 \ldots$) and the weight are irrational numbers in general.

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