IX.2 Measurable calculus and spectral decomposition for normal operators

**Proposition 12** (Lax-Milgram). Let $H$ be a Hilbert space and $B : H \times H \to \mathbb{C}$ a mapping satisfying the following properties.

- $x \mapsto B(x, y)$ is linear for each $y \in H$.
- $y \mapsto B(x, y)$ is conjugate linear for each $x \in H$.
- $\|B\| = \sup \{|B(x, y)|; x, y \in B_H\} < \infty$.

Then there is a unique $T \in L(H)$ such that $B(x, y) = \langle Tx, y \rangle$ for $h, k \in H$. Moreover, $\|T\| = \|B\|$.

**Constructing the spectral measure of a normal operator - Step 1.** Let $H$ be a Hilbert space and let $T \in L(H)$ be a normal operator. Let $f \mapsto \tilde{f}(T)$, $f \in C(\sigma(T))$, be the continuous functional calculus for $T$. For any $x, y \in H$ let $E_{x,y}$ denote the (unique) complex Radon measure on $\sigma(T)$ satisfying

$$\langle \tilde{f}(T)x, y \rangle = \int_{\sigma(T)} f \, dE_{x,y}, \quad f \in C(\sigma(T)).$$

**Proposition 13** (properties of the measures $E_{x,y}$). Using the above notation, the following holds:

- (a) $x \mapsto E_{x,y}$ is linear for each $y \in H$.
- (b) $y \mapsto E_{x,y}$ is conjugate linear for each $x \in H$.
- (c) $E_{x,x}$ is a non-negative measure for each $x \in H$.
- (d) $\|E_{x,y}\| \leq \|x\| \cdot \|y\|$ for $x, y \in H$.
- (e) $E_{x,y} = \frac{1}{4}(E_{x+y,x-y} - E_{x-y,x+y} + iE_{x+y,x+iy} - iE_{x-y,x-iy})$ for $x, y \in H$.

Measurable calculus and the spectral measure. We use the above notation.

- Denote by $A$ the $\sigma$-algebra of all the subsets of $\sigma(T)$ which are $E_{x,y}$-measurable for each $x, y \in H$. (Recall that $A$ is $E_{x,y}$-measurable if and only if there are Borel sets $B, C$ such that $B \subset A \subset C$ and $|E_{x,y}|(B \setminus C) = 0$.) Then $A$ is the $\sigma$-algebra of all the subsets of $\sigma(T)$ which are $E_{x,x}$-measurable for each $x \in H$.
- Let $f : \sigma(T) \to \mathbb{C}$ be a bounded $A$-measurable function By $\tilde{f}(T)$ we denote the operator in $L(H)$ satisfying

$$\langle \tilde{f}(T)x, y \rangle = \int_{\sigma(T)} f \, dE_{x,y}, \quad x, y \in H.$$ 

Its existence and uniqueness is provided by Proposition 12. The assignment $f \mapsto \tilde{f}(T)$ is called the measurable calculus for $T$.

- For $A \in A$ set $E_T(A) = \chi_A(T)$. The assignment $E_T : A \mapsto E_T(A)$ is called the spectral measure of $T$.
- Denote by $\mathcal{N}$ the subfamily of $A$ formed by the sets which are $|E_{x,y}|$-null for each $x, y \in H$. $\mathcal{N}$ is the family of all the sets which are $E_{x,x}$-null for each $x \in H$.
- Denote by $L^\infty(E_T)$ the space of all the bounded $A$-measurable functions on $\sigma(T)$, where we identify the functions which are equal everywhere except on a set from $\mathcal{N}$. Equip $L^\infty(E_T)$ with the norm

$$\|f\| = \text{ess sup}_{\lambda \in \sigma(T)} |f(\lambda)| = \inf \{c > 0; \lambda \in \sigma(T); f(\lambda) > c\} \in \mathbb{N}.$$ 

Then $L^\infty(E_T)$ is a commutative $C^*$-algebra (with the pointwise multiplication and the involution defined as the complex conjugation).
- $\tilde{f}(T)$ is defined exactly for $f \in L^\infty(E_T)$. Moreover, $\tilde{f}(T)$ is then well defined, i.e., $\tilde{f}(T) = \tilde{g}(T)$ whenever $f = g$ except on a set from $\mathcal{N}$.

**Lemma 14** (a consequence of Luzin’s theorem).

- (a) Let $K$ be a compact metric space and let $\mu$ be a non-negative finite Borel measure on $K$. Let $f : K \to \mathbb{C}$ be a bounded $\mu$-measurable function. Then there is a uniformly bounded sequence $(f_n)$ in $C(K)$ such that $f_n \to f$ $\mu$-almost everywhere. In particular, there is a bounded Borel function $g$ on $\sigma(T)$ such that $f = g$ $\mu$-almost everywhere.
- (g) Let $H$ be a separable Hilbert space and let $T \in L(H)$ be a normal operator. Let $f \in L^\infty(E_T)$ Then there is a uniformly bounded sequence $(f_n)$ in $C(\sigma(T))$ such that $f_n \to f$ except on a set from $\mathcal{N}$. In particular, there exists a bounded Borel function $g$ on $\sigma(T)$ such that $f = g$ except on a set form $\mathcal{N}$. 

Theorem 15 (properties of the measurable calculus). Let $H$ be a Hilbert space and $T \in L(H)$ be a normal operator.

(a) $f \mapsto \hat{f}(T)$ is an isometric $*$-isomorphism of $L^\infty(E)$ into $L(H)$.
(b) If $(f_n)$ is a bounded sequence in $L^\infty(E)$ which pointwise converges to a function $f$ (except on a set from $N$), then $f \in L^\infty(E)$ and, moreover, 
\[
\left\langle \hat{f}_n(T)x, y \right\rangle \to \left\langle \hat{f}(T)x, y \right\rangle, \quad x, y \in H.
\]
(c) $\sigma(\hat{f}(T)) = \text{ess \, rng}(f) = \{\lambda \in \mathbb{C}; \forall r > 0 : f^{-1}(U(\lambda, r)) \notin N\}$ for $f \in L^\infty(E)$.
(d) $\hat{f}(T)$ is a normal operator for each $f \in L^\infty(E)$. $\hat{f}(T)$ is self-adjoint if and only if $f$ is essentially real-valued (i.e., $f(\lambda) \in \mathbb{R}$ except on a set from $N$).
(e) $\hat{g}(\hat{f}(T)) = g \circ \hat{f}(T)$ whenever $f \in L^\infty(E)$ and $g$ is continuous on $\sigma(\hat{f}(T))$ (see (c)).
(f) If $S \in L(H)$ commutes with $T$, then $S$ commutes with $\hat{f}(T)$ for each $f \in L^\infty(E)$.

Definition. An abstract spectral measure in a Hilbert space $H$ is a mapping $E$ with the following properties:
(i) The domain of $E$ is a $\sigma$-algebra $\mathcal{A}$ of subsets of $\mathbb{C}$ containing all the Borel sets.
(ii) $E(A)$ is an orthogonal projection on $H$ for each $A \in \mathcal{A}$.
(iii) $E(\emptyset) = 0$, $E(\mathbb{C}) = I$.
(iv) If $A \in \mathcal{A}$ satisfies $E(A) = 0$, then $B \in \mathcal{A}$ (and $E(B) = 0$) for each $B \subset A$.
(v) $E(A \cap B) = E(A)E(B)$ for $A, B \in \mathcal{A}$.
(vi) $E(A \cup B) = E(A) + E(B)$ whenever $A, B \in \mathcal{A}$, $A \cap B = \emptyset$.
(vii) For each pair $x, y \in H$ the mapping $E_{x,y} : A \mapsto (E(A)x, y)$ is a complex Borel measure on $\mathbb{C}$.

The spectral measure $E$ is called compactly supported if there is a compact set $K \subset \mathbb{C}$ such that $E(\mathbb{C} \setminus K) = 0$.

Recall that $\mu$ is a Borel measure if it is a $\sigma$-additive measure defined on a $\sigma$-algebra $\mathcal{A}_\mu$ containing all Borel sets such that for any $A \in \mathcal{A}_\mu$ there are Borel sets $B, C$ such that $B \subset A \subset C$ and $|\mu|(B \setminus C) = 0$.

Lemma 16. If $T \in L(H)$ is a normal operator, then $E_T$ is a compactly supported abstract spectral measure.

Proposition 17 (integral with respect to an abstract spectral measure). Let $E$ be an abstract spectral measure defined on a $\sigma$-algebra $\mathcal{A}$. Let $f : \mathbb{C} \to \mathbb{C}$ a bounded $\mathcal{A}$-measurable function. Then there is a unique $T \in L(H)$ such that 
\[
\langle Tx, y \rangle = \int f \, dE_{x,y}, \quad x, y \in H.
\]

Moreover, $\|T\| \leq \|f\|_\infty$.

Theorem 18. Let $E$ be an abstract spectral measure defined on a $\sigma$-algebra $\mathcal{A}$. Define $N$ and $L^\infty(E)$ in the same way as above (for $E_T$). Then the following holds:
(a) The mapping $\Psi : f \mapsto \int f \, dE$ is an isometric $*$-isomorphism of the $C^*$-algebra $L^\infty(E)$ into $L(H)$.
(b) For each $f \in L^\infty(E)$ the operator $\Psi(f)$ is normal. Moreover, $\Psi(f)$ is self-adjoint if and only if $f$ is real-valued except on a set from $N$ and $\Psi(f)$ is a positive operator if and only if $f \geq 0$ except on a set from $N$.
(c) $\|\Psi(f)x\| = \left(\int |f|^2 \, dE_{x,x}\right)^{\frac{1}{2}}$ for $f \in L^\infty(E)$ and $x \in H$.
(d) If $f \in L^\infty(E)$ and $g \in C(\sigma(\Psi(f)))$, then $\Psi(g \circ f) = \hat{g}(\Psi(f))$.

Lemma 19. Let $E$ be an abstract spectral measure, $f \in L^\infty(E)$ and $T = \int f \, dE$. Then the spectral measure $E_T$ of $T$ is defined by $E_T(A) = E(f^{-1}(A))$.

Corollary 20 (spectral decomposition of a normal operator). Let $H$ be a Hilbert space and $T \in L(H)$ a normal operator. Then there is a unique abstract spectral measure such that $T = \int \text{id} \, dE$. Moreover, this is the measure $E_T$.

Theorem 21. Let $H$ be a Hilbert space and $T \in L(H)$ a normal operator. Then there is a nonnegative measure $\mu$ (defined on some measurable space), a unitary operator $U : H \to L^2(\mu)$ and a function $g \in L^\infty(\mu)$ such that 
\[
Tx = U^*(g \cdot Ux), \quad x \in H.
\]

If $H$ is separable, $\mu$ can be chosen to be $\sigma$-finite.